Chapter 2

Notation. Recall that $\mathcal{F}(\mathbb{R}^3)$ denotes the set of all differentiable real-valued functions $f : \mathbb{R}^3 \to \mathbb{R}$ and $\mathcal{V}(\mathbb{R}^3)$ denotes the set of all differentiable vector fields on $\mathbb{R}^3$.

§2.1 7. The problem in the textbook is not well-posed. This is what you are supposed to show here: given $u \in T_p(\mathbb{R}^n)$ with $||u|| = 1$, show that for every $v \in T_p(\mathbb{R}^n)$ there is a unique pair $v_1, v_2 \in T_p(\mathbb{R}^n)$ such that (i) $v_1 = su$ for some $s \in \mathbb{R}$, (ii) $v = v_1 + v_2$, and (iii) $u \cdot v_2 = 0$; namely $v_1 = (v \cdot u)u$ and $v_2 = v - v_1$. It is easy to show that the given choices for $v_1$ and $v_2$ work. To show that these are the only choices, take the dot product of $v_1$ with both sides of Equation (ii), use Equations (i) and (iii), and solve for $s$.

Note: The condition $v_1 \cdot v_2 = 0$ from the textbook needs to be replaced by $u \cdot v_2 = 0$, so as to avoid the trivial solution $s = 0$, unless $v \cdot u = 0$.

12. The function $\vartheta$ is defined as

$$\vartheta(t) = \vartheta_0 + \int_0^t f(u)g'(u) - g(u)f'(u) \, du,$$

so that $\vartheta(0) = \vartheta_0$ and $\vartheta' = fg' - gf'$. You need to show that $(f(t) - \cos \vartheta(t))^2 + (g(t) - \sin \vartheta(t))^2 = 0$ for all $t$. This is given to be true when $t = 0$. So, you only have to show that this expression is constant. Since $f^2 + g^2 = 1$ and also $\sin^2 \vartheta + \cos^2 \vartheta = 1$, this amounts to showing that $f(t) \cos \vartheta(t) + g(t) \sin \vartheta(t)$ is constant. Take the derivative of this expression, use the chain rule, and use from above that $\vartheta' = fg' - gf'$. When simplifying your answer, group the sine-terms and the cosine-terms and make use of the fact that $ff' = -gg'$, which stems from differentiating $f^2 + g^2 = 1$.

Ponder on how one might come up with a formula like that.

§2.2 3. The answer is in the back of the textbook. Show your work!

Recall the definitions of the hyperbolic trigonometric functions:

$$\sinh t = \frac{e^t - e^{-t}}{2} \quad \text{and} \quad \cosh t = \frac{e^t + e^{-t}}{2}.$$ 

From these definitions you can verify that

$$\sinh' = \cosh \quad \text{and} \quad \cosh' = \sinh.$$
Notice that there is no minus sign in the last identity. It also follows that
\[ \cosh^2 - \sinh^2 = 1. \]
Since \( \sinh' > 0 \), there is an inverse function \( \sinh^{-1} \). (Indeed, you can solve for the inverse and find \( \sinh^{-1}(s) = \log(s + \sqrt{1 + s^2}) \), but this is not relevant for the problem.)
When calculating the unit-speed reparametrization, \( \beta(s) = \alpha(t(s)) \), simplify your answer as much as possible.

6. (c) Notice that \( \alpha'(t) \) is orthogonal to \( \alpha''(t) \) by Problem 2, since \( \|\alpha(t)\| \) is constant.

9. Use Mathematica.

11. (b) We are regarding \( u \) as a parallel vector field along \( \alpha \). First show that \( \|\alpha'(t)\| \geq |\alpha'(t) \cdot u| \). (To show this, use Problem 7 of Section 2.1: \( \alpha' = (\alpha' \cdot u)u + Y \), with \( u \) and \( Y \) orthogonal.) Then compute \( L(\alpha) = \int_a^b \|\alpha'(t)\| \ dt \geq \int_a^b |\alpha'(t) \cdot u| \ dt \geq |\int_a^b \alpha'(t) \cdot u \ dt| = \ldots = ||q - p|| \).
(c) Notice that if there is any \( t \in [a,b] \) with the property that \( \|\alpha'(t)\| > |\alpha'(t) \cdot u| \), then your work in Part (b) shows that \( L(\alpha) > ||q - p|| \), because all functions involved are continuous. Consequently, if \( L(\alpha) = ||q - p|| \), then \( Y \) is constantly zero. (Explain why!) So, \( \alpha'(t) = f(t)u \) with \( f(t) = \alpha'(t) \cdot u \). Show that there is a constant \( c \in \mathbb{R}^3 \) such that \( \alpha(t) \) lies on the line \( su + c \) with \( s \in \mathbb{R} \).

§2.3

2. Find \( T, N, B \) and \( \kappa, \tau \), by hand. Follow the example in your notes.

6. Start with \( \gamma(s) \) as given in the problem. Find \( \gamma'(s) \) and \( \gamma''(s) \). Then evaluate \( \gamma(0), \gamma'(0), \) and \( \gamma''(0) \). Compare these with \( \beta(0), \beta'(0) = T(0), \) and \( \beta''(0) = \kappa(0)N(0) \). This should tell you what to take for \( r, e_1, e_2, \) and \( c \).

Note: A fully rotatable Mathematica animation of the osculating circle for the so-called trefoil knot can be found on the course website.

8. Typos: Beware of the “tildes.”

Just as curves in 3-space have a right-handed Frenet frame \( T, N, B, \) a planar unit speed curve \( \beta : I \to \mathbb{R}^2 \), say \( \beta(t) = (x(t), y(t)) \), can be given a 2-dimensional “right-handed” Frenet frame \( \tilde{T}, \tilde{N} \) by \( \tilde{T} = T = (x', y') \) and \( \tilde{N} = (-y', x') \), that is, a frame which has the same orientation as the natural 2-dimensional frame \( U_1, U_2 \).
Note that $\vec{N} \cdot T = 0$ and $||\vec{N}|| = 1$, so that $\vec{N} = \pm N$ (as $N$ also lies in $\mathbb{R}^2$ with $N \cdot T = 0$ and $||N|| = 1$). Since $T' = \kappa N$, we can therefore define $\tilde{\kappa}$ by $T' = \tilde{\kappa} \tilde{N}$.

(a) If we write the 2-dimensional Frenet apparatus of $\beta$ as $\tilde{T}, \tilde{N},$ and $\tilde{\kappa}$, and follow the logic of the 3-dimensional case, this problems yields the 2-dimensional Frenet formulas:

$$\tilde{T}' = \tilde{\kappa} \tilde{N}$$
$$\tilde{N}' = -\tilde{\kappa} \tilde{T}$$

(b) Since $||\tilde{T}|| = 1$, we can write $\tilde{T}(s) = (\cos \varphi(s), \sin \varphi(s))$ by Problem 12 of Section 2.1. Differentiate this formula using the chain rule and compare with $\tilde{T}' = \tilde{\kappa} \tilde{N}$ to see that $\tilde{\kappa} = \varphi'$. 

(d) Use the definition of $\kappa$.

10. (a) Without loss of generality, you may assume that $c = 0$. Start with $\alpha = (\alpha \cdot T)T + (\alpha \cdot N)N + (\alpha \cdot B)B$. By repeatedly differentiating $\alpha \cdot \alpha = r^2 \equiv constant$, show that $\alpha \cdot T = 0$, that $\alpha \cdot N = -1/\kappa = -\rho$, and that $\rho' = -\tau(\alpha \cdot B)$ (use the Frenet formulas). Conclude that $r^2 = \rho^2 + (\rho')^2$.

(b) As suggested in the text, consider the function

$$\gamma = \alpha + \rho N + \rho' \sigma B.$$  \hspace{1cm} (1)

Differentiate $\gamma$ to show that it is constant, say $\gamma = c$. Once you have shown this, explain why $||\alpha - c|| = r$, so that $\alpha$ lies indeed on a sphere, centered at $c$ with radius $r$.

Note: It is a bit tricky to show that the function $\gamma$, defined in Equation (1), is constant. First differentiate $\gamma$. Then use the Frenet Formulas. After some work, $\gamma'$ will simplify to

$$\gamma' = (\rho\tau + \rho' \sigma' + \rho'' \sigma)B.$$ 

To show that all this is zero, differentiate the equation

$$\rho^2 + (\rho')^2 = r^2 \equiv constant.$$
6. Carefully read the proof of Theorem 4.6.

12. Let \( \beta(s) = \alpha(t(s)) \) be the unit speed parametrization of \( \alpha \). Then \( \alpha(t) = \beta(s(t)) \). Then, using \( \tilde{T}(t) = \tilde{T}_\beta(s(t)) \), \( \tilde{N}(t) = \tilde{N}_\beta(s(t)) \), and \( \tilde{\kappa}(t) = \tilde{\kappa}_\beta(s(t)) \), we get

\[
\alpha'(t) = s'(t)\beta'(s(t)) = v(t)\tilde{T}_\beta(s(t)) = v(t)\tilde{T}(t).
\]

Also,

\[
\tilde{T}'(t) = s'(t)\tilde{T}'_\beta(s(t)) = v(t)\tilde{\kappa}_\beta(t)\tilde{N}_\beta(s(t)) = v(t)\tilde{\kappa}(t)\tilde{N}(t).
\]

Hence,

\[
\alpha''(t) = v'(t)\tilde{T}(t) + v(t)\tilde{T}'(t)
= v'(t)\tilde{T}(t) + v(t)v(t)\tilde{\kappa}(t)\tilde{N}(t)
\]

So, \( \tilde{N} = J(\tilde{T}) = J(\alpha'/\nu) = J(\alpha'/\nu) \) and \( \alpha'' \cdot \tilde{N} = \nu^2 \tilde{\kappa} \). Put all this together.

13. Again, they mean \( \alpha^* = \alpha + (1/\tilde{\kappa})\tilde{N} \). (See Problem 12 and its hint above.) The evolute \( \alpha^* \) is the curve, which connects the centers of curvature. (See Problem 6 of Section 2.3.) In Part (b), make sure to fix the time parameter \( t \) and use a new letter, say \( s \), to express \( \lambda_t(s) \). For Part (c), assume that \( \alpha \) has unit speed and use the 2-dimensional Frenet formulas of Problem 8 of Section 2.3.

To see this problem “in action”, go to the course website or cut and paste the script below (hint to Problem 14) into Mathematica.

14. (a) Here is a Mathematica script, which you can cut and paste into a Mathematica notebook:

```mathematica
x[t_] = 2Cos[t];
y[t_] = Sin[t];
a[t_] = {x[t], y[t]};
evolute[t_] = a[t] + (a'[t].a'[t])/(a''[t].{-y'[t], x'[t]}){-y'[t], x'[t]};
kappaTilde[t_] = (x'[t]y''[t] - x''[t]y'[t])/((x'[t])^2+(y'[t])^2)^(3/2); p=.7;
```
Show[
  ParametricPlot[{a[t], evolute[t]}, {t, 0, 2Pi},
  AspectRatio -> Automatic],
  ParametricPlot[evolute[p] +
  (1/Abs[kappaTilde[p]])*{Cos[t], Sin[t]}, {t, 0, 2Pi}],
  ParametricPlot[t*evolute[p] + (1 - t)*a[p], {t, 0, 1}],
  Graphics[{AbsolutePointSize[5], Point[evolute[p]]}],
  Graphics[{AbsolutePointSize[5], Point[a[p]]}]]

Note: You can add the following line and watch a little movie:

Animate[Show[
  ParametricPlot[{a[t], evolute[t]}, {t, 0, 2 Pi},
  AspectRatio -> Automatic],
  ParametricPlot[
  evolute[p] + (1/Abs[kappaTilde[p]])*{Cos[t], Sin[t]}, {t, 0, 2 Pi}],
  ParametricPlot[t*evolute[p] + (1 - t)*a[p], {t, 0, 1}],
  Graphics[{AbsolutePointSize[5], Point[evolute[p]]}],
  Graphics[{AbsolutePointSize[5], Point[a[p]]}]], {p, 0, 2 Pi}]

15. Here is a Mathematica script for the Frenet apparatus, based on Theorem 4.3. Using this, you can create your projections:

\[a(t) = \{3t - t^3, 3t^2, 3t + t^3\}\]
\[\text{NNorm}[v_] = \sqrt{v.v}\]
\[\text{TT}[t_] = \text{Simplify}[a'[t]/\text{NNorm}[a'[t]]]\]
\[\text{BB}[t_] = \text{Simplify}[\text{Cross}[a'[t], a''[t]]/\]
\[\text{NNorm}[\text{Cross}[a'[t], a''[t]]]\]
\[\text{NN}[t_] = \text{Simplify}[\text{Cross}[\text{BB}[t], \text{TT}[t]]]\]
\[\text{kappa}[t_] = \text{Simplify}[\text{NNorm}[\text{Cross}[a'[t], a''[t]]]/\]
\[\text{NNorm}[a'[t]]^3]\]
\[\text{tau}[t_] = \text{Simplify}[\text{Cross}[a'[t], a''[t]].a'''[t]/\]
\[\text{NNorm}[\text{Cross}[a'[t], a''[t]]]^2]\]

For example, the projection of \(\alpha(t)\) onto the osculating plane spanned by \(T(0)\) and \(N(0)\), can be plotted by

\[\text{eps}=.2\]
\[\text{ParametricPlot[}\{a[t].\text{TT}[0], a[t].\text{NN}[0]\},\{t, -\text{eps}, \text{eps}\},\]
\[\text{AspectRatio->1}\]

Do at least two of the listed examples and describe the output.
18. For the definition of total curvature, see Problem 17.

**Typo:** Problem 18 is stated incorrectly. Assume that the planar curvature $\kappa$ of the unit speed planar closed curve $\alpha : [a, b] \to \mathbb{R}^2$ does not change sign. Show that the total curvature $\int_a^b \kappa(t) \, dt$ of $\alpha$ is an integer multiple of $2\pi$.

**Hint:** $\kappa = |\tilde{\kappa}| = |\varphi'|$ by Problem 8 of Section 2.3.

The point here is that $T = (\cos \varphi, \sin \varphi)$ and that $T(b) = T(a)$ since the curve has to be differentiable all around. Notice that for the planar curve

$$\alpha(t) = (4 \cos 2t + 2 \cos t, 4 \sin 2t - 2 \sin t),$$

$\tilde{\kappa}$ does not change sign, but that its total curvature is equal to $4\pi$. So, the original statement of Problem 18 is not correct.

With a little bit more work (and based on the fact that a circle has total curvature $2\pi$) one can actually prove the following theorem, which you are not asked to prove as part of this exercise:

**Theorem**

Suppose $\alpha : [a, b] \to \mathbb{R}^2$ is a regular closed curve of unit speed and $\alpha(s_1) \neq \alpha(s_2)$ for all $a \leq s_1 < s_2 < b$, then $\int_a^b \kappa(s) \, ds = \pm 2\pi$.

19. Do not call the curve $\tau$, as is suggested in the text. Rather, call it $\alpha$. The symbol $\tau$ is reserved for torsion. Since this curve does not have unit speed, the total curvature has to be computed by $\int_a^b \kappa(t) v(t) \, dt$. For Parts (a) and (b), use the *Mathematica* script given in the hint to Problem 15 above and integrate numerically. For Part (c), multiply the $z$-coordinate of the curve by a sufficiently small fraction.

21. This is cumbersome by hand. You might want to use *Mathematica* to look at the simplified ratio $\tau/\kappa$.

§2.5

1. We took Lemma 5.2 as our definition.

2. (e) **Typo:** They mean $\nabla_V (\nabla_V W)$.

3. Start with $W \cdot W = c$ and calculate $V[W \cdot W]$ using Corollary 5.4.

§2.6

2. First, find the attitude matrix for the cylindrical and spherical frame field, respectively. They can be found in Example 6.2.

Recall that if $v_p = v_1 U_1 + v_2 U_2 + v_3 U_3$ is a tangent vector at $p$, then the matrix product

$$A \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$
yields the coordinates of \( v_p \) in terms of the frame field with attitude matrix \( A \). So, you need to multiply the given natural coordinates with your attitude matrices. To have your answer in terms of the functions \( r, \vartheta, z \) and \( \rho, \vartheta, \varphi \), respectively, you need to rewrite the functions \( x, y, z \) as \( x = r \cos \vartheta \) and \( y = r \sin \vartheta \) in the case of the cylindrical frame field, and as \( x = \rho \cos \vartheta \cos \varphi \), \( y = \rho \sin \vartheta \cos \varphi \), and \( z = \rho \sin \varphi \), in the case of the spherical frame field.

Some of the answers should not come as a surprise. Draw some pictures to illustrate what is happening.

§2.7 3. Do the computations by hand first and then use Mathematica to check your answer. In order to do so, simply download the notebook `Cartan.nb` via the link “Cartan Equations” from our course website and modify it appropriately.

5. **Typo**: Here is what you need to prove:

\[
\nabla_V W = \sum_j \left\{ V[f_j] + \sum_i f_i \omega_{ij}(V) \right\} E_j.
\]

While the main point of this problem is to show how to calculate the covariant derivative \( \nabla \) directly in the moving frame field, we also learn from this equation (and from Lemma 7.1) that the covariant derivative \( \nabla \) and the connection forms \( (\omega_{ij}) \) determine each other.

Here are some hints:

By linearity and the product rule, we have

\[
\nabla_V W = \nabla_V (\sum_{i=1}^{3} f_i E_i) = \sum_{i=1}^{3} V[f_i] E_i + \sum_{i=1}^{3} f_i \nabla_V E_i
\]

Use Theorem 7.2 on \( \nabla_V E_i \). Now comes the tricky part: rename the running index in the first sum from \( i \) to \( j \) and rearrange the order of the two sums at the end. Then factor out \( E_j \).

8. Note: This problem continues onto Page 94.

At the outset of the problem, we assume we have defined the functions \( \kappa \) and \( \tau \) as \( \kappa = T' \cdot N \) and \( \tau = -B' \cdot N \).

It is given that \( T = E_1 \circ \beta, \) \( N = E_2 \circ \beta, \) and \( B = E_3 \circ \beta. \) Since \( T(t) = \beta'(t) \in T_{\beta(t)}(\mathbb{R}^3) \), we can compute, for example,

\[

\nabla_{T(t)} E_2 = \nabla_{\beta'(t)} E_2 = (E_2 \circ \beta)'(t) = N'(t).
\]
This allows us to work out

\[ \omega_{23}(T(t)) = (\nabla_{T(t)} E_2) \cdot E_3(\beta(t)) = N'(t) \cdot B(t). \]

Play with some other combinations! By definition of the connection forms, we have

\[ \nabla_{T(t)} E_2 = \omega_{21}(T(t))E_1(\beta(t)) + \omega_{22}(T(t))E_2(\beta(t)) + \omega_{23}(T(t))E_3(\beta(t)), \]

which boils down to

\[ N'(t) = \omega_{21}(T(t))T(t) + \omega_{22}(T(t))N(t) + \omega_{23}(T(t))B(t). \]

Do you see the connection?

§2.8 1. Repeat the argument of Problem 5 of Section 2.7 using the appropriate product rule:

\[ df_i \wedge \theta_i + f_i d\theta_i. \]

3. (b) Recall that \( V[f] = df(V) \).

(c) This problem needs some clarification on the notation. You should consider \( g = f(r, \vartheta, z) \). Then, by the usual chain rule we get

\[ \frac{\partial g}{\partial x} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial f}{\partial \vartheta} \frac{\partial \vartheta}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x}. \]

Using \( V_p = U_1(p) \) we can write this as

\[ V_p[g] = \frac{\partial f}{\partial r} V_p[r] + \frac{\partial f}{\partial \vartheta} V_p[\vartheta] + \frac{\partial f}{\partial z} V_p[z]. \]  

Since the same can be said about \( V_p = U_2(p) \) and \( V_p = U_3(p) \), we can use linear combinations to argue that the rule in Equation (2) holds for every \( v_p \in T_p(\mathbb{R}^3) \). So, if \( V \in \mathcal{V}(\mathbb{R}^3) \) we have

\[ V[g] = \frac{\partial f}{\partial r} V[r] + \frac{\partial f}{\partial \vartheta} V[\vartheta] + \frac{\partial f}{\partial z} V[z] \]

(so that \( dg = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \vartheta} d\vartheta + \frac{\partial f}{\partial z} dz \)). Use this modified chain rule in conjunction with the other parts of the problem to show that

\[ E_1[g] = \frac{\partial f}{\partial r}, \ E_2[g] = \frac{1}{r} \frac{\partial f}{\partial \vartheta}, \ E_3[g] = \frac{\partial f}{\partial z}. \]

This is what they meant to state.
4. As hinted in the text, put $E_3 = U_3$ and you have a frame going. You are asked to find 2-dimensional analogs of the Cartan formulas, very much like the 2-dimensional Frenet apparatus for planar curves. What does the attitude matrix $A$ equal in this problem?

(a) Calculate the matrix $W$ of connection forms using the formula $W = (dA) \wedge A^T$. (The only non-trivial connection forms will be $\omega_{12} = d\psi$ and $\omega_{21} = -d\psi$.) Then calculate the dual forms by

$$
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{bmatrix} = A
\begin{bmatrix}
dx \\
dy \\
dz
\end{bmatrix}
$$

(b) Show in two different ways, first using Theorem 8.3 and then by differentiating your answers from part (a), that

$$
\begin{align*}
d\theta_1 &= d\psi \wedge \theta_2 \\
d\theta_2 &= -d\psi \wedge \theta_1
\end{align*}
$$

and that

$$
d\omega_{12} = d\omega_{21} = 0.
$$

The point of this problems is that if we now apply these formulas to vector fields of $\mathbb{R}^2$ only, we have 2-dimensional Cartan formulas.