Chapter 1

Notation. Recall that $\mathcal{F}(\mathbb{R}^3)$ denotes the set of all differentiable real-valued functions $f : \mathbb{R}^3 \to \mathbb{R}$ and $\mathcal{V}(\mathbb{R}^3)$ denotes the set of all differentiable vector fields on $\mathbb{R}^3$.

§1.1 4. The point of this problem is not to merely find the answer, but to practice our new notation. Recall that $x, y$ and $z$ are now functions and not variables, and that $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$ denote partial derivatives with respect to the first, second and third coordinate of a function $\mathbb{R}^3 \to \mathbb{R}$, no matter what letter you use to denote these coordinates.

For example, in the stated notation the function $f(x, y, z)$ is equal to the function $f$, because for every $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, we have $f(x(p), y(p), z(p)) = f(p_1, p_2, p_3) = f(p)$. Similarly, the function $\frac{\partial}{\partial x} x^2$ is equal to the function $2x$, because the partial derivative of the function $f = x^2$, given by the rule $f(p_1, p_2, p_3) = p_2^2$, with respect to its first coordinate is the function $g = 2x$, given by the rule $g(p_1, p_2, p_3) = 2p_1$.

To be efficient, use the chain rule. In the old Calculus III notation, you would have written $u = g_1(x, y, z)$, $v = g_2(x, y, z)$, $w = g_3(x, y, z)$, $s = h(u, v, w)$, and used the formula

$$\frac{\partial s}{\partial x} = \frac{\partial s}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial s}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial s}{\partial w} \frac{\partial w}{\partial x}.$$ 

Explain why, in the setting of our textbook, this becomes

$$\frac{\partial f}{\partial x} = \frac{\partial h}{\partial x}(g_1, g_2, g_3) \frac{\partial g_1}{\partial x} + \frac{\partial h}{\partial y}(g_1, g_2, g_3) \frac{\partial g_2}{\partial x} + \frac{\partial h}{\partial z}(g_1, g_2, g_3) \frac{\partial g_3}{\partial x}.$$ 

§1.2 5. (a) Start with $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ and write out the vectors $V_1(p), V_2(p), V_3(p)$. Then use (appropriate) Linear Algebra.

(b) First find $U_1, U_2$ and $U_3$ in terms of $V_1, V_2$ and $V_3$. One way to solve this problem is to use the usual elimination method. Notice that the element $1 + x^2 \in \mathcal{F}(\mathbb{R}^3)$ has a multiplicative inverse, namely $1/(1 + x^2) \in \mathcal{F}(\mathbb{R}^3)$, while the element
$x \in \mathcal{F}(\mathbb{R}^3)$ does not. Alternatively, you can use some Linear Algebra, like Cramer’s Rule.

[An aside: Cramer’s Rule does not only work when the matrices have real-number entries. It works, whenever the entries come from a so-called *commutative ring with identity*, like, for example, $\mathcal{F}(\mathbb{R}^3)$. The coefficient matrix of your problem is

$$\begin{bmatrix}
1 & 0 & x \\
0 & 1 & 0 \\
-x & 0 & 1 \\
\end{bmatrix}$$

This matrix has entries in $\mathcal{F}(\mathbb{R}^3)$. Cramer’s Rule works, whenever the determinant of this matrix has an *inverse* in the sense of the entry items.]

§1.3 4. It might help you to rewrite what you want to prove as

$$V = V[x]U_1 + V[y]U_2 + V[z]U_3.$$ 

To prove this identity, first write $V$ using its Euclidean coordinate functions,

$$V = fU_1 + gU_2 + hU_3,$$

and then show that $f = V[x], g = V[y], h = V[z]$. This can be done by using Formula (1) to evaluate $V[x], V[y], V[z]$. Think about what $U_3[x]$ or $U_2[y]$ might be, for example.

5. By Problem 4, $V = \sum V[x_i]U_i = \sum W[x_i]U_i = W$.

The significance of this problem is the following: If the two vector fields $V$ and $W$ agree as functions $V, W : \mathcal{F}(\mathbb{R}^3) \to \mathcal{F}(\mathbb{R}^3)$, then they agree, in fact, as vector fields.

§1.5 2. Every 1-form $\phi$ can be regarded as a function

$$\phi : \mathcal{V}(\mathbb{R}^3) \to \mathcal{F}(\mathbb{R}^3)$$

which is linear: $\phi(fV + gW) = f\phi(V) + g\phi(W)$ for all $f, g \in \mathcal{F}(\mathbb{R}^3)$ and $V, W \in \mathcal{V}(\mathbb{R}^3)$. We now work out a simple formula for computing the function $\phi(V)$. Let a 1-form $\phi = fdx + gdy + hdz$ with $f, g, h \in \mathcal{F}(\mathbb{R}^3)$ and a vector field $V = uU_1 + vU_2 + wU_3$ on $\mathbb{R}^3$. 
with \( u, v, w \in \mathcal{F}(\mathbb{R}^3) \) be given. Then \( \phi(V) = \phi(uU_1 + vU_2 + wU_3) \)
\[
= u \phi(U_1) + v \phi(U_2) + w \phi(U_3),
\]
by Lemma 5.4. We observe:

\[
\phi(V) = fu + gv + hw
\]

3. Read the above solution to Problem 2 and use this formula for Problem 3. For example, if \( \phi = yzdx + \cos(x + 3y)dz \) and 
\[
V = xyU_1 + (x + z)U_2 - x^3U_3,
\]
then

\[
\phi(V) = (yz) \cdot (xy) + 0 \cdot (x + z) + (\cos(x + 3y)) \cdot (-x^3) = xy^2z - x^3\cos(x + 3y).
\]

7. It’s all about checking linearity of \( \phi \) on any given tangent space \( T_p(\mathbb{R}^3) \); that is, verifying whether or not

\[
\phi(a v_p + b w_p) = a \phi(v_p) + b \phi(w_p)
\]

for all \( a, b \in \mathbb{R} \) and \( v_p, w_p \in T_p(\mathbb{R}^3) \). To do this, you need to fix a point \( p \in \mathbb{R}^3 \) and consider vectors \( v_p = (v_1, v_2, v_3)_p \) of the tangent space \( T_p(\mathbb{R}^3) \) at \( p \). Then the point \( p = (p_1, p_2, p_3) \) of application is fixed and the vector part \( (v_1, v_2, v_3) \) are the variables.

Do not neglect to express every 1-form as \( \sum_{i=1}^{3} f_i \, dx_i \).

§1.6 4. Make use of Problem 3. Also, think about why \( df \wedge df = 0 \) for any function \( f \). (In fact, it is not difficult to argue that if among any collection of forms \( \eta_1, \eta_2, \ldots, \eta_s \) there is an \( r \)-form \( \eta_i \) such that \( r \) is odd and \( \eta_i = \eta_j \) for some \( i \neq j \), then \( \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_s = 0 \).)

6. Simply compute \( dx = d(r \cos \vartheta) = \cos \vartheta \, dr - r \sin \vartheta \, d\vartheta \). Similarly, find \( dy \) and \( dz \). Then wedge them all together:

\[
dx \wedge dy \wedge dz = \ldots = rdr \wedge d\vartheta \wedge dz.
\]

The reason for calling this 3-form a “volume element” will become more clear when we discuss integration of forms in §4.6, in general, and area forms in §6.7, in particular.
7. You are expected to work this out “by hand”, that is, without using Problem 3 and Theorem 6.4(3)—otherwise you could simply repeat the inductive argument in the proof of the last theorem in the handout on differential forms. So, start with a 1-form \( \phi = f dx + g dy + h dz \). First, use Definition 6.3, Corollary 5.5, and the arithmetic rules for the wedge product to express \( d\phi \) in the form

\[
d\phi = u \, dx \wedge dy + v \, dy \wedge dz + w \, dx \wedge dz.
\]

Then compute \( d(d\phi) \). When collecting the terms at the end, remember that

\[
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right).
\]

8. There is a typo in Part (c). It should read “If \( \eta \leftrightarrow V \), then ...”. **Caution:** Be mindful of the order of the functions and of the minus sign in conversion rule (2).

§ 1.7 6. (a) Consider \( F: \mathbb{R} \to \mathbb{R} \) given by \( F(u) = u^3 \).

(b) Since \( F: \mathbb{R}^n \to \mathbb{R}^n \) is assumed to be one-to-one and onto, it has an inverse function \( G: \mathbb{R}^n \to \mathbb{R}^n \). All you need to check is that this inverse function is differentiable at every point \( q \in \mathbb{R}^n \). However, by Theorem 7.10 applied to the point \( p = G(q) \in \mathbb{R}^n \), there is open subset \( U \) of \( \mathbb{R}^n \) containing \( p \) and an open subset \( V \) of \( \mathbb{R}^n \) such that the function \( F|_U : U \to V \) has a differentiable inverse, namely \( G|_V : V \to U \). Since \( p \in U \), then \( q = F(p) \in V \). So, you are done.

7. You want to show that

\[
F_\ast(v_p)[g] = v_p[g \circ F]. \tag{2}
\]

Try not to make this too messy! The cleanest way is to consider any curve \( \alpha : I \to \mathbb{R}^n \) with \( \alpha'(0) = v_p \). (For example, \( \alpha \) could be the straight line \( \alpha(t) = p + tv \).) Put \( \beta(t) = F(\alpha(t)) \). Then \( \beta'(0) = F_\ast(\alpha'(0)) = F_\ast(v_p) \). Plug this into the left-hand side of Equation (2) and use Lemma 4.6.

10. (a) After you have shown that \( F \) is one-to-one and onto, you still have to show that \( F_\ast_p \) is one-to-one at every point \( p \in \mathbb{R}^2 \). (See Problem 6(b).)
(b) Recall that a mapping \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) is called a \textit{diffeomorphism} if it has a differentiable inverse function \( F^{-1} : \mathbb{R}^2 \to \mathbb{R}^2 \).

[An aside: It can be shown that if \( F : \mathbb{R}^n \to \mathbb{R}^m \) is a diffeomorphism, then \( m = n \).]