1. Prove that a topological space $X$ is limit point compact if and only if it does not contain an infinite closed discrete subspace $Y$.

2. Prove that a topological space is countably compact if and only if every nested sequence $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$ of closed nonempty subsets of $X$ has nonempty intersection.

3. For each $i \in \mathbb{R}$ let $X_i$ be a $T_1$ space with more than one element. Prove that $\prod_{i \in \mathbb{R}} X_i$ is not sequentially compact.

4. For each of the following spaces, decide whether or not they are compact, sequentially compact, countably compact, and/or limit point compact:
   
   (a) $[0, 1]^\mathbb{R}$, where $[0, 1]$ is the unit interval in the standard topology
   (b) $S_\Omega$
   (c) $X \times Y$, where $(X, \mathcal{T}_X) = (\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and $(Y, \mathcal{T}_Y) = (\{0, 1\}, \{\emptyset, \{0, 1\}\})$
   (d) The subspace $Y = [0, 1]$ of $\mathbb{R}$

5. Prove or disprove:
   
   (a) If $f : X \rightarrow Y$ is continuous and surjective and $X$ is limit point compact, must $Y$ be limit point compact?
   (b) If $C$ is a closed subset of a limit point compact space $X$, must the subspace $C$ be limit point compact?
   (c) If $C$ is a limit point compact subspace of a Hausdorff space $X$, must $C$ be closed in $X$?

6. Let $X$ be the middle-third Cantor set, that is, the subspace of $\mathbb{R}$ consisting of all points $x \in [0, 1]$ which have a ternary (base 3) expansion $x = 0.a_1a_2a_3 \cdots$ with $a_i \in \{0, 2\}$. Prove the following:

   (a) $X$ is a nonempty compact totally disconnected metric space without isolated points.
   (b) $X$ is homeomorphic to $\{0, 2\}^\mathbb{N}$.
   (c) There is a continuous surjective function $f : X \rightarrow [0, 1]$.
   (d) There is a continuous surjective function $g : X \rightarrow [0, 1] \times [0, 1]$.

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A Challenge Problem.
A compact connected Hausdorff space is called a continuum. Prove that no continuum can be written as a countable union of nonempty disjoint closed proper subsets.