Group theoretic conditions under which closed aspherical manifolds are covered by Euclidean space

Hanspeter Fischer and David G. Wright

September 15, 1999

Abstract

Hass, Rubinstein, and Scott showed that every closed aspherical (irreducible) 3-manifold whose fundamental group contains the fundamental group of a closed aspherical surface, is covered by Euclidean space. This theorem does not generalize to higher dimensions. However, we will prove variations of this theorem in all dimensions by presenting conditions on finitely presented groups that guarantee simple connectivity at infinity. The proofs that we give will all be geometric.

1. Introduction and Statement of Results

Given a closed aspherical manifold $M$, one is interested in conditions on its fundamental group which insure that $M$ is covered by Euclidean space.

Employing least area techniques, Hass, Rubinstein, and Scott [4] showed that this is the case when $M$ is a $P^2$-irreducible 3-manifold whose fundamental group contains a subgroup isomorphic to the fundamental group of a closed surface other than $S^2$ or $P^2$. It is a long-standing conjecture that all irreducible closed aspherical 3-manifolds are covered by Euclidean space.

Davis [1] constructed examples that answered the higher dimensional conjecture in the negative. In fact, Davis’ exotic manifolds illustrate that the Hass-Rubinstein-Scott Theorem does not generalize to higher dimensions. His open contractible manifolds (of any given dimension greater than three) are not homeomorphic to Euclidean space, although each of them covers a closed manifold $M$ whose fundamental group contains a subgroup isomorphic to the fundamental group of a closed codimension-one manifold $N$ which is covered by Euclidean space.

Theorem 1. Let \( 1 \to H \to G \to Q \to 1 \) be a short exact sequence of finitely presented infinite groups. If either \( H \) or \( Q \) is one-ended, then \( G \) is simply connected at infinity.

The universal covering space of a closed aspherical manifold is homeomorphic to Euclidean space if it is simply connected at infinity (provided we assume irreducibility if the manifold is 3-dimensional). Since this is the case precisely if its fundamental group is simply connected at infinity, Theorem 1 bears some relevance to our question. Namely it implies

Theorem 2. Let \( M \) be a closed aspherical \( n \)-manifold (irreducible if \( n = 3 \)). Suppose the fundamental group \( G \) of \( M \) has a normal subgroup \( H \) such that both \( H \) and \( G/H \) are infinite and that either \( H \) or \( G/H \) is one-ended. Then \( M \) is covered by Euclidean space.

We will present a geometric proof of the Houghton-Jackson Theorem. Developing the techniques to do so enabled us to prove the following theorems.

Theorem 3. Let \( M \) be a closed aspherical \( n \)-manifold (irreducible if \( n = 3 \)). Suppose the fundamental group of \( M \) contains a non-trivial cyclic normal subgroup. Then \( M \) is covered by Euclidean space.

We note that such a subgroup must, in fact, be infinite cyclic.

Combining Theorems 2 and 3 one obtains a Hass-Rubinstein-Scott-like result:

Theorem 4. Let \( N \) and \( M \) be closed aspherical manifolds of dimension \( k \) and \( n \), respectively (\( M \) irreducible if \( n = 3 \)). Assume that \( H_n(N; \mathbb{Z}_2) = 0 \) (for example if \( k < n \)). If \( \pi_1(N) \) is isomorphic to a normal subgroup of \( \pi_1(M) \), then \( M \) is covered by Euclidean space.

We will then relax the normality condition and prove

Theorem 5. Let \( H \) be a finitely presented subgroup of a finitely presented group \( G \). Suppose that the index of \( H \) in its normalizer \( N_G(H) \) in \( G \) is infinite. If both \( H \) and \( G \) are one-ended and the pair \((G, H)\) is two-ended, then \( G \) is simply connected at infinity.

Applied to the setting of aspherical manifolds, Theorem 5 implies the following alternative to Theorem 4.

Theorem 6. Let \( N \) and \( M \) be closed orientable aspherical manifolds of dimension \((n - 1)\) and \( n \), respectively (\( M \) irreducible if \( n = 3 \)). If \( \pi_1(N) \) is isomorphic to a subgroup of \( \pi_1(M) \) which has infinite index in its normalizer in \( \pi_1(M) \), then \( M \) is covered by Euclidean space.
Note that the fundamental group of a compact manifold is always finitely presented. The fact that above assumptions insure the two-endedness of the pair $(\pi_1(M), \pi_1(N))$, follows from a theorem by Swarup [10].

In the last section of this article, we will analyze Davis’ examples from the viewpoint of Theorem 6. We will show that the situation is prototypical of the obstruction which one encounters by verifying that $\pi_1(N)$ equals its normalizer in $\pi_1(M)$ in these examples.

2. Definitions

We begin by reviewing some basic definitions. Recall that if $p : \tilde{X} \to X$ is a covering map (of connected, locally path connected topological spaces), then the group $\text{Aut}(\tilde{X} \to X)$ of covering transformations is isomorphic to $\text{N}_G(H)/H$, where $H = p_\#(\pi_1(\tilde{X})), G = \pi_1(X)$, and $\text{N}_G(H)$ denotes the normalizer of $H$ in $G$. We will always suppress base points. The action of $\text{Aut}(\tilde{X} \to X)$ on $\tilde{X}$ is properly discontinuous and fixed-point free. (Recall that the action of a group $Q$ on a topological space $Y$ is called properly discontinuous, if $\{g \in Q \mid g(C) \cap C \neq \emptyset\}$ is finite for every compact subset $C \subseteq Y$.) If $X$ has a universal covering space, we will denote it by $\tilde{X}$.

Conversely, if a group $G$ acts on a connected, locally path connected topological space $Y$ properly discontinuously and fixed-point free, then the quotient map $Y \to Y/G$ is a regular covering projection with automorphism group isomorphic to $G$.

We will call the action of a group $H$ on a topological space $X$ cocompact if there is a compact subset $C \subseteq X$ such that $H(C) = X$. Here, and later, $H(E)$ is defined to be $\bigcup \{h(E) \mid h \in H\}$ for subsets $E \subseteq X$.

A non-compact topological space $Y$ is called one-ended, if for every compact set $A \subseteq Y$ there is a compact set $B \subseteq Y$ such that $A \subseteq B$ and every pair of points in $Y \setminus B$ is joined by a path in $Y \setminus A$. A one-ended space $Y$ is called simply connected at infinity, if for every compact set $A \subseteq Y$ there is a compact set $B \subseteq Y$ such that $A \subseteq B$ and loops in $Y \setminus B$ contract in $Y \setminus A$. In the next section, we will extend these definitions to groups. Two-endedness of pairs of groups will be defined in Section 7.

A topological space $Y$ is called locally simply connected if for every $y \in Y$ and every neighborhood $U$ of $y$ in $Y$ there is a neighborhood $V$ of $y$ in $Y$ such that $V \subseteq U$ and loops in $V$ contract in $U$.

A manifold is called aspherical if its universal covering space is contractible. We note that all open contractible manifolds of dimension at least two are one-ended. Moreover, if an open contractible manifold is simply connected at infinity, then it is homeomorphic to Euclidean space [3] [9], provided we assume the manifold to be irreducible in case it is 3-dimensional. Clearly, all one-dimensional and two-dimensional closed aspherical manifolds are covered by Euclidean space.

We add to this list of definitions some relative notions of connectivity. Given a triple of topological spaces $C \subseteq D \subseteq Y$, we say that $C$ is one-ended in $D$ with respect
to \( Y \) if \( C \) is not contained in a compact subset of \( Y \) and for every compact set \( A \subseteq Y \) there is a compact set \( B \subseteq Y \) such that \( A \subseteq B \) and every pair of points in \( C \setminus B \) is joined by a path in \( D \setminus A \). Similarly, \( C \) is simply connected at infinity in \( D \) with respect to \( Y \) if \( C \) is not contained in a compact subset of \( Y \) and for every compact set \( A \subseteq Y \) there is a compact set \( B \subseteq Y \) such that \( A \subseteq B \) and loops in \( C \setminus B \) contract in \( D \setminus A \). Whenever the ambient space \( Y \) is understood we drop the reference to it. For brevity, we will from now on say that \( C \) is path connected in \( D \) if every pair of points in \( C \) is joined by a path in \( D \). Similarly, if all loops in \( C \) contract in \( D \), we will call \( C \) simply connected in \( D \).

3. Some Tools

For this section, we fix two topological spaces \( X \) and \( Y \) which are connected, locally path connected, locally compact, and Hausdorff. Suppose \( H \) is a subgroup of a group \( G \) and assume that \( H \) and \( G \) act properly discontinuously on the spaces \( X \) and \( Y \), respectively. Suppose, further, that the action of \( H \) on \( X \) is cocompact. (Notice that, under these hypotheses, \( H \) is necessarily finitely generated.) We will also assume that \( Y \) is locally simply connected (although this is irrelevant for Lemmas 1, 3, and 4).

We state the following lemma for the record, its proof is immediate.

**Lemma 1.** For every compact set \( C \subseteq Y \) there is a compact set \( D \subseteq Y \) such that \( C \subseteq D \) and \( C \) is path connected in \( D \).

**Lemma 2.** Suppose \( Y \) is simply connected. Then for every compact set \( C \subseteq Y \) there is a compact set \( D \subseteq Y \) such that \( C \subseteq D \) and \( C \) is simply connected in \( D \).

**Proof.** Choose open subsets \( U_0, U_1, \cdots, U_k, V_0, V_1, \cdots, V_k \) of \( Y \) such that \( C \subseteq \bigcup \{ U_i \mid i = 0, 1, \cdots, k \} \) and, for each \( i \), \( U_i \) is simply connected in \( V_i \) and \( V_i \) has compact closure. One can then find a finite collection \( W \) of open path connected subsets of \( Y \) such that \( C \subseteq \bigcup W \) and with the property that for each pair \( W_1, W_2 \in W \) with \( W_1 \cap W_2 \neq \emptyset \), there is a \( U_i \) with \( W_1 \cup W_2 \subseteq U_i \).

For example, one could use the following partition of unity to find \( W \): choose a compact set \( D \subseteq Y \) with \( C \subseteq \text{int } D \subseteq D \subseteq \bigcup \{ U_i \mid i = 0, 1, \cdots, k \} \) and continuous functions \( \phi_i : D \to [0, 1] \) such that \( \phi_i^{-1}((0, 1]) \subseteq U_i \) for all \( i = 0, 1, \cdots, k \) and \( \sum_{i=0}^{k} \phi(x) = 1 \) for each \( x \in D \). Define a map \( f \) from \( D \) to a \( k \)-simplex \( \sigma_k = (v_0, v_1, \cdots, v_k) \) by defining \( f(x) = \sum_{i=0}^{k} \phi_i(x)v_i \) for \( x \in D \). Let \( W' \) be a covering of \( \sigma_k \) by finitely many open sets so that for each pair \( W_1', W_2' \in W' \) with \( W_1' \cap W_2' \neq \emptyset \), \( W_1' \cup W_2' \) lies in the open star \( S_1 \) of \( v_i \) in \( \sigma_k \) for some \( i \). Since \( f^{-1}(S_i) \subseteq U_i \), one can now select the desired collection \( W \) from the path components of the sets \( f^{-1}(W') \cap \text{int } D \), \( W' \in W' \).

Define a finite graph \( \Gamma \) as follows. For each \( W \in W \) take a vertex \( v(W) \). Join two distinct vertices \( v(W) \) and \( v(W') \) by an edge \( e(W; W') \) whenever \( W \cap W' \neq \emptyset \).
Choose a map \( \mu : \Gamma \to Y \) such that \( \mu(v(W)) \in W \) and \( \mu(e(W,W')) \subseteq W \cup W' \) for all \( W,W' \in W \). Since \( Y \) is simply connected, there is a homotopy from \( \mu \) to a constant map. Choose a compact set \( E \) such that it contains the closure of each \( V_i \) and the image of this homotopy. A loop \( \alpha \) in \( C \) can now be subdivided into paths \( \alpha_i \) so that each \( \alpha_i \) lies in an element \( W_i \in W \). If we connect the endpoints of each \( \alpha_i \) to \( \mu(v(W_i)) \) with a path in \( W_i \), we produce a bootstrap pattern between \( \alpha \) and \( \Gamma \) whose loops lie alternately in a member of \( W \) and in the union of two intersecting members of \( W \). This allows us to homotope \( \alpha \) into \( \mu(\Gamma) \) within \( E \). From there we can contract it to a point within \( E \).

**Lemma 3.** For every compact set \( C \subseteq Y \) there is a compact set \( D \subseteq Y \) such that \( C \subseteq D \) and \( H(C) \) is path connected in \( H(D) \).

The proof is similar to but simpler than the proof of

**Lemma 4.** Suppose \( X \) is one-ended. Then for every compact set \( C \subseteq Y \) there is a compact set \( D \subseteq Y \) such that \( C \subseteq D \) and \( H(C) \) is one-ended in \( H(D) \).

**Proof.** Let a compact set \( C \subseteq Y \) be given. Choose a compact set \( E \subseteq X \) so that \( H(\text{int}E) = X \). Choose a compact set \( D' \subseteq Y \) such that \( C \subseteq D' \) and \( g_1(D') \cap g_2(D') \neq \emptyset \) whenever \( g_1, g_2 \in H \) and \( g_1(E) \cap g_2(E) \neq \emptyset \). (This is possible since the set \( \{ g \in H \mid E \cap g(E) \neq \emptyset \} \) is finite.) Choose a compact set \( D \subseteq Y \) such that \( D' \subseteq D \) and \( D \) is path connected in \( D \).

Now, let \( A \subseteq Y \) be compact. Choose a compact set \( L \subseteq X \) such that \( X \setminus L \) is path connected in \( X \setminus \bigcup \{ g(E) \mid g \in H, g(D) \cap A \neq \emptyset \} \). Define the compact set \( B = \bigcup \{ g(D) \mid g \in H \text{ and either } g(E) \cap L \neq \emptyset \text{ or } g(D) \cap A \neq \emptyset \} \).

If \( a \in H(C) \setminus B \), then there are \( g_a, g_b \in H \) such that \( a \in g_a(D') \) and \( b \in g_b(D') \). Hence, \( g_a(D) \cap A = \emptyset \), \( g_a(E) \cap L = \emptyset \), \( g_b(D) \cap A = \emptyset \), and \( g_b(E) \cap L = \emptyset \). Pick a point \( a' \in g_a(E) \), a point \( b' \in g_b(E) \), and choose a path \( \gamma' : [0,1] \to X \setminus \bigcup \{ g(E) \mid g \in H, g(D) \cap A \neq \emptyset \} \) with \( \gamma'(0) = a' \) and \( \gamma'(1) = b' \). Choose \( n \in N \) such that for each \( i \in \{0,1,\ldots,n-1\} \) there is a \( g_i \in H \) such that \( \gamma'([\frac{i}{n},\frac{i+1}{n}]) \subseteq g_i(\text{int}E) \). Then we have \( g_a(D') \cap g_b(D') \neq \emptyset \), \( g_b(D') \cap g_{n-1}(D') \neq \emptyset \), and \( g_i(D') \cap g_{i+1}(D') \neq \emptyset \) for all \( i \in \{0,1,\ldots,n-2\} \), but \( g_i(D) \cap A = \emptyset \) for all \( i \in \{0,1,\ldots,n-1\} \). Since \( D' \) is path connected in \( D \), there is a path \( \gamma : [0,1] \to H(D) \setminus A \) with \( \gamma(0) = a \) and \( \gamma(1) = b \).

**Lemma 5.** Suppose both \( X \) and \( Y \) are simply connected. Then for every compact set \( C \subseteq Y \) there is a compact set \( D \subseteq Y \) such that \( C \subseteq D \) and \( H(C) \) is simply connected in \( H(D) \).

**Proof.** Let \( C \subseteq Y \) be compact. Choose a compact set \( C' \subseteq Y \) such that \( C \subseteq \text{int}C' \). Choose a compact set \( E \subseteq X \) such that \( H(\text{int}E) = X \) and \( g_1(E) \cap g_2(E) \neq \emptyset \) whenever \( g_1, g_2 \in H \) and \( g_1(C') \cap g_2(C') \neq \emptyset \). Choose a compact set \( E' \subseteq X \) such that \( E \subseteq E' \) and \( E \) is path connected in \( E' \). Choose a compact set \( F \subseteq Y \) such that \( C' \subseteq F \).
and $\bigcap \{g(F) \mid g \in S\} \neq \emptyset$ whenever $S \subseteq H$ and $\bigcap \{g(E') \mid g \in S\} \neq \emptyset$. Choose a compact set $F' \subseteq Y$ such that $F \subseteq F'$ and $F$ is path connected in $F'$. Put $F'' = \bigcup \{g(F') \mid g \in H, F' \cap h(F') \neq \emptyset\}$, and $h(F') \cap g(F') \neq \emptyset$ for some $h \in H$.

By Lemma 2, there is a compact subset $D \subseteq Y$ such that $F'' \subseteq D$ and $F''$ is simply connected in $D$. For $n \in \mathbb{N}$ and $x \in [0, 1]^2$ define the sets $\mathcal{G}(n) = \{\{ \frac{i}{n} \} \times [\frac{i}{n}, \frac{i+1}{n}] \mid i \in \{0, 1, \ldots, n-1\}\} \cup \{[\frac{i}{n}, \frac{i+1}{n}] \times \{ \frac{i}{n} \} \mid i \in \{0, 1, \ldots, n-1\}\}$, $\mathcal{B}(n) = \{P \in \mathcal{G}(n) \mid P \subseteq \partial[0, 1]^2\}$, $\mathcal{T}(n) = \mathcal{G}(n) \setminus \mathcal{B}(n)$, $N(x, n) = \{P \in \mathcal{G}(n) \mid x \in P\}$, and $D(n) = \{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\} \times \{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\}$.

Let $\gamma : \partial[0, 1]^2 \to H(C)$ be a loop. Since $\gamma(\partial[0, 1]^2) \subseteq H(\text{int } C')$, there is an $n \in \mathbb{N}$ such that for all $P \in \mathcal{B}(n)$ there is a $g_P \in H$ with $\gamma(P) \subseteq g_P(\text{int } C')$. Choose $\gamma' : \partial[0, 1]^2 \to X$ such that $\gamma'(P) \subseteq g_P(\text{int } C')$ for all $P \in \mathcal{B}(n)$. Since $X$ is simply connected, we can extend $\gamma'$ to $f' : [0, 1]^2 \to X$. Since $H(\text{int } E') = X$, there is an $m \in \mathbb{N}$ ($m \geq 2$) such that for all $P \in \mathcal{G}(nm)$ there is a $g_P \in H$ with $f'(P) \subseteq g_P(\text{int } E')$.

Extend $\gamma$ to a map $f : [0, 1]^2 \cup D(nm) \to H(F)$ such that $f(x) \in \bigcap \{g_P(F) \mid P \in \mathcal{N}(x, nm)\}$ for all $x \in D(nm)$. Next, extend $f$ to a map $\bigcup \mathcal{G}(nm) \to H(F')$ such that for all $P \in \mathcal{T}(nm)$ with $P \subseteq \text{int } [0, 1]^2$ we have $f(P) \subseteq g_P(F')$, and for all $P \in \mathcal{T}(nm)$ with $P \cap \text{int } [0, 1]^2$ we have $f(P) \subseteq g_P(F') \cup g_Q(F')$ for some $Q \in \mathcal{B}(n)$ with $P \cap Q \neq \emptyset$. Finally, since for all $i, j \in \{0, 1, \ldots, nm - 1\}$ there is a $g \in H$ such that $f(\partial([\frac{i}{nm}, \frac{i+1}{nm}] \times [\frac{j}{nm}, \frac{j+1}{nm}])) \subseteq g(F''\{0\})$, we can, by choice of $D$, extend $f$ to a map $[0, 1]^2 \to H(D)$. Hence, $\gamma$ contracts in $H(D)$.

Similarly, we have

**Lemma 6.** Suppose $X$ is simply connected at infinity and $Y$ is simply connected.

Then for every compact set $C \subseteq Y$ there is a compact set $D \subseteq Y$ such that $C \subseteq D$ and $H(C)$ is simply connected at infinity in $H(D)$.

**Proof.** We have to change the proof of Lemma 5 only slightly. Using the same setup, let $A \subseteq Y$ be compact. Choose a compact set $L \subseteq X$ so that $X \setminus L$ is simply connected in $X \setminus \bigcup \{g(E') \mid g \in H, g(D) \cap A \neq \emptyset\}$. Set $B = A \cup \bigcup \{g(C') \mid g \in H, g(E') \cap L \neq \emptyset\}$. Now, if $\gamma : \partial[0, 1]^2 \to H(C) \setminus B$ is a loop, then $\gamma(\partial[0, 1]^2) \subseteq X \setminus L$ (where $\gamma'$ is as before) and $\gamma'$ extends to a map $f' : [0, 1]^2 \to X \setminus \bigcup \{g(E') \mid g \in H, g(D) \cap A \neq \emptyset\}$. This yields an extension $f : [0, 1]^2 \to H(D) \setminus A$ of $\gamma$.

Let us recall the following standard terminology. The group $H$ is called **one-ended** if $X$ is one-ended, and $H$ is called **simply connected at infinity** if $X$ is locally simply connected, simply connected and simply connected at infinity. It is well-known, and also follows from Lemma 4 and Lemma 6, that these definitions do not depend on the choice of $X$, but rather are invariants of the group $H$. (For if we further assume the action of $H$ on $Y$ to be cocompact, the conclusions of Lemma 4 and Lemma 6 are equivalent to $Y$ being one-ended and simply connected at infinity, respectively.)
Note that admissible choices for $X$ include (appropriate) locally finite CW-complexes and topological manifolds.

**Lemma 7.** Suppose both $X$ and $Y$ are simply connected, that the action of $G$ on $Y$ is also cocompact, and that $H$ is a normal subgroup of $G$. Then for every compact set $A \subseteq Y$ there is a compact set $B \subseteq Y$ with $A \subseteq B$ such that if $\gamma$ is a loop in $Y \smallsetminus H(B)$ and $h \in H$, we can homotope $\gamma$ to $h \circ \gamma$ by a homotopy missing $H(A)$.

**Proof.** Let the compact set $A \subseteq Y$ be given. Choose a compact set $C \subseteq Y$ such that $G(\text{int } C) = Y$. By Lemma 3, there is a compact set $D \subseteq Y$ such that $C \subseteq D$ and $H(C)$ is path connected in $H(D)$. Define $E = \bigcup \{ g(D) \mid g \in G, D \cap g(D) \neq \emptyset \}$. Then $E$ is a compact subset of $Y$ containing $D$. Using Lemma 5, we may choose a compact set $F \subseteq Y$ such that $E \subseteq F$ and $H(E)$ is simply connected in $H(F)$. Put $B = \bigcup \{ g(F) \mid g \in G, A \cap g(F) \neq \emptyset \}$.

Now, let $\gamma : [0,1] \to Y \smallsetminus H(B)$ be a map with $\gamma(0) = \gamma(1)$ and $h \in H$. Choose $n \in \mathbb{N}$ such that for all $i \in \{0,1,\ldots,n-1\}$ there is a $g_i \in G$ with $\gamma\left([\frac{i}{n}, \frac{i+1}{n}]\right) \subseteq g_i(\text{int } C)$. Then both $\gamma\left([\frac{i}{n}, \frac{i+1}{n}]\right)$ and $h \circ \gamma\left([\frac{i}{n}, \frac{i+1}{n}]\right)$ are contained in $g_i H(C)$ for all $i \in \{0,1,\ldots,n-1\}$. For each $i \in \{0,1,\ldots,n-1\}$ we connect $\gamma\left(\frac{i}{n}\right)$ to $h \circ \gamma\left(\frac{i}{n}\right)$ with a path in $g_i H(D)$. For $i \in \{0,1,\ldots,n-1\}$ and $j = i+1$, or $i = n$ and $j = 0$, we have $g_i(D) \cap g_j(D) \neq \emptyset$. Therefore, for all $h' \in H$ there is an $h'' \in H$ such that $g_i h''(D) \cap g_j h'(D) \neq \emptyset$, because $H$ is normal in $G$. Consequently, the two paths joining $\gamma\left(\frac{i}{n}\right)$ to $h \circ \gamma\left(\frac{i}{n}\right)$ and $\gamma\left(\frac{2}{n}\right)$ to $h \circ \gamma\left(\frac{2}{n}\right)$, respectively, both lie in $g_i H(E)$. By our choices of $F$ and $B$ we can fill in a homotopy from $\gamma$ to $h \circ \gamma$ missing $H(A)$.

Later, we will also need

**Lemma 8.** Let $L$ be a locally finite simplicial complex, $r : \hat{L} \to L$ the universal covering, $U$ a connected subcomplex of $L$ such that the inclusion induced homomorphism $\pi_1(U) \to \pi_1(L)$ is surjective, and $P$ a finite subcomplex of $L$. Then for every compact set $A \subseteq \hat{L}$ there is a compact set $B \subseteq \hat{L}$ with $A \subseteq B$ and such that loops in $r^{-1}(U \cup P) \smallsetminus B$ can be homotoped into $r^{-1}(U)$ with a homotopy missing $A$.

**Proof.** Since $\pi_1(U) \to \pi_1(L)$ is surjective and $U$ is connected, there is a homotopy $H$ that takes the 1-skeleton of $L$ into $U$ leaving the 1-simplices of $U$ fixed. For each 1-simplex $\sigma$ of the finitely many 1-simplices of $P$ which do not lie in $U$, choose a compact subset $E_\sigma$ of $\hat{L}$ which contains a lift of the given homotopy that takes $\sigma$ into $U$. Put $E = \bigcup E_\sigma$. Let a compact set $A \subseteq \hat{L}$ be given. Choose a compact set $B' \subseteq \hat{L}$ with $A \subseteq B'$ such that every translate of $E$ (under a covering translation) which intersects $A$, lies in $B'$. Choose a compact set $B \subseteq \hat{L}$ with $B' \subseteq B$ such that every simplex of $\hat{L}$ which meets $B'$, lies in $B$. Let $\alpha$ be a loop in $r^{-1}(U \cup P) \smallsetminus B$. We can homotope $r \circ \alpha$ to a loop that lies in the union of $U$ and the 1-skeleton of $P$ such that during the homotopy points do not leave the top dimensional simplex containing them. The lift of this homotopy lies in $\hat{L} \smallsetminus B'$; call its end $\alpha'$. Now, $r \circ \alpha'$ can be
Covering spaces of closed aspherical manifolds

homotoped into $U$ using the homotopy $H$. We lift this homotopy to a homotopy of $\alpha'$ and call its end $\alpha''$. If a point is moved during this final homotopy, then it must lie in a translate of $E$. Hence, the track of such a point must miss $A$. \hfill}

4. Proof of Theorem 3

From [12] we quote

**Lemma 9.** [The Orbit Lemma I] Suppose $W$ is an open contractible $n$-manifold, $n \geq 3$. Let $h$ be a non-trivial homeomorphism of $W$ onto itself so that the group $H$ of homeomorphisms generated by $h$ acts without fixed-points and properly discontinuously on $W$. If $C$ is a compact subset of $W$, then loops of $W$ can be homotoped off $H(C)$. Furthermore, given a compact set $A$ there is a compact set $B$, which contains $A$, so that loops in $W \setminus B$ can be homotoped off $H(C)$ by a homotopy that lies in $W \setminus A$.

**Proof.** (of Theorem 3) Let $H = \langle h \rangle$ be a non-trivial cyclic normal subgroup of $\pi_1(M)$. Note that $h$ must have infinite order. (Otherwise $s = |\langle h \rangle| < \infty$ and $\tilde{M}/\langle h \rangle$ is a finite dimensional $K(\mathbb{Z}s, 1)$, which contradicts the fact that $\mathbb{Z}s$ has infinite cohomological dimension.) We assume that $n \geq 3$ and wish to show that $\tilde{M}$ is simply connected at infinity.

Let $C \subseteq \tilde{M}$ be compact. Use Lemma 7 to choose a compact set $E \subseteq \tilde{M}$ such that $C \subseteq E$ and loops in $\tilde{M} \setminus H(E)$ can be homotoped to any $H$-translate via a homotopy missing $H(C)$. By Lemma 9, there is a compact set $D \subseteq \tilde{M}$ such that $E \subseteq D$ and loops in $\tilde{M} \setminus D$ can be homotoped into $\tilde{M} \setminus H(E)$ by a homotopy missing $E$ (and hence $C$). Now, let $\gamma$ be a loop in $\tilde{M} \setminus D$. By choice of $D$ we may already assume that $\gamma$ lies in $\tilde{M} \setminus H(E)$. Since $\tilde{M}$ is simply connected, $\gamma$ can be contracted to a point in $\tilde{M}$. Let $\eta$ be the image of this contraction. Choose $n \in \mathbb{N}$ such that $h^n(\eta) \cap C = \emptyset$. We then homotope $\gamma$ to $h^n \circ \gamma$ with a homotopy missing $H(C)$, where it contracts missing $C$. \hfill}

5. A Geometric Proof of Theorem 1

We now want to use our tools of Section 3 to give a geometric proof of the Houghton-Jackson Theorem. To this end, let $K$ be a finite simplicial complex with fundamental group $G$. We identify $H$ with a subgroup of $G$ and $Q$ with $G/H$. Let $p : \tilde{K} \to K$ be the universal covering and identify $G \equiv \text{Aut}(\tilde{K} \to K)$. Put $K = \tilde{K}/H$ with quotient (and covering) map $q : \tilde{K} \to K$, so that $H \equiv \text{Aut}(\tilde{K} \to K)$. We also identify $Q = G/H$ with $\text{Aut}(\tilde{K} \to K)$. Note that $H$ is isomorphic to the fundamental group
of a finite simplicial complex, because it is finitely presented. Since \( H \) acts properly discontinuously and cocompactly on the universal cover of this complex, we are in the setting of Section 3 with two simply connected spaces.

Choose a finite subcomplex \( C \) of \( \tilde{K} \) such that \( G({\text{int}} \ C) = \tilde{K} \). \( H(C) \) is path connected, and the inclusion induced homomorphism \( \pi_1(q(C)) \to \pi_1(\tilde{K}) \cong H \) is surjective.

We inductively define the following subsets of \( G/H \). Put \( B_0 = \{H\} \) and \( B_n = \{gH \in G/H \mid gH(C) \cap g'H(C) \neq \emptyset \text{ for some } g'H \in B_{n-1}\} \). Then each \( B_n \) is finite and \( G/H = \bigcup B_n \). Finally, put \( T_n = \bigcup \{gH(C) \mid gH \in B_n \setminus B_{n-1}\} \).

**Lemma 10.** \( G \) is one-ended.

**Proof.** Let \( D \subseteq \tilde{K} \) be a compact set. Choose \( n \in \mathbb{N} \) such that \( D \subseteq \bigcup_{j \leq n} T_j \) and \( D \cap T_n = \emptyset \). Choose a compact set \( E \subseteq \tilde{K} \) with \( D \subseteq E \) and such that for every \( g_1H, g_2H \in B_n \) every point of \( g_1H(C) \setminus E \) can be joined to some point of \( g_2H(C) \) by a path missing \( D \).

Now, let \( a, b \in \tilde{K} \setminus E \). These points may or may not lie in \( \bigcup_{j \leq n} T_j \). Without loss of generality, say \( a \not\in \bigcup_{j \leq n} T_j \) and \( b \in g_1H(C) \) for some \( g_1H \in B_n \). Connect \( a \) to a point \( p_1 \in g_2H(C) \) for some \( g_2H \in B_n \setminus B_{n-1} \) with a path in \( \tilde{K} \setminus \bigcup_{j \leq n} T_j \). Connect \( b \) to a point \( p_2 \in g_2H(C) \) by a path missing \( D \). Finally, connect \( p_1 \) and \( p_2 \) in \( g_2H(C) \). This yields a path from \( a \) to \( b \) missing \( D \).

**Lemma 11.** Suppose \( Q \) is one-ended and a compact set \( E \subseteq \tilde{K} \) is given. Then for every compact set \( A \subseteq \tilde{K} \) there is a compact set \( B \subseteq \tilde{K} \) such that \( A \subseteq B \) and loops in \( \tilde{K} \setminus B \) can be homotoped off \( H(E) \) by a homotopy that lies in \( \tilde{K} \setminus A \).

**Proof.** Since \( \tilde{K} \) is one-ended, there is a finite subcomplex \( P \) of \( \tilde{K} \) and a connected subcomplex \( U \) of \( \tilde{K} \) such that \( U \cap q(E) = \emptyset \) and \( \tilde{K} = U \cup P \). Now, \( Q \) is infinite so that \( v(q(C)) \cap P = \emptyset \) for some \( v \in Q \). Hence, \( v(q(C)) \subseteq U \). Since \( \pi_1(q(C)) \to \pi_1(\tilde{K}) \) is surjective, then so is \( \pi_1(v(q(C))) \to \pi_1(\tilde{K}) \). Hence \( \pi_1(U) \to \pi_1(\tilde{K}) \) is surjective. The result follows now at once from Lemma 8.

**Lemma 12.** If \( Q \) is one-ended, then \( G \) is simply connected at infinity.

**Proof.** Since \( H \) is infinite, we can repeat the argument used in the last paragraph of the proof of Theorem 3, substituting Lemma 11 for Lemma 9 and an appropriate element of \( H \) for \( h^n \).

**Lemma 13.** If \( a, b \in T_n \) lie in the same component of \( \tilde{K} \setminus \bigcup_{j \leq n} T_j \), then \( a \) and \( b \) can be joined by a path in \( T_n \).

**Proof.** Join \( a \) to \( b \) via two paths, one lying in \( \tilde{K} \setminus \bigcup_{j \leq n} T_j \) and the other in \( \bigcup_{j \leq n} T_j \).
Since $\tilde{K}$ is simply connected this loop contracts. The result follows now from the fact that $T_n$ separates $\bigcup_{j \geq n} T_j$ from $\bigcup_{j < n} T_j$. //

If $H$ is one-ended we can choose a finite subcomplex $D$ of $\tilde{K}$ such that $C \subseteq D$ and $H(C)$ is one-ended in $H(D)$, by Lemma 4. Let us also arrange for $H(D)$ to be path connected. We then put $T_n' = \bigcup \{gH(D) \mid gH \in B_n \setminus B_{n-1}\}$.

**Lemma 14.** Suppose $H$ is one-ended. Then for every compact set $A \subseteq \tilde{K}$ there is a compact subset $B \subseteq \tilde{K}$ with $A \subseteq B$ such that if $a, b \in T_n \setminus B$ lie in the same component of $T_n$, then $a$ and $b$ can be joined by a path in $T_n' \setminus A$.

**Proof.** By Lemma 4, there is a compact set $B' \subseteq \tilde{K}$ with $A \subseteq B'$ such that for all the finitely many $gH \in B_n \setminus B_{n-1}$, $gH(C) \setminus B'$ is path connected in $gH(D) \setminus A$. Choose a compact set $B \subseteq \tilde{K}$ with $B' \subseteq B$ such that for every $g_1H, g_2H \in B_n \setminus B_{n-1}$ with $g_1H(C)$ and $g_2H(C)$ in the same component of $T_n$, every point of $g_1H(C) \setminus B$ can be joined to some point of $g_2H(C)$ by a path in $T_n \setminus B'$.

Now suppose that $a, b \in T_n \setminus B$ lie in the same component of $T_n$. Say $b \in gH(C)$ with $gH \in B_n \setminus B_{n-1}$. Join $a$ to a point $p_0 \in gH(C)$ by a path in $T_n \setminus B'$. Then join $p_0$ to $b$ by a path in $gH(D) \setminus A$. This yields a path from $a$ to $b$ in $T_n' \setminus A$. //

**Lemma 15.** If $H$ is one-ended, then $G$ is simply connected at infinity.

**Proof.** Let a compact set $A \subseteq \tilde{K}$ be given. By Lemma 7, there is a compact set $E \subseteq \tilde{K}$ with $A \subseteq E$ so that loops in $\tilde{K} \setminus H(E)$ can be homotoped to any $H$-translate via a homotopy missing $H(A)$. Choose $n \in \mathbb{N}$ such that $E \subseteq \bigcup_{j \leq n} T_j$ and $E \cap T_n' = \emptyset$. Pick any $sH \in B_n \setminus B_{n-1}$. Since $\pi_1((sH(q(C))) \to \pi_1(\tilde{K})$ is surjective, we can use Lemma 8 to find a compact set $F \subseteq \tilde{K}$ with $E \subseteq F$ such that loops in $\bigcup_{j \leq n} T_j \setminus F$ can be homotoped into $sH(D)$ by a homotopy missing $E$ (and hence missing $A$). By Lemma 14, there is a compact set $B \subseteq \tilde{K}$ with $F \subseteq B$ such that points in $T_n \setminus B$ that are in the same component of $T_n$ can be joined by paths in $T_n' \setminus F$.

Now, let $\gamma$ be a loop in $\tilde{K} \setminus B$. We wish to show that $\gamma$ contracts in $\tilde{K} \setminus A$. We may assume that $\gamma$ is an edge path in the 1-skeleton of $\tilde{K}$. First we argue that we may assume, without loss of generality, that $\gamma$ either lies in $\bigcup_{j \leq n} T_j \setminus F$ or in $\tilde{K} \setminus H(E)$. For if $\gamma$ intersects $\bigcup_{j \leq n} T_j$ and intersects $\bigcup_{j \geq n} T_j$ we can cut it into finitely many subpaths that either lie in $\bigcup_{j \leq n} T_j \setminus B$ or in $\bigcup_{j \geq n} T_j \setminus B$ and whose endpoints are in $T_n \setminus B$. The latter kind has its endpoints in the same component of $T_n$, by Lemma 13. We join these endpoints by paths in $T_n' \setminus F$. This leaves us with the problem of contracting finitely many loops that either lie in $\bigcup_{j \leq n} T_j' \setminus F$ or in $\tilde{K} \setminus H(E)$ with homotopies that miss $A$. Since the first kind can be homotoped into $sH(D) \subseteq \tilde{K} \setminus H(E)$ with a homotopy that misses $A$, we are actually left with only loops of the second kind.

So, we now assume that $\gamma$ is a loop in $\tilde{K} \setminus H(E)$ to be contracted missing $A$. We do this as before. Since $\tilde{K}$ is simply connected, we can contract $\gamma$ to a point in $\tilde{K}$. //
Let $\eta$ be the image of that contraction. Since $H$ is infinite, there is an $h \in H$ with $h(\eta) \cap A = \emptyset$. By choice of $E$ we can homotope $\gamma$ to $h \circ \gamma$ by a homotopy missing $H(A)$, where it contracts missing $A$. //

6. Proof of Theorem 4

Since we may assume by Theorem 3 that $N$ is at least 2-dimensional, we have that $\tilde{N}$ is one-ended. Also, $H = \pi_1(N)$ is infinite. Finally, both $\tilde{M}/H$ and $N$ are $K(H,1)$’s and are thus homotopy equivalent. Then $H_n(\tilde{M}/H; \mathbb{Z}_2) = H_n(N; \mathbb{Z}_2) = 0$. Therefore $\tilde{M}/H$ is not compact. If we denote $\pi_1(M)$ by $G$, this implies that $G/H$, whose cardinality equals the number of sheets of the covering $\tilde{M}/H \to M$, is infinite. Now apply Theorem 1 to complete the proof.

7. Proof of Theorem 5

Let $H$ be a subgroup of a group $G$. Suppose $L$ is a finite simplicial complex with regular covering $\hat{L} \to L$ whose automorphism group is isomorphic to $G$. Denote the quotient $\hat{L}/H$ by $L$. We say that the pair $(G, H)$ is two-ended if for every compact set $A \subseteq \hat{L}$ there is a compact set $B \subseteq \hat{L}$ with $A \subseteq B$ such that $\tilde{L} \setminus B$ has two components both of which are unbounded. It can be shown that this notion is independent of the choice of $\hat{L} \to L$. (See [7] and [8] for a more general discussion of ends of pairs of groups.)

Proof. (of Theorem 5) Let $\tilde{K}$, $\hat{K}$, $K$, $p$, and $q$ be as in Section 5. Let $A \subseteq \hat{K}$ be a compact set. We will find a compact set $\tilde{B} \subseteq \hat{K}$ such that $A \subseteq B$ and $\tilde{K} \setminus B$ is simply connected in $\tilde{K} \setminus A$. Since $(G, H)$ is two-ended, we may choose a finite subcomplex $C_1 \subseteq \hat{K}$ such that $A \subseteq C_1$ and $\tilde{K} \setminus q(C_1)$ has two components both of which are unbounded. We also arrange for the inclusion induced map $\pi_1(q(C_1)) \to \pi_1(\tilde{K})$ to be surjective and for $H(C_1)$ to be path connected. (Note that $\pi_1(\tilde{K}) \simeq H$ is finitely generated.) Use Lemmas 4 and 5 to choose a finite subcomplex $C_2 \subseteq \hat{K}$ such that $C_1 \subseteq C_2$, $H(C_1)$ is one-ended in $H(C_2)$, and $H(C_1)$ is simply connected in $H(C_2)$. Again, we may assume that $\tilde{K} \setminus q(C_2)$ has two components both of which are unbounded. Since the infinite group $\mathcal{N}_G(H)/H \simeq \text{Aut}(\hat{K} \to K)$ acts properly discontinuously on $\hat{K}$ and $(gH)(q(T)) = q(g(T))$ for all $g \in \mathcal{N}_G(H)$ and $T \subseteq K$, there are elements $g_1, \cdots, g_5 \in \mathcal{N}_G(H)$ such that the collection $\{q(g_i(C_2)) \mid i = 1, \cdots, 5\}$ is pairwise disjoint and such that $q(g_i(C_2))$ lies in the bounded component of $\tilde{K} \setminus (q(g_{i-1}(C_2)) \cup q(g_{i+1}(C_2)))$ for $i = 2, 3, 4$. We take $g_3 = 1$.

Let $D$ be a finite subcomplex of $\hat{K}$ such that $q(D)$ equals the complement of the two unbounded components of $\tilde{K} \setminus (q(g_1(C_2)) \cup q(g_5(C_2)))$. By Lemma 8, there is a compact set $C_3 \subseteq \hat{K}$ with $A \subseteq C_3$ and such that loops in $q^{-1}(q(D)) \setminus C_3 = H(D) \setminus C_3$
can be homotoped into \( q^{-1}(q(g_2(C_1))) = g_2H(C_1) \) missing \( A \). Finally, we choose a compact set \( B \subseteq \tilde{K} \) with \( C_3 \subseteq B \) such that \( g_1H(C_1) \setminus B \) and \( g_5H(C_1) \setminus B \) are path connected in \( g_1H(C_2) \setminus C_3 \) and \( g_3H(C_2) \setminus C_3 \), respectively.

Now, let \( \gamma \) be a loop in \( \tilde{K} \setminus B \). We may assume that \( \gamma \) is an edge path in the 1-skeleton of \( \tilde{K} \). If \( \gamma \) lies outside of \( H(D) \), we first contract it in \( \tilde{K} \). Since \( g_1H(C_1) \) is simply connected in \( g_1H(C_2) \), we can cut off this singular disk at \( g_2H(C_1) \cup g_4H(C_1) \) and cap it off at \( g_2H(C_2) \cup g_4H(C_2) \). Hence, in this case, \( \gamma \) contracts missing \( A \).

If \( \gamma \) has subpaths which lie outside of \( H(D) \) and whose endpoints are in \( g_1H(C_1) \) or \( g_3H(C_1) \), we connect the endpoints of each such subpath by a path in \( g_1H(C_2) \setminus C_3 \) or \( g_3H(C_2) \setminus C_3 \), respectively. Since we can deal with these newly formed loops as in the previous case, we may now assume that \( \gamma \) lies entirely in \( H(D) \setminus C_3 \).

If \( \gamma \) lies in \( H(D) \setminus C_3 \), we can homotope it into \( g_2H(C_1) \) missing \( A \), where it contracts within \( g_2H(C_2) \), still missing \( A \).

8. Davis’ Examples

In this section we will analyze the examples of Davis mentioned in the introduction, and discover that \( \pi_1(N) \) equals its normalizer in \( \pi_1(M) \) in these examples.

A Coxeter system \( \Gamma = \langle V \mid v^2 = 1, (uv)^{m(u,v)} = 1 \forall u, v \in V \rangle \) (a group defined in terms of finitely many generators and specific relations) is called right angled if \( m(u,v) \in \{\infty, 2\} \) for all \( u \neq v \). Its nerve is defined to be the abstract simplicial complex \( N(\Gamma, V) \) consisting of all non-empty subsets of \( V \) which generate a finite subgroup of \( \Gamma \), where incidence is by inclusion. Fix a right angled Coxeter system \( (\Gamma, V) \) whose nerve \( N(\Gamma, V) \) is the first barycentric subdivision of a non-simply connected PL homology \((n - 1)\)-sphere. (Such examples exist in all dimensions \( 3 \) and higher.) The generating set \( V \) of \( \Gamma \) is identified with the vertex set of \( N(\Gamma, V) \). Two such generators commute exactly if they are joined by an edge in \( N(\Gamma, V) \). Let \( C \) be the unique compact contractible \( n \)-manifold with boundary \( N(\Gamma, V) \); it will serve as a basic chamber. Denote the dual cell of a vertex \( v \) in \( N(\Gamma, V) \) by \( C_v \) (i.e. \( C_v \) is the star of \( v \) in a further barycentric subdivision of \( N(\Gamma, V) \)). Put \( \mathcal{M}(\Gamma) = \Gamma \times C / \sim \) where \( (g, x) \sim (h, y) \iff x = y \) and \( g^{-1}h \in \langle v \mid x \in C_v \rangle \). Then \( \Gamma \) acts properly discontinuously and cocompactly on \( \mathcal{M}(\Gamma) \) by left multiplication on the first coordinate.

In [1], Davis shows that \( \mathcal{M}(\Gamma) \) is an open contractible manifold, which is not homeomorphic to Euclidean space. Notice that the commutator subgroup \( [\Gamma, \Gamma] \) of \( \Gamma \) is torsion free and of finite index in \( \Gamma \). It therefore acts fixed-point free and cocompactly on \( \mathcal{M}(\Gamma) \). The quotient \( \mathcal{M}(\Gamma)/[\Gamma, \Gamma] \) is our manifold \( M \).

Fix an element \( v \in V \). Then \( v \) acts as a reflection on \( \mathcal{M}(\Gamma) \) through the fixed-point set \( \text{Fix}(v) = \{ p \in \mathcal{M}(\Gamma) \mid v(p) = p \} \). Put \( \tilde{V} = V \cap \text{lk}(v, N(\Gamma, V)) \) and let \( (\tilde{\Gamma}, \tilde{V}) \) be the induced right angled Coxeter system. Then the nerve of \( (\tilde{\Gamma}, \tilde{V}) \) is the PL sphere \( \text{lk}(v, N(\Gamma, V)) \) and \( \text{Fix}(v) = \{ gC_v \mid g \in \tilde{\Gamma} \} \). In fact, we can identify \( \text{Fix}(v) \) with \( \mathcal{M}(\tilde{\Gamma}) \), where \( C_v \) takes on the role of the basic chamber. (See, for example, [2].) Now,
C_v is a ball, so that \( \text{Fix}(v) = \mathcal{M}(\tilde{\Gamma}) \) is homeomorphic to \((n-1)\)-dimensional Euclidean space. Since the commutator subgroups satisfy \([\Gamma, \tilde{\Gamma}] = [\Gamma, \Gamma] \cap \tilde{\Gamma} \), the covering map \( p : \mathcal{M}(\Gamma) \to \mathcal{M}(\Gamma)/[\Gamma, \Gamma] = M \) restricts to \( p|_{\text{Fix}(v)} : \text{Fix}(v) \to \text{Fix}(v)/[\tilde{\Gamma}, \Gamma] \). The quotient \( \mathcal{M}(\tilde{\Gamma})/[\Gamma, \Gamma] \) is our manifold \( N \). Clearly \( \pi_1(N) = [\Gamma, \tilde{\Gamma}] \leq [\Gamma, \Gamma] = \pi_1(M) \). We will now verify that in these examples \( \mathcal{N}_{[\Gamma, \Gamma]}([\Gamma, \Gamma]) = [\Gamma, \tilde{\Gamma}] \).

Recall that Coxeter groups have a very simple solution to the word problem [11]. A word (finite sequence of generators) is reduced if and only if it cannot be shortened by a combination of the following two operations: (i) the obvious cancellation of a subword of the form \( uu \), and (ii) replacement of a subword of the form \( uuwuwu \cdots \) (of length \( m \)) by \( wu(wu) \cdots \) (of length \( m \)), where \( m \) is the order of the element \( uw \) in the group. This becomes especially easy to check in a right angled Coxeter group!

Let \( g \in [\Gamma, \tilde{\Gamma}] \) with \( g[\tilde{\Gamma}, \Gamma]g^{-1} = [\tilde{\Gamma}, \Gamma] \). Express \( g = u_1u_2 \cdots u_n \), reduced with all \( u_i \in V \). We will show that \( u_i \in \tilde{V} \) for all \( i \), so that \( g \in [\tilde{\Gamma}, \tilde{\Gamma}] \). Choose the maximal index \( i_0 \) with \( u_{i_0} \notin \tilde{V} \) (if there is such an index). Since \([\tilde{\Gamma}, \tilde{\Gamma}]\) is a normal subgroup of \( \tilde{\Gamma} \), we have \( u_1u_2 \cdots u_{i_0}[\tilde{\Gamma}, \Gamma]u_{i_0} \cdots u_2u_1 = [\tilde{\Gamma}, \tilde{\Gamma}] \). We now show that \( u_{i_0}x = xu_{i_0} \) for all \( x \in \tilde{V} \). Let \( x \in \tilde{V} \). Choose \( y \in V \) with \( xyy \neq 1 \). (The easiest way of seeing that such a \( y \) always exists, is to take \( v \) to be a barycenter of a top-dimensional simplex in the original triangulation of the homology sphere.) Then \( u_1u_2 \cdots u_{i_0}xyxyu_{i_0} \cdots u_2u_1s_1s_2 \cdots s_p = 1 \) for some \( s_i \in V \). Applying the right angled reduction scheme to this equation, we conclude that \( u_{i_0}x = xu_{i_0} \). Since \( u_{i_0}x = xu_{i_0} \) for all \( x \in \tilde{V} \), we must have \( u_{i_0} = v \), because \( v \) is the only vertex of \( N(\Gamma, V) \) which is joined to all vertices of its link. Inductively, we conclude that all \( u_i \in \{v\} \cup \tilde{V} \).

Since the word \( u_1u_2 \cdots u_n \) is reduced and contains every generator an even number of times, we have in fact \( u_i \in \tilde{V} \) for all \( i \).

References

2. H. Fischer, Boundaries of right angled Coxeter groups with manifold nerves. Preprint.


Department of Mathematics
Brigham Young University
Provo, Utah 84602
U.S.A.

fischer@math.byu.edu  wright@math.byu.edu