Chapter 5

Section 5.1:

#17 Consider the cases (1) the sequence ends in ...1; (2) the sequence ends in ...10; and (3) the sequence ends in ...00.

#32 Every new great circle that you draw must intersect all previous ones twice.

#56–62 The book defines $\nabla^{k+1}$ by $\nabla^{k+1}a_n = \nabla^k a_n - \nabla^k a_{n-1}$. Notice that this is the same as $\nabla^{k+1}a_n = \nabla (\nabla^k a_n)$ since this operator $\nabla$ is linear.

#57 There is an obvious typo in this question. They mean to refer to Exercise 56.

#60 Carry out an induction on $k$.

The case $k = 1$ is taken care of by Problem 58.

Inductively, assume that $a_{n-(k-1)}$ is a (linear) combination of the terms $a_n$, $\nabla a_n$, $\nabla^2 a_n$, $\cdots$, $\nabla^{k-1} a_n$. Then $a_{n-k+1}$ is a (linear) combination of $a_n$, $\nabla a_n$, $\nabla^2 a_n$, $\cdots$, $\nabla^{k-1} a_n$ and $\nabla a_{n-k+1}$ is a (linear) combination of $\nabla a_n$, $\nabla^2 a_n$, $\nabla^3 a_n$, $\cdots$, $\nabla^k a_n$.

Now, $a_{n-k} = a_{n-k+1} - \nabla a_{n-k+1}$ by the very definition of $\nabla a_{n-k+1}$.

#62 This really follows from Problem 60.

The point of all this is that studying recurrence relations of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

comes down to the same as studying difference equations of the form

$$a_n = d_1 \nabla a_n + d_2 \nabla^2 a_n + d_k \nabla^k a_n.$$  

In this form, the problem at hand looks a lot like a differential equation. In fact, many of the methods used for differential equations can be used in the theory of difference equations.
Observe the following analogy: if \( f(x) \) is the function with values \( f(n) = a_n \) for all \( n \) and equal to a straight line segment on every interval \([n-1,n]\). Then

\[
f'(x) = \frac{f(n) - f(n-1)}{n - (n-1)} = a_n - a_{n-1} = \nabla a_n.
\]

for all \( x \in (n-1,n) \). In other words, the operator \( \nabla \) measures the sequence of slopes of the line segments “connecting the dots” of the sequence \( a_n \).

Section 5.5: #13 Compute how many bit strings of length eight do contain six consecutive 0’s by considering the sets \( A_1, A_2, \) and \( A_3 \) consisting of all bit strings of length eight that have six consecutive 0’s beginning at position 1, 2, and 3, respectively. Use the inclusion-exclusion principle.

Section 5.6: #6 Observe that a positive integer is divisible by the square of a positive integer if and only if it is divisible by the square of a prime.

#11 Since the most difficult job necessarily goes to the most experienced employee, there are only six more jobs to assign. Split your analysis into two cases: either the most experienced employee gets more jobs, or s/he does not.

#18 Say the first object goes to position \( i \in \{2, 3, \ldots, n\} \). Consider two cases: either the object originally in position \( i \), call it object \( i \), goes to position 1, or it does not. In the first case, object 1 and object \( i \) swap places, and the remaining \( n-2 \) objects get deranged among each other. In the second case object 1 goes to position \( i \), and object \( i \) goes to a position \( j \neq 1 \). This is the same as deranging the objects \( \{1, 2, \ldots, i-1, i+1, \ldots, n\} \) while sending object 1 to position \( j \), and then making the adjustment “object 1 goes to position \( i \)” and “object \( i \) goes to position \( j \)”.

#19 Use mathematical induction and Problem #18. Also notice that \( D_0 = 1 \) by definition.

#21 Re-discover the formula of Theorem 2 by reiterating the recurrence relation of Problem #19.

#25 This is a derangement of \( \{1, 2, 3\} \) and a derangement of \( \{4, 5, 6\} \).