1. Which of the following subspaces of $\mathbb{R}^2$ are connected?
   (a) $(\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q})$
   (b) $(\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{Q} \times \mathbb{Q})$

2. A nonempty connected subset $C$ of a topological space $X$ is called a component of $X$ if there is no connected subset $D$ of $X$ with $C \subseteq D$ and $C \neq D$.
   Prove that the collection $\mathcal{C}$ of all components partitions $X$ into closed subsets.

3. Let $X$ be a topological space. Prove that the following are equivalent:
   (a) $X$ is connected.
   (b) For every collection $\mathcal{U}$ of open subsets of $X$ with $X = \bigcup \mathcal{U}$ and every two points $x, y \in X$, there are finitely many $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that $x \in U_1, U_i \cap U_{i+1} \neq \emptyset$ for all $1 \leq i < n$, and $y \in U_n$.

4. Let $X$ be a connected topological space with finitely many elements.
   Prove that for every $x, y \in X$ there is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

5. A topological space $X$ is called zero-dimensional, if for every $x \in X$ and every neighborhood $U$ of $x$ in $X$, there is a neighborhood $V$ of $x$ in $X$ such that $x \in V \subseteq U$ and $\text{bdy}(V) = \emptyset$. A topological space $X$ is called totally disconnected if its only nonempty connected subspaces are one-point sets.
   Prove the following:
   (a) Every discrete space is zero-dimensional.
   (b) Both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are zero-dimensional, but $\mathbb{R}$ is not zero-dimensional.
   (c) Every metric space with countably many elements is zero-dimensional.
   (d) Every zero-dimensional $T_1$ space is totally disconnected.

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**A Challenge Problem.** Construct a countable connected Hausdorff space $X$ with an element $x_0 \in X$ such that $X \setminus \{x_0\}$ is totally disconnected but not zero-dimensional.