Chapter 4

§4.1 4. It will help to use Mathematica to graph these mappings (along with some parameter curves) as follows:

\[
\text{ParametricPlot3D}\{u, u*v, v\}, \{u, -5, 1\}, \{v, -1, 2\}\]

\[
\text{ParametricPlot3D}\{u^2, u^3, v\}, \{u, -1, 1\}, \{v, -1, 1\}\]

\[
\text{ParametricPlot3D}\{u, u^2, v+v^3\}, \{u, -1, 1\}, \{v, -1, 1\}\]

\[
\text{ParametricPlot3D}\{(\cos(2\pi*u), \sin(2\pi*u), v), \{u, -2, 2\}, \{v, -1, 1\}\}
\]

Recall that Mathematica allows you to enlarge and rotate the output graph with the mouse.

A patch \(x : D \to \mathbb{R}^3\) is a one-to-one regular mapping, where \(D \subseteq \mathbb{R}^2\) is an open subset.

(a) Prove that this mapping is one-to-one and regular.

(b) Prove that this mapping is one-to-one. However, it is not regular: there are points \((u, v)\) for which \(x_{(u,v)}\) is not injective.

(c) Prove that this mapping is one-to-one and regular.

(d) Prove that \(x_{(u,v)}\) is injective for all \((u, v)\). However, the mapping \(x(u, v)\) is clearly not one-to-one.

12. It is more convenient to think of the plane containing the curve \(C\) as being the \(xz\)-plane, with the \(z\)-axis being the (drawn) axis of revolution. So, rotate the given figure by 90° clockwise and consider \(f(x^2, z) = c\), instead. Let \(M\) be the surface of revolution about this axis.

We assume that \(f(x^2, z) = c\) implicitly defines a curve \(C\) in the \(xz\)-plane. For reference, the fundamental theorem for implicitly defined curves in the plane (from Calculus III) is stated below (and a proof in the spirit of MATHS 445 is provided).
First of all, recall that we showed in class that $M$ is a surface at every point, except possibly at the point(s) where the axis of revolution pokes through $M$. In this problem, you are supposed to establish that there is also a proper patch around such a point, making all of $M$ a surface.

Step 1: Verify that $C$ is symmetric with respect to the $z$-axis. (This is simple: $f(x^2, z) = f((-x)^2, z)$.)

Explain why that makes the tangent to the curve $f(x^2, z) = c$ at the point(s) on the $z$-axis horizontal. Therefore, near such a point, the curve $C$ must be the graph of a function $z = g(x)$. (See theorem in box below.)

Step 2: By Step 1, $M$ is the graph of a function $z = F(x, y)$ near the point in question, namely $F(x, y) = g(\sqrt{x^2 + y^2})$. It remains to show that this function $F$ is differentiable, as defined in Calculus III. (That will give us a proper patch around such a point.)

Explain why $F_x(0, 0) = \frac{\partial F}{\partial x}(0, 0) = 0$ and $F_y(0, 0) = \frac{\partial F}{\partial x}(0, 0) = 0$. You wish to find functions $\epsilon_1$ and $\epsilon_2$ such that

$$F(x, y) - F(0, 0) = \Delta z = F_x(0, 0)\Delta x + F_y(0, 0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

and $\epsilon_1 \to 0$, $\epsilon_2 \to 0$ as $(x, y) \to (0, 0)$. To this end, observe that $\Delta x = x - 0 = x$ and that $\Delta y = y - 0 = y$. Then write

$$\Delta z = \left(\pm \frac{\Delta z}{|x| + |y|}\right)\Delta x + \left(\pm \frac{\Delta z}{|x| + |y|}\right)\Delta y$$

and observe that

$$|\epsilon_1| = |\epsilon_2| \leq \left| \frac{\Delta z}{\sqrt{x^2 + y^2}} \right| = \left| \frac{g(t) - g(0)}{t - 0} \right|$$

with $t = \sqrt{x^2 + y^2} \to 0$. 
Theorem. [Implicitly defined curves]
Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be a differentiable function and suppose that the set \( C = \{ p \in \mathbb{R}^2 \mid g(p) = 0 \} \neq \emptyset \). Suppose, further, that for every \( p \in C \), \( dg : T_p(\mathbb{R}^2) \to \mathbb{R} \) is not identically zero. Then, for every \( p \in C \), there is an open region \( R \subseteq \mathbb{R}^2 \) such that \( C \cap R \) is the graph of a differentiable function of the form \( y = f(x) \) or \( x = h(y) \).

In either case, we can parameterize \( C \cap R \) by a curve \( \alpha(t) = (x(t), y(t)) \).

Note: Observe the striking similarity to Theorem 1.4. This theorem is usually not proved in Calculus. However, we can give the exact same proof as for Theorem 1.4.

Proof. Let \( p = (p_1, p_2) \in C \). Since \( dg = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy \), then one of the partial derivatives of \( g \) at \( p \) must be non-zero. Say \( \frac{\partial g}{\partial y}(p) \neq 0 \).

Consider the mapping \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( F(u, v) = (u, g(u, v)) \).

Then \( \text{det } DF(p) = \begin{vmatrix} \frac{\partial g}{\partial x}(p) & \frac{\partial g}{\partial y}(p) \\ 1 & 0 \end{vmatrix} \neq 0 \), then by the Inverse Function Theorem, there is an open set \( R \subseteq \mathbb{R}^2 \) about \( p \) and an open set \( G \subseteq \mathbb{R}^2 \) about \( F(p) = (p_1, g(p_1, p_2)) = (p_1, 0) \) such that the mapping \( F|_R : R \to G \) has a differentiable inverse \( (F|_R)^{-1} : G \to R \). Define an open subset of \( \mathbb{R} \) by \( D = \{ u \in \mathbb{R} \mid (u, 0) \in G \} \) and consider \( f(u) = y \circ F^{-1}(u, 0) \). Then \( F^{-1}(u, 0) = (u, f(u)) \). Consequently, \( C \cap R \) is the graph of \( f : D \to \mathbb{R} \), that is,

\( C \cap R = \{ (u, v) \in \mathbb{R}^2 \mid u \in D \text{ and } v = f(u) \} \).

§4.2 1. For a definition of parametrization, see Definition 2.3. Then consult Example 2.4 and Figure 4.14. (Caution: In part (c) you need to change your strategy if you want to parametrize the entire surface with one mapping; parametrizations need to be regular!)

2. Note that this problem continues onto Page 146. While the solution to the stated problem is straightforward, here is a short comment on its relevance:
Suppose $M$ is a surface and $\mathbf{x} : D \to M$ a patch. Let $(u_0, v_0) \in D$ and put $p = \mathbf{x}(u_0, v_0)$. By Lemma 3.6, a basis for $T_p(M)$ is given by $\{x_u(u_0, v_0), x_v(u_0, v_0)\}$. So, if $v_p, w_p \in T_p(M)$, say

$$v_p = a_1 x_u(u_0, v_0) + a_2 x_v(u_0, v_0)$$

and

$$w_p = b_1 x_u(u_0, v_0) + b_2 x_v(u_0, v_0),$$

then

$$v_p \cdot w_p = (a_1 x_u(u_0, v_0) + a_2 x_v(u_0, v_0)) \cdot (b_1 x_u(u_0, v_0) + b_2 x_v(u_0, v_0))$$

$$= a_1 b_1 E + a_1 b_2 F + a_2 b_1 F + a_2 b_2$$

$$= \begin{bmatrix} a_1 a_2 \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

In other words, the differentiable functions $E, F, G : D \to \mathbb{R}$ completely describe the inner product on the tangent space $T_p(M)$ for all $p \in x(D)$. They do so, using the coordinates in the basis $\{x_u, x_v\}$, but without direct reference to $\mathbb{R}^3$. This observation will become important later in the course.

Some textbooks, denote this matrix of real-valued functions with domain $D$ by

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}.$$

3. Background: As described in Definition 2.6, a ruled surface is a surface which can be parameterized as

$$\mathbf{x}(u, v) = \beta(u) + v\delta(u),$$

with some differentiable curves $\beta, \delta : I \to \mathbb{R}^3$. That is, a surface which consists of all points that lie on lines $L(u)$ through the (distinct) points $\{\beta(u), \beta(u) + \delta(u)\}$. As you vary $u$, you are varying the lines. As you are varying the $v$, you are running up and down a fixed line $L(u)$. You can think of the lines $L(u)$ as rulers who are marked by the parameter $v$. The notation $\beta$ stands for base curve and $\delta$ for directional curve.
Your generalized cone has all its lines running through a fixed point \( p \) and a variable point \( \delta(u) \). Notice that \( x(u, v) \) equals the cone point \( p \) exactly if \( v = 0 \).

Compute \( ||x_u \times x_v|| \).

5. This is a ruled surface \( x(u, v) = \beta(v) + u\delta(v) \) with \( \beta(v) = (0, 0, bv) \) and \( \delta(v) = (\cos v, \sin v, 0) \). Notice the role of \( u \) and \( v \) are switched when compared to the definition of a ruled surface. (Compare with hints to Problem 3.) So, we have rulers (lines) \( L(v) \) that are marked by the parameter \( u \).

(a) Use Problem 2.
(b) Describe both parameter curves \( x(t, v_0) \) as well as \( x(u_0, t) \).
(c) Start with

\[
\frac{y}{x} = \frac{u \sin v}{u \cos v}
\]

and

\[ z = bv. \]

Combine this into one equation, containing only \( x, y, z \), that has no denominators, and is of the form \( g(x, y, z) = 0 \). Then every point on the surface satisfies \( g(x, y, z) = 0 \). Conversely, explain why every point \( p = (p_1, p_2, p_3) \in \mathbb{R}^3 \) with \( g(p) = 0 \) lies on the surface.

(d) Restrict the parameters to \( 0 \leq v \leq 2\pi \) and \(-1 \leq u \leq 1 \) and use \( b = 1/2 \). The Mathematica command is

```
ParametricPlot3D[{u*Cos[v], u*Sin[v], v/2}, {u, -1, 1}, {v, 0, 2Pi}]
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7. (b) Use \( \beta(u) = (\cos u, \sin u, 0) \) and \( U_3 = (0, 0, 1) \).
(c) First show that every point \( x(u, v) \) satisfies

\[ g(x(u, v)) = 0 \]

with \( g = x^2 + y^2 - z^2 - 1 \). Then verify that every point \( p = (p_1, p_2, p_3) \in \mathbb{R}^3 \) with \( g(p) = 0 \) lies on the surface parameterized by \( x : \mathbb{R}^2 \to \mathbb{R}^3 \). When you do this, draw \( (p_1, p_2) \) in the \( xy \)-plane. It must lie outside the unit circle. Why? Draw the two tangents through the point \( (p_1, p_2) \) to the unit circle in the \( xy \)-plane. The length of the segment from \( (p_1, p_2) \)
to the origin is $\sqrt{x^2(p) + y^2(p)}$. The length of the segment from the origin to either point of tangency is 1. Notice that the length of the segment from $(p_1, p_2)$ to either point of tangency is $\sqrt{p_1^2 + p_2^2 - 1} = |p_3|$. Compare with Figure 4.20. Since $z(p) = \pm \sqrt{x^2(p) + y^2(p)} - 1$, you know that $(p_1, p_2, p_3)$ lies either on the ruler through one point of tangency or the other. Explain!

(d) The sign in front of $U_3$ in Part (a) changes to a minus sign. How does this affect Part (c)? The term *doubly ruled*, simply means that there is more than one way to describe the surface as a ruled surface.

§4.3 4. (a) Actually, it is not necessary for $x : D \to M$ to be one-to-one or regular. Fix $p = x(u_0, v_0)$.

Approach 1: [Preferred Method]
Consider $\alpha(t) = (t, v_0)$, and show that $\alpha'(u_0) = U_1(u_0, v_0)$. Then $x_*(U_1(u_0, v_0)) = x_*(\alpha'(u_0)) = (x \circ \alpha)'(u_0) = \cdots$.

Approach 2: [Alternative Method]
Let $q = (u_0, v_0)$, then $p = x(q)$. You wish to show that $x_*(U_1(q)) = x_*(q)$. Recall that $x_*(w_q) = (Aw)_x(q)$, where $A$ is the derivative matrix of $x : D \to M$ at $q$ and $w_q \in T_q(R^2)$.

(b) Check your class notes.

5. (a) Ignore the hint in the back of your textbook!
Instead, consider the Monge patch $x : D \to M$ given by

$$x(u, v) = (u, v, f(u, v))$$

Compute $x_u$ and $x_v$. Now, consider a point $p = x(u_0, v_0) \in M$ and a vector $v_p \in T_p(R^3)$, say $v_p = (v_1, v_2, v_3)_p$. Then $v_p \in T_p(M)$ if an only if

$$v_p = c_1 x_u(u_0, v_0) + c_2 x_v(u_0, v_0)$$

for some $c_1, c_2 \in \mathbb{R}$. Write out what that means.

(b) Write $v_p = c_1 x_u(u_0, v_0) + c_2 x_v(u_0, v_0) + c_3 (x_u(u_0, v_0) \times x_v(u_0, v_0))$ and consider $v_p \cdot (x_u(u_0, v_0) \times x_v(u_0, v_0))$

8. Find $x_u$ and $x_v$ and make sure you use Part (a) to do Part (b).
Here are some useful *Mathematica* commands for Part (c):
x[u_, v_] = {v*Cos[u], v*Sin[u], v}
Cone = ParametricPlot3D[x[u, v], {u, 0, 2 Pi}, {v, 0, 1}]
Curve = ParametricPlot3D[x[Sqrt[2]*t, Exp[t]],
{t, -5, 1}, PlotStyle -> Thickness[.01]]
Show[Cone, Curve]

§ 4.4 1. (a) Let \( v_p \in T_p(M) \), say \( v_p = \alpha'(0) \) for some curve \( \alpha : I \rightarrow M \). Then
\[
d(fg)(v_p) = v_p[fg] = ((fg)\circ \alpha)'(0) = ((f\circ \alpha)(g\circ \alpha))'(0) = \cdots
\]
(b) Here is one way to do Part (b); another way is shown in the textbook. Put \( \eta = f\phi \). Then on any patch \( x : D \rightarrow M \) we have
\[
d\eta = (f\phi(x_u))d\hat{u} + (f\phi(x_v))d\hat{v} = \phi(x_u)d\hat{u} + \phi(x_v)d\hat{v}, \text{ so that}
\]
\[
d\eta = d(f\phi(x_u))\wedge d\hat{u} + d(f\phi(x_v))\wedge d\hat{v},
\]
with \( f, \phi(x_u), \phi(x_v) \in F(x(D)) \). Now use Part (a).

8. You are asked to draw a sketch which shows what the parametric curves of equidistant horizontal and vertical lines in the rectangular domain of \( x : D \rightarrow M \) turn into. Label the resulting curves on \( M \) like a coordinate system.

(c) The geographical patch \( x \) for the sphere is described in Example 2.2 on Page 140.

§ 4.5 3. Let \( p = x(u_0, v_0) \in M \). Since \( y : D \rightarrow N \) might not be a patch, we choose an actual patch \( w : E \rightarrow N \) with \( q = F(p) \in w(E) \). Then, by Definition 5.1, \( F : M \rightarrow N \) will be differentiable at \( p \) if and only if \( w^{-1} \circ F \circ x \) is differentiable at \( (u_0, v_0) \). Now, \( w^{-1} \circ F \circ x = w^{-1} \circ y \). However, since \( y : D \rightarrow N \) is assumed to be differentiable at \( (u_0, v_0) \), then \( w^{-1} \circ y \) is differentiable at \( (u_0, v_0) \), as we proved in class.

4. Start with the patch \( x(u, v) : \mathbb{R}^2 \rightarrow H \) that you found in Exercise 5 of Section 4.2. Also review Example 2.5 on Page 144, which describes a parametrization \( y(u, v) : \mathbb{R}^2 \rightarrow T \). Define \( F : H \rightarrow T \) as in Problem 3. (Make sure that the rulings of \( H \) are parametrized by \( u \) and that the meridians of \( T \) are the \( u \)-parameter curves of \( T \). Verify that
\[ \mathbf{x}^{-1}(p_1, p_2, p_3) = (p_1 \cos \frac{p_3}{b} + p_2 \sin \frac{p_3}{b}, p_3) \]

by checking that \( \mathbf{x}^{-1} \circ \mathbf{x}(u, v) = (u, v) \) for all \((u, v) \in \mathbb{R}^2 \) and use it to find \( F = \mathbf{y} \circ \mathbf{x}^{-1} \).

7. To get from a parametrization to a patch, we need to restrict the domain of \( \mathbf{x} : D \to M \) to small enough a domain \( G \subseteq D \), so that \( \mathbf{x} : G \to M \) is injective. To this end, let \((u_0, v_0) \in D \) and choose an actual patch \( \mathbf{y} : E \to M \) with \( \mathbf{x}(u_0, v_0) = \mathbf{p} \in \mathbf{y}(E) \). First, we can assume without loss of generality that we have already made \( D \) small enough so that \( \mathbf{x}(D) \subseteq \mathbf{y}(E) \). Then \( \mathbf{y} \circ (\mathbf{y}^{-1} \circ \mathbf{x}) \circ (\mathbf{x}^{-1}) = (\mathbf{y} \circ \mathbf{y}^{-1} \circ \mathbf{x}) = \mathbf{x} \) is injective. So, \((\mathbf{y}^{-1} \circ \mathbf{x})_* \) must be injective. Therefore, by the Inverse Function Theorem, there is an open region \( G \subseteq D \) about \((u_0, v_0) \) such that the restriction of \( \mathbf{y}^{-1} \circ \mathbf{x} \) to \( G \) has a differentiable inverse. Consequently, \( \mathbf{x} \) must be injective on \( G \).

9. (a) Prove that the definition of \( F_* \) does not depend on the choice of curve: given \( \mathbf{v}_p = \alpha'(0) \in T_p(M) \) and a mapping \( F : M \to N \), we have defined \( F_*(\mathbf{v}_p) = (F \circ \alpha)'(0) \in T_{F(p)}(N) \). You need to show that if \( \mathbf{v}_p = \beta'(0) \) for some other curve \( \beta : I \to M \), then \((F \circ \alpha)'(0) = (F \circ \beta)'(0) \). To this end, choose a patch \( \mathbf{x} : D \to M \) about \( \mathbf{p} \) and denote the coordinate functions of your curves by \( \mathbf{x}^{-1} \circ \alpha(t) = (a_1(t), a_2(t)) \) and \( \mathbf{x}^{-1} \circ \beta(t) = (b_1(t), b_2(t)) \). Then \( \alpha(t) = \mathbf{x}(a_1(t), a_2(t)) \) and \( \beta(t) = \mathbf{x}(b_1(t), b_2(t)) \). Differentiate with respect to \( t \) at \( t = 0 \) and use the fact that \( \mathbf{x}_a \) and \( \mathbf{x}_v \) constitute a basis, to show that \( a'_i(0) = b'_i(0) \) for \( i = 1, 2 \). Then differentiate

\[ F \circ \alpha = (F \circ \mathbf{x}) \circ (\mathbf{x}^{-1} \circ \alpha) \]

and

\[ F \circ \beta = (F \circ \mathbf{x}) \circ (\mathbf{x}^{-1} \circ \beta) \]

to show that the two derivatives agree at \( t = 0 \).
4.6 2. When you evaluate \( \phi \), recall that \( \phi(\mathbf{v}_p) = p_2^2 v_1 + 2p_1p_2v_2 \), for \( p = (p_1, p_2) \) and \( \mathbf{v}_p = (v_1, v_2)_p \).

4. To show that \( \psi \) is closed, prove that \( d\psi = 0 \). To show that \( \psi \) is not exact, show there is no function \( f \) with \( df = \psi \). [Note that if \( \psi = df \), then \( \int_\alpha \psi = f(q) - f(p) = 0 \) for all closed curves in \( \mathbb{R}^2 \setminus \{0\} \). However, if \( \alpha \) goes around the unit circle once, \( \int_\alpha \psi \neq 0 \).]
10. Although it might not seem that way, all the necessary information is given. For example, if \( \alpha : [0, 1] \to M \) is the bottom curve of the rectangle \( R = [0, 1] \times [0, 1] \), i.e. \( \alpha(t) = x(t, 0) \), then we get \( \phi(\alpha'(t)) = \phi(x_u(t, 0)) = t + 0 \).

If you wonder what the given hint is all about, observe that Definition 4.4 only applies to patches \( x \). The fact that this formula works in general is listed as Problem 6 in Section 4.4 and proved as a Corollary on your handout “Differential forms on surfaces” (at the very end). The given hint, namely that \( x^*d\phi = d(x^*\phi) \), is used in the proof of that Corollary.

13. Let the textbook, or its hints, not distract you!

I suggest you prove Stokes’s Theorem in the following version:

**Theorem** [Stokes]

Let \( x : D \to M \) be a parametrization of a surface. Suppose \([a, b] \times [c, d] \subseteq D \) and let a vector field \( V = v_1 U_1 + v_2 U_2 + v_3 U_3 \) on \( \mathbb{R}^3 \) be given. Then

\[
\int_c^d \int_a^b (\text{curl } V) \cdot (x_u \times x_v) \, du \, dv = \sum_{i=1}^4 \int_{a_i}^{b_i} V(\alpha(t)) \cdot \alpha'(t) \, dt,
\]

where \( a = a_1 = b_3, b = a_3 = b_1, c = a_2 = b_4, d = a_4 = b_2 \) and \( \alpha \) is the curve which results from restricting \( x \) to the counterclockwise boundary of the rectangle \([a, b] \times [c, d]\).

Proceed as follows:

Use the 1-form \( \eta \) defined by \( \eta(w_p) = V(p) \cdot w_p \). Observe that \( \eta = \eta(U_1)dx + \eta(U_2)dy + \eta(U_3)dz = v_1 dx + v_2 dy + v_3 dz \) is a 1-form on \( \mathbb{R}^3 \), namely the 1-form which corresponds to the vector field \( V \) by way of Rule (1) from Problem 8 of Section 1.6.

So, you may compute \( d\eta \) in terms of \( dx \wedge dy, dx \wedge dz \) and \( dy \wedge dz \), by way of \( d\eta = dv_1 \wedge dx + dv_2 \wedge dy + dv_3 \wedge dz = \ldots \) and then show that \( d\eta(x_u, x_v) = \ldots = (\text{curl } V) \cdot (x_u \times x_v) \). Finally, use Theorem 6.5 and our computational rules from class.