1. Let $X$ be a Hausdorff space. Prove that if $X$ has infinitely many elements, then $X$ has infinitely many pairwise disjoint open subsets.

2. Consider the set $X = (\mathbb{N} \times \mathbb{N}) \cup \{\ast\}$. For each function $f : \mathbb{N} \to \mathbb{N}$ we define a subset $B(f)$ of $X$ by $B(f) = \{\ast\} \cup \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid f(m) \leq n\}$.

   (a) Show that the collection of all subsets $B$ of $X$, which are either of the form $B = \{(m, n)\}$ with $m, n \in \mathbb{N}$ or of the form $B = B(f)$ for some function $f : \mathbb{N} \to \mathbb{N}$, form a basis for some topology $T$ on $X$.

   (b) Prove that $(X, T)$ has no countable neighborhood basis at the point $\ast$.

   (c) Is the following true: if a topological space has countably many elements, then it must have a countable neighborhood basis at every point?

**Definition.**
Let $X$ be a topological space, let $x \in X$ and let $A \subseteq X$. We call $x$ an *uncountable limit point* of $A$ if for every neighborhood $U$ of $x$, the set $U \cap (A \setminus \{x\})$ is uncountable.

3. Let $X$ be a topological space with a countable basis $\mathcal{B}$ and let $A \subseteq X$ be an uncountable subset. Prove that all but countably many points of $A$ are uncountable limit points of $A$.

4. Let $Y$ be a subspace of both a Hausdorff space $X_1$ and a Hausdorff space $X_2$ such that $\text{cl}_{X_1}(Y) = X_1$ and $\text{cl}_{X_2}(Y) = X_2$.

Suppose that the identity function $i : Y \to Y$ can be extended to a continuous function $f : X_1 \to X_2$ as well as to a continuous function $g : X_2 \to X_1$.

Prove that $X_1$ and $X_2$ are homeomorphic.

5. (a) Prove that the following statement is equivalent to the axiom of choice: given an indexed collection $\{X_i \mid i \in I\}$ of nonempty sets with $I \neq \emptyset$, the Cartesian product $\prod_{i \in I} X_i$ is not empty.

(b) Let $I$ be an uncountable index set. For each $i \in I$, let $X_i$ be a topological $T_1$ space with more than one element. Prove that the topological product $\prod_{i \in I} X_i$ does not have a countable neighborhood basis at any of its points.

(c) Prove that a topological product of uncountably many metric spaces, each having more than one element, is never metrizable.