Chapter 2

Section 2.1: #18 Notice that since $D_3 \leq S_3$ and both groups have 6 elements, we actually have $D_3 = S_3$.

#24 On the circle, label the points at “12:00 o’clock”, “9:00 o’clock”, “6:00 o’clock”, and “3:00 o’clock”, by 1, 2, 3, and 4, respectively. The symmetries of this figure are: identity, reflection across the vertical line of symmetry, reflection across the horizontal line of symmetry, and $180^\circ$ rotation. Write down these four elements as members of $S_4$. For example, the reflection across the vertical line is given by

$$
\sigma = \begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 4 & 3 & 2
\end{pmatrix}.
$$

Now prove that your four elements really form a group. It turns out that your group is isomorphic to the group $V$ in Table 1.4.11 on page 67 of your text. Do you see an isomorphism?

#40 Show that this set might not be a group. Careful though, if $A$ is a finite set, then this set is always a group. (See Problem 41.) So, take an infinite set, like $A = \mathbb{Z}$. Find a subset $B \subseteq A$ and a permutation $\sigma$ of $A$ such that $\sigma[B] \subseteq B$ but $\sigma^{-1}[B] \nsubseteq B$. This will show that the given set

$$
H = \{ \sigma \in S_A \mid \sigma[B] \subseteq B \}
$$

does not contain all the inverses of its elements.

#43 Label the corners of the cube 1 through 8. The corner labeled 1 must, after you put the cube back, goes to one of the 8 corners. For each of these 8 possibilities, the corner labeled 2 has how many options? Once these two corners are placed, the cube is in the box: game over. So, how many rigid motions are there? When you work on the remainder of the problem, you can refer to elements of this group by using its permutation on the vertices.

#45 If $\sigma$ is not the identity, then there is an element $a \in \{1, 2, \ldots, n\}$ such that $\sigma(a) = b \neq a$. Choose a third element $c \in \{1, 2, \ldots, n\}$ such that $c \neq a$ and $c \neq b$. Write down a permutation $\gamma$, which has
the property that $\gamma(a) = a$ and $\gamma(b) = c$. Argue that $\sigma\gamma \neq \gamma\sigma$ by showing that the two sides do different things to a certain element.

#46 Since these orbits are really equivalence classes, your prove will look similar to the one you saw in class, where we proved that equivalence classes never overlap, unless they are equal.

Section 2.2: #16 Think about how many different disjoint cycle decompositions are possible, and use the rule from Problem 13(e).

#24 (a) First write the permutation in disjoint cycle decomposition and then break down each cycle as suggested on page 111, right after Definition 2.2.11. Now, carefully count how many transposition total you can get at the most.

(b) This should follow from your considerations in Part a.

(c) All this is really saying, is that once you can write a permutation as the product of some even/odd number of transpositions, then you can do it with any larger even/odd number of transpositions. Why should that be so?

#26 If you can find a single odd permutation in $H$, then you can use the same argument as when we showed that half the elements of $S_n$ are even.

#32 First do some examples to see that $\sigma^r$ need not be a cycle, even if $\sigma$ is a cycle.

We are working in $S_n$.

The sentence ends in ... gcd($n, r$) = 1. Prove that theorem!

Put $\tau = \sigma^r$. Argue that

$$\tau^i(1) = 1 \text{ if and only if } n \mid ri. \quad (1)$$

(Use the usual division algorithm argument: $ri = qn + j$ with $q, j \in \mathbb{Z}$ and $0 \leq j < n$. Also make use of the fact that $\sigma$ is a cycle of length $n$.)

Suppose gcd($n, r$) = 1. Then Statement (1) above turns into “$\tau^i(1) = 1 \text{ if and only if } n \mid i.$” How big does this make the $\tau$-orbit of 1? What does that say about $\tau$?

Conversely, suppose that $\tau$ is a cycle. Argue that $\tau = \sigma^r$ either moves every element or none at all. Being a cycle, $\tau$ has exactly
one orbit of size $\geq 2$. Argue that if this orbit had size $< n$, then $\tau$ would only move some elements and not others. Contradiction. So, $\tau$ is an $n$-cycle. Now, if $d = \gcd(n, r) \neq 1$, then $i = n/d$ is less than $n$. However, $n|ri$. What’s wrong with that?

Section 2.3: #6 I suggest you use our notation from class, that is consider the subgroup $H = \{e, sr\}$ of $D_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$. For arithmetic use $s^2 = e$, $r^4 = e$ and $sr = r^{-1}s = r^3s$.

#7 As in Problem 6, please identify the elements of $D_4$ by $s^i r^j$.

#33 What are the possible orders for a subgroup of $G$?

#37 First, read Theorem 2.3.14. Then read the hint given in the problem.

You need to show that the collection

$$\{a_ib_jK \mid i = 1, 2, \ldots, r \text{ and } j = 1, 2, \ldots, s\}$$

consists of distinct left cosets of $K$ in $G$ and that this collection contains all left cosets of $K$ in $G$. After you have shown this, you know that this number is $r \times s = [G : H] \cdot [H : K]$.

Justify the steps in the following outline:

If $a_ib_jK = a_pb_qK$, then $(a_ib_j)^{-1}(a_pb_q) \in K$. So, $b_j^{-1}a_i^{-1}a_pb_q = k$ for some $k \in K$. Since $b_j$ and $b_q$ are in $H$, and $K \subseteq H$, you should be able to argue that $a_i^{-1}a_p \in H$. So, $a_iH = a_pH$. Now, carefully reread the definition of the elements $a_1, a_2, \ldots, a_r$, to argue that $a_i = a_p$. Why does this imply that $b_j = b_q$? Now you know that these cosets are distinct.

In order to show that above collection contains all left cosets of $K$ in $G$, let an arbitrary element $g \in G$ be given. You need to locate $a_i$ and $b_j$ with $g \in a_ib_jK$. Do this in two steps. Place $g$ into its left $H$-coset in $G$, in order to write it as $g = a_ih$ for some $h \in H$. Now place $h$ into its left $K$-coset in $H$. That will allow you to place $g$ into one of the set $a_ib_jK$.

#38 You know that $G = \{H, aH\}$ and that $G = \{H, Hb\}$ for some $a, b \in G$. You should be able to argue that $aH = Hb$, without ever using that the group is finite. For example, to show that every element of $aH$ is in $Hb$ consider the alternative and prove that it cannot hold. Then argue that $gH = Hg$ for all $g \in G$. 
Recall: \( x \in xH \) for all \( x \in G \). Moreover, \( x \in yH \) if and only if \( xH = yH \). Similar statements hold for the right cosets.

#45 Consider the cyclic group \( G = \mathbb{Z}_n \). For every (positive) divisor \( d \) of \( n \) there is exactly one subgroup \( H \) of \( G \) of order \( d \). This subgroup is isomorphic to \( \mathbb{Z}_d \), and therefore has exactly \( \varphi(d) \) generators. Now, every element of \( G \) generates some subgroup. Put all this information together and argue that \( \varphi(d_1) + \varphi(d_2) + \cdots + \varphi(d_k) = n \), where \( d_1, d_2, \cdots, d_k \) is the list of all (positive) divisors of \( n \).

Section 2.4: #29 It is OK if you only go up to column labeled 6. The other entries are

<table>
<thead>
<tr>
<th>( n )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of groups</td>
<td></td>
<td></td>
<td></td>
<td>15</td>
<td>22</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#42 What they call torsion coefficients are simply the invariant factors. So, you are asked to rewrite the given groups in invariant factor form.