   20. Note that we need $-1 \leq x + y \leq 1$.
   24. This is the surface $z = x^2$.
   32. Focus on distinguishing features of the given formulas for $f(x, y)$ and the shown graphs, such as
      “for what $x$ and $y$ is $f(x, y) = 0$?”
      “is there a maximum/minimum value for $f(x, y)$?”
      “does $f(x, y)$ fluctuate between positive and negative values?”
      etc.
   48. Put $f(x, y) = k$ for various choices of $k$. Sketch each resulting curve in the $x$-$y$-input plane and label it with its number $k$. (Compare with Examples 10 and 11.)

§14.2. 10. If the limit
   \[ \lim_{(x,y) \to (0,0)} f(x, y) = \lim_{(x,y) \to (0,0)} \frac{5y^4 \cos^2 x}{x^4 + y^4} \]
   exists, you should get the same limit whether you approach $(0,0)$ along the $x$-axis, that is, whether you compute
   \[ \lim_{x \to 0} f(x,0) = \ldots \]
   or along the $y$-axis:
   \[ \lim_{y \to 0} f(0,y) = \ldots \]
   Are you getting the same answer?
   11. Try approaching the point $(0,0)$ along the $x$-axis, along the $y$-axis, and along the line $y = x$. Recall that $\frac{\sin t}{t} \to 1$ as $t \to 0$.
   15. After approaching $(0,0)$ along the axes and along the line $y = x$, try approaching it along the parabola $x = y^2$. 
16. When making the denominator smaller, the fraction gets larger. So, you can estimate as follows:

\[ 0 \leq \left| \frac{xy^4}{x^4 + y^4} \right| = \frac{|x|y^4}{x^4 + y^4} \leq \frac{|x|y^4}{0 + y^4} = |x|. \]

You can now either base an \( \epsilon-\delta \)-proof on this inequality (along the lines of \( |x| \leq \sqrt{x^2 + y^2} < \delta = \epsilon \)) or you can use the squeeze theorem.

17. Try this:

\[
\frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} = \frac{(x^2 + y^2)}{(\sqrt{x^2 + y^2 + 1} - 1)} \cdot \frac{(\sqrt{x^2 + y^2 + 1} + 1)}{(\sqrt{x^2 + y^2 + 1} + 1)} = \ldots
\]

23. Enter the following into Mathematica:

\[
\text{Plot3D}[(2*x^2+3*x*y+4*y^2)/(3*x^2+5*y^2),\{x,-2,2\},\{y,-2,2\},\text{PlotPoints}\to40]
\]

Then left-click and drag the picture around to view it from various angles.

37. At points \((x, y) \neq (0, 0)\) you can use the fact that combinations of continuous functions are continuous. At \((0, 0)\), you can either calculate the limit as \((x, y) \to (0, 0)\) and compare it with the function value at \((0, 0)\), which equals 1. If you do, use the fact that

\[ 0 \leq \left| \frac{x^2y^3}{2x^2 + y^2} \right| \leq \left| \frac{x^2y^3}{2x^2 + 0} \right|. \]

Alternatively, you could argue that if the function was continuous at \((0, 0)\), then the limit as \((x, y) \to (0, 0)\) would have to equal 1, no matter how \((x, y)\) approaches \((0, 0)\), and find a problem with that.

\section*{14.3.}

4. Follow the introductory example to Section 14.3 in your textbook.

5. Think about what \( f_x = \frac{\partial f}{\partial x} \) means. Then look at the picture and decide if \( f_x(1, 2) \) is positive or negative. Do the same for \( f_y(1, 2) \).

10. This is just like Problem 4(b), only that you have a contour map rather than a table.
49. Follow Example 5 in the book.

74. You are supposed to decide if these quantities are positive or negative. When considering \( f_{xy} \), for example, you need to decide how \( f_x \) changes as \( y \) varies up and down.

97. If there were such a function \( f(x, y) \), what would \( f_{xy} \) and \( f_{yx} \) work out to be? Is that possible?

105. The point of this problem is to show that there are functions where \( f_{xy} \neq f_{yx} \). How can this be? Carefully read the statement of Clairaut’s Theorem. (This theorem is also called Euler’s Theorem).

\[ \text{§14.4.} \]

26. Find \( \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial y} \) and plug them into \( du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \).

30. Since \( L = L(x, y, z) \) is a function of three variables, its differential reads

\[ dL = \frac{\partial L}{\partial x} dx + \frac{\partial L}{\partial y} dy + \frac{\partial L}{\partial z} dz. \]

34. First, find the formula for the volume \( V = f(H, D) \) of a solid cylinder of height \( H \) and diameter \( D \). Notice that you are supposed to use the differential \( dV = \frac{\partial V}{\partial H} dH + \frac{\partial V}{\partial D} dD \) to estimate the drop in volume \( \Delta V = f(9.8, 3.9) - f(10, 4) \) from the solid outer to the solid inner cylinder. Use \( H_0 = 10, \Delta H = -0.2, D_0 = 4 \) and \( dD = \Delta D = -0.1 \).

39. This is similar to Problem 34. There are however three differences. The function \( R = R(R_1, R_2, R_3) \) is a function of three variables, so that the differential formula is as in Problem 30. The partial derivatives \( \frac{\partial R}{\partial R_1}, \frac{\partial R}{\partial R_2} \) and \( \frac{\partial R}{\partial R_3} \) are best found using implicit differentiation. Also, it is given in the problem that \( dR_1 = \Delta R_1 = \frac{5}{1000} R_1 \), \( dR_2 = \Delta R_2 = \frac{5}{1000} R_2 \) and \( dR_3 = \Delta R_3 = \frac{5}{1000} R_3 \).

44. You need to show that

\[ f(x, y) - f(x_0, y_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = ... \]

can be written in the form

\[ ... = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \quad (1) \]
with \(\epsilon_1 \to 0\) and \(\epsilon_2 \to 0\) as \((x, y) \to (x_0, y_0)\). First, find \(f_x(x_0, y_0)\) and \(f_y(x_0, y_0)\). Then calculate
\[
f(x, y) - f(x_0, y_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = \left((x_0 + \Delta x)(y_0 + \Delta y) - 5(y_0 + \Delta y)^2\right) - \left(x_0 y_0 - 5y_0^2\right).
\]
Sort out the terms, bring it all into the form of Equation (1) above, identify what the \(\epsilon_1\) and \(\epsilon_2\) are and show that \(\epsilon_1 \to 0\) and \(\epsilon_2 \to 0\) as \((\Delta x, \Delta y) \to (0, 0)\).

§14.5. 14. Use the chain rule:
\[
\frac{\partial R}{\partial s} = \frac{\partial R}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial R}{\partial v} \frac{\partial v}{\partial s}.
\]
So,
\[
R_s(1, 2) = G_u(u(1, 2), v(1, 2))u_s(1, 2) + G_v(u(1, 2), v(1, 2))v_s(1, 2).
\]
35. Use the chain rule:
\[
\frac{\partial T}{\partial t} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt}.
\]
38. Start with \(V = \pi r^2 h/3\) and use the chain rule.
49. Recall that
\[
\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial z}{\partial t}\right) = z_u
\]
and
\[
\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x}\right) = z_{xx}
\]
are second derivatives of \(z(x, t) = f(u(x, t)) + g(v(x, t))\) where \(u(x, t) = x + at\) and \(v(x, t) = x - at\).
Note, for example, that \(\frac{\partial z}{\partial t} = f'(u)\) and that \(\frac{\partial}{\partial t} f'(u) = f''(u) \frac{\partial u}{\partial t}\).
51. You are asked to find \(z_{sr}\). So, start with
\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}.
\]
and differentiate your answer with respect to $r$ using the product rule and the chain rule. Note that since $\frac{\partial z}{\partial x} = f_x(x, y)$ is again a function of $x$ and $y$, we have, for example, that

$$\frac{\partial}{\partial r} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial z_x}{\partial r} = z_{xx} \frac{\partial x}{\partial r} + z_{xy} \frac{\partial y}{\partial r}.$$  

Your answer will be in terms of $r$, $s$, and derivatives of $z$ with respect to $x$ and $y$. When simplifying your answer, remember that $z_{xy} = z_{yx}$.

54. This is similar to Problem 51. Start with

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t},$$

and differentiate one more time with respect to $t$ using both the product rule and the chain rule. Be sure to combine mixed partials, like $z_{xy} = z_{yx}$.

§14.6.

5. Use Formula [6].

18. Observe that $\nabla f(2, 2) \cdot \mathbf{u}$ is the scalar projection of $\nabla f(2, 2)$ onto the unit vector $\mathbf{u}$.

29. Set this up as $\nabla f(x, y) = s(i + j)$.

35. Notice that they are giving you $f_x(1, 3)$ and $f_y(1, 3)$.

45. Use Formula [19].

49. Since $\nabla f(3, 2)$ will be perpendicular to the tangent line to the curve at $(3, 2)$, the generic point on the tangent line can be calculated from the equation $\nabla f(3, 2) \cdot ((x, y) - (3, 2)) = 0$.

51. Use Formula [19] and simplify your answer using the fact that $(x_0, y_0, z_0)$ satisfies the ellipsoid equation.

55. Consider $F(x, y, z) = x^2 - y^2 - z^2$. On one hand, a normal for the tangent plane to the surface $F(x, y, z) = 1$ at the point $P(x_0, y_0, z_0)$ is given by $\nabla F(x_0, y_0, z_0)$. On the other hand, a normal to the plane $x + y - z = 0$ is given by $n = (1, 1, -1)$. For the planes to be parallel, the two normals have to be parallel. Is it possible that $\nabla F(x_0, y_0, z_0) = s(1, 1, -1)$ for some $s$? (Work it out and try substituting it into the equation $x^2 - y^2 - z^2 = 1$.)
3. First find the critical points. Do this by setting $f_x(x, y) = 0$ and $f_y(x, y) = 0$. Then use the decision process of the Second Derivative Test on page 947 to determine, for each critical point, whether it is a local minimum, local maximum, or saddle point.

8. When you set $f_x$ and $f_y$ equal to zero, remember that the exponential function is never equal to zero.

15. Explain why there are no critical points at all. (Again, the exponential function is never zero.)

43. Instead of minimizing the distance
$$d((x, y, z), (4, 2, 0)) = \sqrt{(x - 4)^2 + (y - 2)^2 + (z - 0)^2},$$
minimize its square:
$$d^2 = (x - 4)^2 + (y - 2)^2 + (z - 0)^2 = (x - 4)^2 + (y - 2)^2 + x^2 + y^2$$
as a function of $x$ and $y$.

Note: Two different arguments can be made for why the local minimum that you found is an absolute minimum:

1. The second derivative matrix is a matrix of constants.

2. Alternatively, you could argue that there is a radius $R$ such that all points $Q(x, y, z)$ on the cone $z^2 = x^2 + y^2$ with $x^2 + y^2 \geq R^2$ are farther from the point $P(4, 2, 0)$ than some particular point on the cone $z^2 = x^2 + y^2$ with $x^2 + y^2 < R^2$. On the remaining closed and bounded domain $x^2 + y^2 \leq R^2$ an absolute minimum must occur which is, in particular, a local minimum on the domain $x^2 + y^2 < R^2$; the one you found.

45. You want $x + y + z = 100$ with $xyz$ as large as possible. So, maximize $f(x, y) = xyz = xy(100 - x - y)$ on the (triangular) domain $x > 0$, $y > 0$, and $x + y < 100$. You will find one local maximum value.

Note: Since the continuous function $f(x, y)$ must have a maximum on the closed and bounded domain $x \geq 0$, $y \geq 0$ and $x + y \leq 100$, and since this maximum does not occur on the boundary of this domain (where the function equals zero), your local maximum value is the absolute maximum value.
49. You want to maximize \( V = xyz \) when \( z = (6 - x - 2y)/3 \). Proceed as in Problem 45: Maximize \( f(x, y) = xy(6 - x - 2y)/3 \) on the domain \( x > 0, y > 0 \) and \( x + 2y < 6 \).

50. The surface area is given by \( S = 2xy + 2xz + 2yz = 64 \). Solve for \( z \) to get a function \( f(x, y) = xyz = xy^{32-xy}/x+y \) and maximize it as in Problem 43. (Pull out common factors from \( f_x(x, y) = 0 \) and \( f_y(x, y) = 0 \), and subtract the equations to find \( y = x \).)

**Note:** There is actually no need for a second derivative test here. Just as in Problem 43, you can restrict your attention to a closed and bounded domain on which \( f \) is continuous with a maximum value that does not occur on the boundary of the domain. Specifically, start with the region \( x > 0, y > 0, z > 0 \). (The latter inequality is equivalent to \( y < 32/x \).) While the boundary of this region is given by \( x = 0, y = 0 \) and \( y = 32/x \), the region is not bounded (or closed). However, when \( x \leq 1/32 \), we have \( f(x,y) \leq \frac{1}{32}y^{32-0}/y+0 = 1 \). Similarly, when \( y \leq 1/32 \), we have \( f(x,y) \leq 1 \). Since the value 1 is less than the value of \( f \) you found above, you can restrict your attention to the closed and bounded domain \( D \) given by \( x \geq 1/32, y \geq 1/32 \) and \( y \leq 32/x \). (When \( y = 32/x \) we have \( f(x,y) = 0 \).) The continuous function \( f \) must have a maximum value on \( D \), which is then the absolute maximum for \( f \). Since the values of \( f \) on the boundary of \( D \) are not maximal, this maximum must occur in the interior of \( D \); hence be a critical value. Since you found only one critical value, you must have found the absolute maximum of \( f \).

59. Notice that \((x_1, y_2), (x_2, y_2), \ldots \) are constants. You are asked to minimize the function

\[
f(m, b) = \sum_{i=1}^{n} (y_i - (mx_i + b))^2
\]

as a function of \( m \) and \( b \). So, calculate \( f_m(m, b) \) and \( f_b(m, b) \) and set them equal to zero. That should yield the two equations you are looking for. To prove that this is indeed an absolute minimum, apply the second derivative test. Since the second derivative matrix is a matrix of constants, the test will actually be a global test.
In applying the second derivative test, you might find the following computation helpful:

\[
2n \left( \sum_{i=1}^{n} x_i^2 \right) - 2 \left( \sum_{i=1}^{n} x_i \right)^2 = 2 \sum_{j=1}^{n} \left( \sum_{i=1}^{n} x_i^2 \right) - 2 \left( \sum_{j=1}^{n} x_j \right) \left( \sum_{i=1}^{n} x_i \right)
\]

\[
= \sum_{j=1}^{n} \left( \sum_{i=1}^{n} x_i^2 \right) + \sum_{j=1}^{n} \left( \sum_{i=1}^{n} x_i^2 \right) - 2 \left( \sum_{j=1}^{n} x_j \right) \left( \sum_{i=1}^{n} x_i \right)
\]

\[
= \sum_{j=1}^{n} \sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{n} \sum_{i=1}^{n} x_j^2 - \sum_{j=1}^{n} \sum_{i=1}^{n} 2x_j x_i = \sum_{j=1}^{n} \sum_{i=1}^{n} (x_i^2 + x_j^2 - 2x_j x_i)
\]

\[
= \sum_{j=1}^{n} \sum_{i=1}^{n} (x_i - x_j)^2 > 0.
\]

§14.8. 9. Since you are dealing with functions of three variables, namely \( f(x, y, z) = xy^2z \) and \( g(x, y, z) = x^2 + y^2 + z^2 \), you need to set up four equations:

\[
f_x(x, y, z) = \lambda g_x(x, y, z)
\]

\[
f_y(x, y, z) = \lambda g_y(x, y, z)
\]

\[
f_z(x, y, z) = \lambda g_z(x, y, z)
\]

\[
g(x, y, z) = 4
\]

17. Read the comments to Problem 18.

18. Notice that in this problem, there are two constraint functions, namely \( g(x, y, z) = x^2 + y^2 - z^2 \) and \( h(x, y, z) = x + y + z \), with respect to which you want to maximize/minimize the function \( f(x, y, z) = z \). The level surfaces \( g = 0 \) and \( h = 24 \) meet in some curve \( C \). (There is no need to find the curve \( C \).) Since both \( \nabla g \) and \( \nabla h \) are perpendicular to their level surfaces, they are both perpendicular to \( C \), everywhere on \( C \). Now, if \( f \) has an extreme value on \( C \), then \( \nabla f \) it is also perpendicular to \( C \) there. Therefore, you are looking for a point \( (x_0, y_0, z_0) \) where all three \( \nabla f(x_0, y_0, z_0), \nabla g(x_0, y_0, z_0) \) and \( \nabla h(x_0, y_0, z_0) \) are in a
plane perpendicular to $C$ at $(x_0, y_0, z_0)$. (See Figure 5 in the text.)

Hence, you have

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

with two Lagrange multipliers $\lambda$ and $\mu$. In summary, there are five equations:

\[
\begin{align*}
    f_x(x, y, z) &= \lambda g_x(x, y, z) + \mu h_x(x, y, z) \\
    f_y(x, y, z) &= \lambda g_y(x, y, z) + \mu h_y(x, y, z) \\
    f_z(x, y, z) &= \lambda g_z(x, y, z) + \mu h_z(x, y, z) \\
    g(x, y, z) &= 0 \\
    h(x, y, z) &= 24
\end{align*}
\]

30. Use the given hint: maximize

$$f(x, y, z) = s(s - x)(s - y)(s - z)$$

where $s = p/2$ is a constant and $x, y, z$ are the side lengths, under the constraint that $g(x, y, z) = x + y + z = p$.

45. You wish to find min and max of the function

$$f(x, y, z) = (x - 0)^2 + (y - 0)^2 + (z - 0)^2$$

subject to the two constraints $x + y + 2z = 2$ and $x^2 + y^2 - z = 0$. Read the comments to Problem 18.

49. (a) It’s not as bad as it looks. Maximize

$$f(x_1, x_2, x_3, \ldots, x_n) = (x_1 x_2 x_3 \cdots x_n)^{1/n}$$

with $g(x_1, x_2, x_3, \ldots, x_n) = x_1 + x_2 + x_3 + \cdots + x_n = c$. That is, solve

\[
\begin{align*}
    f_{x_1}(x_1, x_2, x_3, \ldots, x_n) &= \lambda g_{x_1}(x_1, x_2, x_3, \ldots, x_n) \\
    f_{x_2}(x_1, x_2, x_3, \ldots, x_n) &= \lambda g_{x_2}(x_1, x_2, x_3, \ldots, x_n) \\
    f_{x_3}(x_1, x_2, x_3, \ldots, x_n) &= \lambda g_{x_3}(x_1, x_2, x_3, \ldots, x_n) \\
    & \vdots \\
    f_{x_n}(x_1, x_2, x_3, \ldots, x_n) &= \lambda g_{x_n}(x_1, x_2, x_3, \ldots, x_n) \\
    g(x_1, x_2, x_3, \ldots, x_n) &= c
\end{align*}
\]
Since the first \( n \) equations are very similar, you should not have much difficulty showing that \( n\lambda x_1 = n\lambda x_2 = \cdots = n\lambda x_n \). Since all \( x_1, x_2, x_3, \ldots, x_n \) are positive, we have \( \lambda \neq 0 \) from the very first equation. Therefore, \( x_1 = x_2 = \cdots = x_n \). Plug this into the last equation.

(b) Simply use the maximum value for \( f \) that you found in Part (a).