#1. Do Exercise S-4.4 on page 98 of the text.
(c) Do Exercise 4.4 on page 77 of the text.

#2. Prove that the set of all compact surfaces $X$, together with the binary operation $X \# Y$, does not form a group.

#3. Let $X$ be a compact surface. Suppose that $\xi(X)$ is a function which assigns a real number to every tiling $T$ of $X$ with $V$ vertices, $E$ edges, and $F$ faces. Suppose, further, that

(i) $\xi(X)$ is a linear function of the variables $V$, $E$, and $F$;
(ii) $\xi(X)$ is not affected by subdividing an edge with a vertex;
(iii) $\xi(X)$ is not affected by subdividing a face with an edge.

Prove that there is an $\alpha \in \mathbb{R}$ such that $\xi(X) = \alpha \chi(X)$ for all tilings $T$ of $X$.

#4. Sketch a handlebody decomposition on $T^2 \# T^2$ with three 0-handles, six 1-handles, and one 2-handle.

#5. There is a very natural way to turn a plane model of a compact surface into a nice handlebody decomposition:

Draw small basic neighborhoods around the vertices and connect the boundaries of these neighborhoods with lines that are parallel to the edges. Then the neighborhoods around those vertices that are to be identified form 0-handles and the region next to every edge is exactly half of a 1-handle. (The other half can be found next to the matching edge.) The middle region defines a 2-handle.

Illustrate this process by turning the following plane model of a compact surface into a handlebody decomposition. $X : e \ d \ e \ f \ a^{-1} b \ c^{-1} d^{-1} f \ a \ c^{-1} b$.

#6. Consider the handlebody decomposition of the compact surface $X$ of the previous problem. Recall that from an earlier homework problem you know that $X$ is topologically equivalent to $P^2 \# P^2 \# P^2 \# P^2 \# P^2$. Reestablish this fact by taking this example through the proof we discussed in class for characterizing surfaces represented as handlebodies.

[See the reverse side for hints!]
Hints:

2. First ponder on, whether or not, the following holds for all compact surfaces $X, Y$ and $Z$:

   (i) $(X \# Y) \# Z = X \# (Y \# Z)$ (associativity);
   (ii) $X \# Y = Y \# X$ (commutativity);
   (iii) $X \# S^2 = X$ (identity element).

At this point you might think that there is an underlying abelian group structure. However, that would require the existence of inverses. Show that, given any compact surface $X$, other than $S^2$ itself, there is no compact surface $Y$ such that $X \# Y = S^2$. [Why is this problem on this week’s exercise set?]

3. By (i) there are real numbers $a, b$, and $c$ such that $\xi(X) = aV + bE + cF$ for every tiling $T$ of $X$ with $V$ vertices, $E$ edges, and $F$ faces. Find the effect on this formula of subdividing an edge or a face. Deduce a relationship between $a$, $b$, and $c$.

   **Note:**
   This problem shows that there is only one possible formula for the Euler characteristic, if we want a linear function that is the same for all compact surfaces, not affected by subdivision, and insist that $\xi(S^2) = 2$.

4. **To maximize the fun, try not to read this hint!**

   Find two subsets $A$ and $B$ of $X = T^2 \# T^2$ each of which is topologically equivalent to an annulus (i.e. a “band”) and have the property that $X \setminus (A \cup B)$ is topologically equivalent to $S^2$ with four holes. (Illustrate this with a sequence of sketches.) Notice that each annulus is a 0-handle with a 1-handle attached. Place a disk $D$ (another 0-handle) on $X$ which is disjoint from $A$ and $B$. Then $X \setminus (A \cup B \cup D)$ is topologically equivalent to $S^2$ with five holes. Now run four more 1-handles through this sphere with five holes so that the remainder consists of one single disk, which is your 2-handle. Draw all your handles into a depiction of $X$, including the 2-handle.

5. First determine which vertices get identified. Those are your 0-handles. When you attach the 1-handles, make sure that they are twisted correctly: following the boundary curve around the handles must spell out a word that is consistent with that of the plane model. According to our convention with 2-dimensional handlebodies, suppress the 2-handle(s) in the final sketch. (The 2-handle(s) necessarily have to be attached to the remaining boundary circle(s).)