The integers

The lack of solutions to the equation $1 + x = 0$ within the system of natural numbers $\mathbb{N}$ prompts an extension of this system to that of the integers $\mathbb{Z}$. In this note we will discuss the details of such an extension, the verification of which are left as exercises. We will work out some of these exercises in class, others will be assigned as homework.

**Theorem 1**

Consider the relation $R$ on the set $\mathbb{N} \times \mathbb{N} = \{(x, y) ~|~ x, y \in \mathbb{N}\}$ defined by $(x_1, y_1)R(x_2, y_2)$ if and only if $y_1 + x_2 = y_2 + x_1$. Then $R$ is an equivalence relation.

**Exercises**

1. Prove Theorem 1.

We denote the set of equivalence classes of $R$ by $\mathbb{Z}$. The equivalence class $[(n, 0)]$ of $(n, 0)$ will be denoted by $-n$. Similarly, we will write $+n$ for the equivalence $[(0, n)]$. (Notice that the only ambiguity here is the equivalence class $[(0, 0)]$, which we can denote by either $+0$ or $-0$.) For example, $-3 = [(3, 0)] = [(4, 1)] = [(6, 3)]$, $+3 = [(0, 3)] = [(1, 4)] = [(3, 6)]$, and $+0 = -0 = [(0, 0)] = [(3, 3)] = [(5, 5)]$. We call such an equivalence class an integer.

**Definition**

On $\mathbb{Z}$ we define

1. $[(x_1, y_1)] + [(x_2, y_2)] = [(x_1 + x_2, y_1 + y_2)]$
2. $[(x_1, y_1)] \cdot [(x_2, y_2)] = [(x_1 \cdot y_2 + x_2 \cdot y_1, x_1 \cdot x_2 + y_1 \cdot y_2)]$
3. $[(x_1, y_1)] < [(x_2, y_2)]$ if and only if $y_1 + x_2 < y_2 + x_1$
4. $-[(x, y)] = [(y, x)]$

**Exercises**

2. Suppose $n \in \mathbb{N}$ and $a, b \in \mathbb{Z}$. Explain the difference in the use of the symbol “+” when forming the expressions $a + b$ and $+n$. Explain the difference in the use of the symbol “-” when forming the expressions $-a$ and $-n$.

3. Show that addition, multiplication, order, and negation of integers is well-defined. That is, show that these “definitions” do not depend on the particular choice of representatives.

We now want to check out the basic properties of our new number system. When working on the following exercises, keep in mind that every integer $a \in \mathbb{Z}$ is to be represented as an equivalence class $a = [(x_1, y_1)]$ with $x_1, y_1 \in \mathbb{N}$. 
Exercises

4. Let $a, b, c \in \mathbb{Z}$. Show that
   
   (i) $a + b = b + a$
   (ii) $(a + b) + c = a + (b + c)$
   (iii) $a \cdot b = b \cdot a$
   (iv) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
   (v) $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$

5. $a + (+0) = a$ for all $a \in \mathbb{Z}$.

6. $a + (−a) = +0$ for all $a \in \mathbb{Z}$.

7. $a \cdot (+1) = a$ for all $a \in \mathbb{Z}$.

8. For all $a, b \in \mathbb{Z}$, $a < b$ if and only if $+0 < b + (−a)$.

9. For each $a \in \mathbb{Z}$ exactly one of the following is true: either $+0 < a$, or $+0 < −a$, or $+0 = a$.

10. If $a, b \in \mathbb{Z}$ with $+0 < a$ and $+0 < b$, then $+0 < a + b$ and $+0 < a \cdot b$.

As it turns out, all other properties of the integers can be deduced from the above seven basic properties. That is, if you want, you can give a proof of each of the following without using representatives for equivalence classes or a direct reference to the original definitions of $+$, $\cdot$, $<$, $−$ on $\mathbb{Z}$. Try it out!

Exercises

11. $a \cdot (+0) = +0$ for all $a \in \mathbb{Z}$.

12. $−(−a) = a$ for all $a \in \mathbb{Z}$.

13. $(−1) \cdot a = −a$ for all $a \in \mathbb{Z}$.

14. If $a \in \mathbb{Z}$ with $a \not= +0$, then $+0 < a \cdot a$.

15. If $a, b, c \in \mathbb{Z}$ with $a < b$, then
   
   (i) $a + c < b + c$
   (ii) $a \cdot c < b \cdot c$, provided $+0 < c$
   (iii) $b \cdot c < a \cdot c$, provided $c < +0$

16. $\leq$ is a linear order on $\mathbb{Z}$ (where $a \leq b$ stands for “$a < b$ or $a = b$”).

17. If $a, b, c \in \mathbb{Z}$ with $a + c = b + c$, then $a = b$.

18. If $a, b, c \in \mathbb{Z}$ with $a \cdot c = b \cdot c$ and $c \not= +0$, then $a = b$. 
We want to conclude our discussion of the integers with a theorem that says that, for all intents and purposes, the subset \( \{ +n \mid n \in \mathbb{N} \} \subseteq \mathbb{Z} \) of non-negative integers is the same as the set \( \mathbb{N} \) of natural numbers. Consequently, in the future we will no longer distinguish between the equivalence class \( +n = [(0, n)] \) (which is a set of pairs of natural numbers) and the natural number \( n \).

**Theorem 2**

The function \( f : \mathbb{N} \to \mathbb{Z} \) given by \( f(n) = +n \) has the following properties:

(i) If \( n, m \in \mathbb{N} \) with \( n < m \), then \( f(n) < f(m) \). In particular, \( f \) is injective.

(ii) \( f(n + m) = f(n) + f(m) \) for all \( n, m \in \mathbb{N} \).

(iii) \( f(n \cdot m) = f(n) \cdot f(m) \) for all \( n, m \in \mathbb{N} \).

In short, the function \( f \) is a bijection between the set \( \mathbb{N} \) of natural numbers and the set \( \{ +n \mid n \in \mathbb{N} \} \) of non-negative integers which preserves all relevant properties.

**Exercises**


20. Writing \( a - b \) for \( a + (-b) \) when \( a, b \in \mathbb{Z} \), show that \( [(x, y)] = (+y) - (+x) \) for all \( x, y \in \mathbb{N} \). Does this problem shed some light on the definition of addition and multiplication of integers above?