Structure of Julia sets of polynomial semigroups with bounded finite postcritical set

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Abstract

Let G be a semigroup of complex polynomials (under the operation of composition of functions) such that there exists a bounded set in the plane which contains any finite critical value of any map $g \in G$. We discuss the dynamics of such polynomial semigroups as well the structure of the Julia set of G. In general, the Julia set of such a semigroup G may be disconnected, and each Fatou component of such G is either simply connected or doubly connected. In this paper, we show that for any two distinct Fatou components of certain types (e.g., two doubly connected components of the Fatou set), the boundaries are separated by a Cantor set of quasicircles (with uniform dilatation) inside the Julia set of G. Furthermore, we provide results concerning the (semi) hyperbolicity of such semigroups as well as discuss various topological consequences of the postcritically boundedness condition.

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1 Introduction

A **rational semigroup** is a semigroup generated by non-constant rational maps on the Riemann sphere $\overline{\mathbb{C}}$ with the semigroup operation being the composition of maps. We denote by $\langle h_\lambda : \lambda \in \Lambda \rangle$ the rational semigroup generated by the family of maps $\{h_\lambda : \lambda \in \Lambda\}$. A **polynomial semigroup** is a semigroup generated by non-constant polynomial maps. Research on the dynamics of rational semigroups was initiated by A. Hinkkanen and G.J. Martin in [2], who were interested in the role of the dynamics of polynomial semigroups while studying various one-complex-dimensional moduli spaces for discrete groups. Also, F. Ren, Z. Gong, and W. Zhou studied such semigroups from the perspective of random dynamical systems (see [18], [1]).

The polynomial maps $f_c(z) = z^2 + c$ for c in the Mandelbrot set are such that the orbit of the sole critical point $\{f_c^n(0)\}$ is bounded, which in turn leads to many important dynamic and structural properties. It is then natural to look at the more general situation of polynomial semigroups with bounded postcritical set. We discuss the dynamics of such polynomial semigroups as well the structure of their Julia sets. For some properties of polynomial semigroups with bounded finite postcritical set, also see [12], [14].

Note that the research of polynomial semigroups is deeply related to the research of random dynamics of polynomials (See [16]).

Definition 1 *Let G be a rational semigroup. We set*

$$F(G) = \{z \in \overline{\mathbb{C}} \mid G \textit{ is normal in a neighborhood of } z\} \textit{ and } \ J(G) = \overline{\mathbb{C}} \setminus F(G).$$

We call $F(G)$ the **Fatou set** *of G and $J(G)$ the* **Julia set** *of G. The Fatou set and Julia set of the semigroup generated by a single map g is denoted by $F(g)$ and $J(g)$, respectively.*

From the definition, we get that $F(G)$ is **forward invariant** under each element of G, i.e., $g(F(G)) \subset F(G)$ for all $g \in G$, and thus $J(G)$ is **backward invariant** under each element of G, i.e., $g^{-1}(J(G)) \subset J(G)$ for all $g \in G$ (see [2], p. 360). The sets $F(G)$ and $J(G)$ are, however, not necessarily completely invariant under the elements of G. This is in contrast to the case of single function dynamics, i.e., the dynamics of semigroups generated by a single rational function. For a treatment of alternatively defined *completely* invariant Julia sets of rational semigroups the reader is referred to [4], [5], [6] and [7].

Note that $J(G)$ contains the Julia set of each element of G. Moreover, the following result due to Hinkkanen and Martin holds.

Theorem 2 ([2], Corollary 3.1) *For rational semigroups G with $\sharp(J(G)) \geq 3$, we have*

$$J(G) = \overline{\bigcup_{f \in G} J(f)}.$$

The **backward orbit** of z is given by $G^{-1}(z) = \cup_{g \in G} g^{-1}(\{z\})$ and the **forward orbit** of z is given by $G(z) = \cup_{g \in G} g(\{z\})$. For any subset A of $\overline{\mathbb{C}}$, we set $G^{-1}(A) = \cup_{g \in G} g^{-1}(A)$.

For any polynomial g, we set $K(g) := \{z \in \mathbb{C} \mid \cup_{n\in\mathbb{N}} g^n(\{z\}) \text{ is bounded in } \mathbb{C}\}$, which is known as the **filled in Julia set** of g. We note that $J(g) = \partial K(g)$ and that $K(g)$ is the polynomial hull of $J(g)$. The appropriate extension (to our situation with polynomial semigroups) of the concept of filled in Julia set is as follows.

Definition 3 *For a polynomial semigroup G, we set*

$$\hat{K}(G) := \{z \in \mathbb{C} \mid G(z) \text{ is bounded in } \mathbb{C}\}.$$

Remark 4 *We note that for all $g \in G$, we have $\hat{K}(G) \subset K(g)$ and $g(\hat{K}(G)) \subset \hat{K}(G)$.*

Definition 5 *The* **postcritical set** *of a rational semigroup G is defined by*

$$P(G) = \overline{\bigcup_{g\in G} \{\text{all critical values of } g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}\}} \ (\subset \overline{\mathbb{C}}).$$

*We say that G is **hyperbolic** if $P(G) \subset F(G)$ and we say that G is **subhyperbolic** if both $\#\{P(G) \cap J(G)\} < +\infty$ and $P(G) \cap F(G)$ is a compact set.*

For research on (semi-)hyperbolicity and Hausdorff dimension of Julia sets of rational semigroups see [8, 9, 10, 15, 11].

Definition 6 *The **finite postcritical set** (or, the planar postcritical set) of a polynomial semigroup G is defined by*

$$P^*(G) = P(G) \setminus \{\infty\}.$$

Definition 7 *Let $\mathcal{G}$ be the set of all polynomial semigroups G with the following properties:*

- *each element of G is of degree at least two, and*
- *$P^*(G)$ is bounded in $\mathbb{C}$.*

Moreover, we set $\mathcal{G}_{dis} := \{G \in \mathcal{G} \mid J(G) \text{ is disconnected}\}$.

Remark 8 *Since $P(G)$ is forward invariant under G, we see that $G \in \mathcal{G}$ implies $P^*(G) \subset \hat{K}(G)$, and thus $P^*(G) \subset K(g)$ for all $g \in G$.*

Remark 9 *For a polynomial g of degree two or more, it is well known that $\langle g \rangle \in \mathcal{G}$ implies $J(g)$ is connected. (Hence, for any $g \in G \in \mathcal{G}$, we have that $J(g)$ is connected.) We note, however, that the analogous result for polynomial semigroups does not hold as there are many examples where $G \in \mathcal{G}$, but $J(G)$ is not connected (see [17]). See also [13] for an analysis of the number of connected components of $J(G)$ involving the inverse limit of connected components of the realizations of the nerves of finite coverings $\mathcal{U}$ of $J(G)$, where $\mathcal{U}$ consists of backward images of $J(G)$ under maps in G.*

The aim of this paper is to investigate what can be said about the structure of the Julia sets and the dynamics of semigroups $G \in \mathcal{G}$? We begin by examining the structure of the Julia set and note that a natural order (that is respected by the backward action of the maps in G) can be placed on the components of $J(G)$, which then leads to implications on the connectedness of Fatou components.

Definition 10 *For a polynomial semigroup $G \in \mathcal{G}$, we denote by $\mathcal{J} = \mathcal{J}_G$ the set of all connected components of $J(G)$ which do not include ∞.*

Definition 11 *We place a partial order on the space of all non-empty compact connected sets in $\mathbb{C}$ as follows. For any connected compact sets K_1 and K_2 in $\mathbb{C}$, "$K_1 \leq K_2$" indicates that $K_1 = K_2$ or K_1 is included in a bounded component of $\mathbb{C} \setminus K_2$. Also, "$K_1 < K_2$" indicates $K_1 \leq K_2$ and $K_1 \neq K_2$. We call $\leq$ the surrounding order and read $K_1 < K_2$ as "K_1 is surrounded by K_2".*

Theorem 12 ([12]) *Let $G \in \mathcal{G}$ (possibly infinitely generated). Then*

1. *$(\mathcal{J}, \leq)$ is totally ordered.*
2. *Each connected component of $F(G)$ is either simply or doubly connected.*
3. *For any $g \in G$ and any connected component J of $J(G)$, we have that $g^{-1}(J)$ is connected. Let $g^*(J)$ be the connected component of $J(G)$ containing $g^{-1}(J)$. If $J \in \mathcal{J}$, then $g^*(J) \in \mathcal{J}$. If $J_1, J_2 \in \mathcal{J}$ and $J_1 \leq J_2$, then both $g^{-1}(J_1) \leq g^{-1}(J_2)$ and $g^*(J_1) \leq g^*(J_2)$.*

With this order and the following notation we will then be able to state our main results.

Let $h_1, \cdots, h_m$ be rational functions on $\overline{\mathbb{C}}$. Let $\Sigma_m = \{1, \cdots, m\}^{\mathbb{N}}$ be the one-sided shift space and let $\sigma : \Sigma_m \to \Sigma_m$ be the shift map, i.e., $\sigma(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$. Let $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the map defined by $\tilde{f}(x, y) = (\sigma(x),\ h_{x_1}(y))$, where $x = (x_1, x_2, \cdots) \in \Sigma_m$. This is called the skew product map associated with $\{h_1, \cdots, h_m\}$. Let $\pi : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m$ and $\pi_{\overline{\mathbb{C}}} : \Sigma_m \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be the natural projections. We set $\tilde{f}_x^n := \tilde{f}^n|_{\pi^{-1}x} : \pi^{-1}x \to \pi^{-1}\sigma^n(x) \subset \Sigma_m \times \overline{\mathbb{C}}$ and we denote by $F_x(\tilde{f})$ the set of points $y \in \pi^{-1}x$ which has a neighborhood U in $\pi^{-1}x$ such that $\{\tilde{f}_x^n : U \to \Sigma_m \times \overline{\mathbb{C}}\}_{n \in \mathbb{N}}$ is normal. Furthermore, we set $J_x(\tilde{f}) := \pi^{-1}x \setminus F_x(\tilde{f})$.

Remark 13 *Note that $\pi_{\overline{\mathbb{C}}}(J_x(\tilde{f}))$ is equal to the set of points $z \in \overline{\mathbb{C}}$ where the sequence of rational functions $\{h_{x_n} \circ \cdots \circ h_{x_1}\}_{n \in \mathbb{N}}$ is not normal. This is sometimes called the Julia set along the trajectory (sequence) $x \in \Sigma_m$.*

Theorem 14 *Let $G \in \mathcal{G}$ and let A and B be disjoint subsets of $\overline{\mathbb{C}}$. Suppose that we have one of the following conditions:*

- *A and B are doubly connected components of $F(G)$.*
- *A is a doubly connected component of $F(G)$ and B is a connected component of $F(G)$ with $\infty \in B$.*

- *$A = \hat{K}(G)$ and B is a doubly connected component of $F(G)$.*

Then $\partial A \cap \partial B = \emptyset$. Furthermore, $\overline{A}$ and $\overline{B}$ are separated by a Cantor family of quasicircles with uniform dilatation which all lie in $J(G)$. More precisely, there exist two elements $\alpha_1, \alpha_2 \in G$ satisfying all of the following.

1. *There exists an open set U in $\overline{\mathbb{C}}$ with $\alpha_1^{-1}(\overline{U}) \cap \alpha_2^{-1}(\overline{U}) = \emptyset$ and $\alpha_1^{-1}(\overline{U}) \cup \alpha_2^{-1}(\overline{U}) \subset U$.*
2. *$H = \langle \alpha_1, \alpha_2 \rangle$ is hyperbolic.*
3. *Let $\tilde{f} : \Sigma_2 \times \overline{\mathbb{C}} \to \Sigma_2 \times \overline{\mathbb{C}}$ be the skew product map associated with $\{\alpha_1, \alpha_2\}$. Then*
   (a) *$J(H) = \bigcup_{x \in \Sigma_2} \pi_{\overline{\mathbb{C}}}(J_x(\tilde{f}))$ (disjoint union),*
   (b) *for any component J of $J(H)$, there exists an $x \in \Sigma_2$ with $J = \pi_{\overline{\mathbb{C}}}(J_x(\tilde{f}))$ and*
   (c) *there exists a constant $K \geq 1$ such that any component J of $J(H)$ is a K-quasicircle.*
4. *$\{\pi_{\overline{\mathbb{C}}}(J_x(\tilde{f}))\}_{x \in \Sigma_2}$ is totally ordered with $\leq$, consisting of mutually disjoint subsets of $J(H)$. Furthermore, for each $x \in \Sigma_2$, the set $\pi_{\overline{\mathbb{C}}}(J_x(\tilde{f}))$ separates $\overline{A}$ and $\overline{B}$.*

Remark 15 *It should be noted that in the above theorem, the quasicircles $\pi_{\overline{\mathbb{C}}}(J_x(\tilde{f}))$ are all disjoint components of $J(H)$, but may all lie in the same component of $J(G)$.*

Example 16 *We give an example of a semigroup $G \in \mathcal{G}$ such that $J(G)$ is a Cantor set of round circles. Let $f_1(z) = az^k$ and $f_2(z) = bz^j$ for some positive integers k and j. Then $J(f_1)$ and $J(f_2)$ are both circles centered at the origin. Let A denote the closed annulus between $J(f_1)$ and $J(f_2)$. For positive integers m_1 and m_2 large enough, we see that the iterates $g_1 = f_1^{m_1}$ and $g_2 = f_2^{m_2}$ will yield $A_1 = g_1^{-1}(A) \cup g_2^{-1}(A) \subset A$ where $g_1^{-1}(A) \cap g_2^{-1}(A) = \emptyset$. Now iteratively define $A_{n+1} = g_1^{-1}(A_n) \cup g_2^{-1}(A_n)$ and note that for $G = \langle g_1, g_2 \rangle$ we have that $J(G) = \cap_{n=1}^{\infty} A_n$, since $J(G)$ is the smallest closed backward invariant (under each element of G) set which contains three or more points.*

The next results concern the (semi-)hyperbolicity of polynomial semigroups in $\mathcal{G}$, and in particular show how one can build larger (semi-)hyperbolic polynomial semigroups in $\mathcal{G}$ from smaller ones by including maps with certain properties. For this result we need to note the existence of a minimal element in $\mathcal{J}$ and state a few of its properties.

Theorem 17 ([12]) *Let G be a polynomial semigroup in $\mathcal{G}_{dis}$. Then there is a unique element $J_{\min} \in \mathcal{J}$ such that $J_{\min}$ meets (and therefore contains) $\partial \hat{K}$. Furthermore, we have the following*

- *$J_{\min} \leq J$ for all $J \in \mathcal{J}$.*
- *$P^*(G)$ is contained in the polynomial hull of $J_{\min}$.*

Definition 18 *A rational semigroup H is **semi-hyperbolic** if for each $z \in J(H)$ there exists a neighborhood U of z and a number $N \in \mathbb{N}$ such that for each $g \in H$ we have $\deg(g : V \to U) \leq N$ for each connected component V of $g^{-1}(U)$.*

Theorem 19 *Let $H \in \mathcal{G}$ and let $G = \langle H, h_1, \ldots, h_n \rangle$ be a polynomial semigroup generated by H and $h_1, \cdots, h_n$. Suppose*
(1) $G \in \mathcal{G}_{dis}$,
(2) $J(h_j) \cap J_{\min}(G) = \emptyset$ for each $j = 1, \cdots, n$, and
(3) H is semi-hyperbolic.
Then, G is semi-hyperbolic.

Remark 20 *Theorem 19 would not hold if we were to replace both instances of the word semi-hyperbolic with the word hyperbolic (see Example 37). However, with an additional hypothesis we do get the following:*

Theorem 21 *Let $H \in \mathcal{G}$ and let $G = \langle H, h_1, \ldots, h_n \rangle$ be a polynomial semigroup generated by H and $h_1, \cdots, h_n$. Suppose*
(1) $G \in \mathcal{G}_{dis}$,
(2) $J(h_j) \cap J_{\min}(G) = \emptyset$ for each $j = 1, \cdots, n$,
(3) $P^*(H) \cap J(H) = \emptyset$. and
(4) For each $j = 1, \ldots, n$, the critical values of h_j do not meet $J_{\min}(G)$.
Then, $P^*(G) \cap J(G) = \emptyset$.

Remark 22 *We note that if in Theorem 21 we replace hypothesis (3) with the hypothesis "H is hyperbolic", then the conclusion becomes "G is hyperbolic". This follows immediately as one can show that $\infty \in F(H)$ implies $\infty \in F(G)$ (see the proof of Theorem 19 for more details).*

The rest of this paper is organized as follows. In Section 2 we give the necessary background and tools required, in Section 3 we give the proof of Theorem 14, and in Section 4 we give the proofs of Theorems 19 and 21 along with Example 37.

2 Background and Tools

Although not all connected compact sets in $\mathbb{C}$ are comparable in the surrounding order, we do have the following lemma whose proof we leave to the reader.

Lemma 23 *Given two connected compact sets A and B in $\mathbb{C}$ we must have exactly one of the following:*

1. $A < B$
2. $B < A$

3. $A \cap B \neq \emptyset$

4. *A and B are **outside of each other**, i.e., A is a subset of the unbounded component of $\mathbb{C} \setminus B$ and B is a subset of the unbounded component of $\mathbb{C} \setminus A$.*

For various sets of interest in this paper the last case listed above is not possible (see Corollary 33). We now proceed to show this through a series of results which we will also find useful later in the paper.

Lemma 24 *Let A and B be compact connected subsets of $\mathbb{C}$ and suppose there exists a point $p \in \mathbb{C}$ such that one of the following holds*

(i) $\{p\} < A$ *and* $\{p\} < B$

(ii) $p \in A$ *and* $\{p\} < B$

(iii) $\{p\} < A$ *and* $p \in B$

(iv) $p \in A$ *and* $p \in B$.

Then exactly one of the following holds:

1. $A < B$
2. $B < A$
3. $A \cap B \neq \emptyset$.

Proof: Cases (ii), (iii), and (iv) are trivial, so we detail only case (i). Suppose $A \cap B = \emptyset$. Call V_A and V_B the bounded components of $\mathbb{C} \setminus A$ and $\mathbb{C} \setminus B$, respectively, which contain p. Since $V_A \neq V_B$ (else A and B are not disjoint) we must have $\partial V_B \cap V_A \neq \emptyset$ or $\partial V_A \cap V_B \neq \emptyset$. But $\partial V_B \cap V_A \neq \emptyset$ implies $B \cap V_A \neq \emptyset$ which implies $B \subset V_A$ (since B is connected), and thus $B < A$. Similarly, $\partial V_A \cap V_B \neq \emptyset$ implies $A < B$. **QED**

Definition 25 *For a compact set $A \subset \mathbb{C}$ we define the **polynomial hull** $PH(A)$ of A to be the union of A and all bounded components of $\mathbb{C} \setminus A$.*

Remark 26 *With regards to the Lemma 24, we note that if $PH(A)$ and $PH(B)$ share a common point p, then one of the cases $(i) - (iv)$ must hold, and therefore so does the conclusion.*

Lemma 27 *Let $g \in G \in \mathcal{G}$ and suppose $P^*(G) \subset PH(A)$ where $A \subset \mathbb{C}$ is compact. Then $P^*(G) \subset PH(g^{-1}(A))$.*

Proof: Suppose $z_0 \in P^*(G)$ is in the unbounded component U of $\overline{\mathbb{C}} \setminus g^{-1}(A)$. Let γ be a curve in U connecting z_0 to ∞. Then $\Gamma = g \circ \gamma$ is a curve in $\overline{\mathbb{C}} \setminus A$ which connects $g(z_0)$ to ∞ which shows that $g(z_0) \notin PH(A)$. Since $P^*(G)$ is forward invariant we have that $g(z_0) \in P^*(G) \setminus PH(A)$ which contradicts our hypothesis. **QED**

Corollary 28 *Let $f, g \in G \in \mathcal{G}$. If A is of the form $J \in \mathcal{J}, J(f), g^{-1}(J)$, or $g^{-1}(J(f))$, then $P^*(G) \subset PH(A)$.*

Proof: Since by Theorem 17 we have $P^*(G) \subset PH(J_{\min})$ and $J_{\min} \leq J$, we must also have $P^*(G) \subset PH(J)$. By Remark 8 any point $p \in P^*(G)$ must lie in $K(f) = PH(J(f))$. The other cases then follow from Lemma 27. **QED**

An important consequence of Lemma 3.1 in [12] is the following.

Corollary 29 ([12]) *Let $G \in \mathcal{G}$. If $\epsilon > 0$ and $J \in \mathcal{J}$, then there exists $g \in G$ such that $J(g) \subset B(J, \epsilon)$, where $B(J, \epsilon) = \cup_{z \in J} B(z, \epsilon)$ denotes the ϵ neighborhood of J.*

Lemma 30 ([3]) *Let X be a compact metric space and let $f : X \to X$ be a continuous open map. Let K be a compact connected subset of X. Then for each connected component B of $f^{-1}(K)$, we have $f(B) = K$.*

Lemma 31 *Let g be a polynomial with $d = \deg(g) \geq 1$ and let $K \subset \mathbb{C}$ be a connected compact set such that the unbounded component U of $\overline{\mathbb{C}} \setminus K$ contains no critical values of g other than ∞. Then $g^{-1}(K)$ is connected. Further, if K_1 is a connected compact set such $K < K_1$, then $g^{-1}(K) < g^{-1}(K_1)$.*

Proof: Set $V = g^{-1}(U)$ and note that V contains no finite critical points of g. Thus by the Riemann-Hurwitz relation we have $\chi(V) + \delta_g(V) = d\chi(U)$, where $\chi(\cdot)$ denotes the Euler characteristic and $\delta(\cdot)$ is the deficiency. Since the hypotheses on U imply $\delta_g(V) = d - 1$ and $\chi(U) = 1$, we see that $\chi(V) = 1$. Hence the open and connected set V is simply connected and thus ∂V is connected. Let $A = \overline{\mathbb{C}} \setminus \overline{V}$ and note that $\partial A \subset \partial V$. Suppose B is a component of $g^{-1}(K)$ which does not meet ∂V. Then $B \subset A$. We note that by Lemma 30 we have $g(B) = K \supset \partial U$. Since $g(B) \supset \partial U$ and $B \subset A$ we see that $g(A) \cap \partial U \neq \emptyset$. Thus $g(A) \cap U \neq \emptyset$ by the Open Mapping Theorem, which contradicts the fact that A and $V = g^{-1}(U)$ are disjoint. Thus, since the connected set $\partial V \subset g^{-1}(K)$ meets all the connected components of $g^{-1}(K)$, we conclude that $g^{-1}(K)$ is connected.

Let $V_1 = g^{-1}(U_1)$ where U_1 is the unbounded component of $\overline{\mathbb{C}} \setminus K_1$. Since $K < K_1$ implies $K \cap K_1 = \emptyset$, we get that $g^{-1}(K) \cap g^{-1}(K_1) = \emptyset$ which implies $\partial V \cap \partial V_1 = \emptyset$ since $\partial V \subset g^{-1}(K)$ and $\partial V_1 \subset g^{-1}(K_1)$. Since $\infty \in V \cap V_1$, it is easy to show then that $\partial V < \partial V_1$. Since $g^{-1}(K) \supset \partial V$ and $g^{-1}(K_1) \supset \partial V_1$, it quickly follows that $g^{-1}(K) < g^{-1}(K_1)$. **QED**

Corollary 32 *Let $g, h \in G \in \mathcal{G}$ and $J \in \mathcal{J}$. Then $g^{-1}(J)$ and $g^{-1}(J(h))$ are connected. Furthermore, $J_1 < J_2$ for $J_1, J_2 \in \mathcal{J}$ implies $g^{-1}(J_1) < g^{-1}(J_2)$, and $J(h_1) < J(h_2)$ for $h_1, h_2 \in G$ implies $g^{-1}(J(h_1)) < g^{-1}(J(h_2))$.*

Proof: The result follows immediately from Lemma 31 once it is shown that the unbounded components U and U' of $\overline{\mathbb{C}} \setminus J$ and $\overline{\mathbb{C}} \setminus J(h)$, respectively, contain no critical values of g other than ∞. Suppose U contains a finite critical value z_0 of g. Let γ be a path in U connecting z_0 to ∞ and let $\epsilon = dist(\gamma, J)$. By Corollary 29 we see that $B(J, \epsilon/2)$ contains $J(f)$ for some $f \in G$. Since $J(f)$ cannot meet γ, the point z_0 lies in the unbounded component of $\mathbb{C} \setminus J(f)$ and thus $f^n(z_0) \to \infty$, which contradicts the hypothesis that $P^*(G)$ is bounded. Hence U contains no critical values of g other than ∞. The condition on U' follows from the fact that $P^*(g) \subset P^*(G) \subset K(h)$. **QED**

Corollary 33 *Let $f, g \in G \in \mathcal{G}$. For any two sets A and B of the form $J \in \mathcal{J}$, $J(f)$, $g^{-1}(J)$, or $g^{-1}(J(f))$, exactly one of the following must hold:*

1. *$A < B$*
2. *$B < A$*
3. *$A \cap B \neq \emptyset$.*

Proof: This follows immediately from Corollary 32, Corollary 28 and Remark 26. **QED**

3 Proof of Theorem 14

Definition 34 *For compact connected sets K_1 and K_2 in $\mathbb{C}$ such that $K_1 < K_2$ we define $Ann(K_1, K_2) = U \cap V$ where U is the bounded component of $\mathbb{C} \setminus K_2$ which contains K_1, and V is the unbounded component of $\overline{\mathbb{C}} \setminus K_1$. Thus $Ann(K_1, K_2)$ is the open doubly connected region "between" K_1 and K_2.*

Remark 35 *For any connected compact set $A \subset Ann(K_1, K_2)$ we immediately see that $A < K_2$ and, by Lemma 23, either K_1 and A are outside of each other or $K_1 < A$.*

Lemma 36 *Let $f, g \in G \in \mathcal{G}$ be such that $J(f)$ and $J(g)$ lie in different components of $J(G)$ with $J(f) < J(g)$. Then for any fixed $n, m \in \mathbb{N}$ there exists $h, k \in G$ such that $f^{-(n+1)}(J(g)) < J(h) < f^{-n}(J(g))$ and $g^{-m}(J(f)) < J(k) < g^{-(m+1)}(J(f))$.*

Proof: We claim that $g^{-1}(J(f)) > J(f)$. If $g^{-1}(J(f)) \cap J(f) \neq \emptyset$, then the connected set $\overline{\cup_{n=0}^{\infty} g^{-n}(J(f))}$, which lies in $J(G)$, meets both $J(g)$ and $J(f)$, thus contradicting the hypothesis that $J(f)$ and $J(g)$ lie in different components of $J(G)$. If $g^{-1}(J(f)) < J(f)$, then one could easily show that $g^{-1}(K(f)) \subset K(f)$ which would imply that $J(g) \subset K(f)$, thus contradicting the hypothesis that $J(f) < J(g)$. Hence, we conclude by Corollary 33 that $g^{-1}(J(f)) > J(f)$.

Let $X = g^{-1}(J(f)), A = g^{-m}(J(f))$ and $B = g^{-(m+1)}(J(f))$ and note that $J(f) < A < B$ from Lemma 31. Choose $\ell \in \mathbb{N}$ large enough so that $f^{-\ell}(B) \subset \overline{Ann(J(f), X)}$. Then $g^{-m}(f^{-\ell}(\overline{Ann(A, B)})) \subset g^{-m}(Ann(J(f), X)) \subset Ann(A, B) \subset \overline{Ann(A, B)}$ which implies that $k = f^{\ell} \circ g^m \in G$ is such that $J(k) \in Ann(A, B)$. Since by Corollary 33 we must have either $J(k) < A$ or $A < J(k)$, we see by construction that $A < J(k)$ must hold.

The other result is proved similarly. **QED**

Proof of Theorem 14: We give a proof in the case that A and B are doubly connected components of $F(G)$. We assume without loss of generality that A is contained in the bounded component of $\overline{\mathbb{C}} \setminus B$. Let Γ_A denote the component of ∂A which meets the bounded component of $\overline{\mathbb{C}} \setminus A$ and let Γ denote the other component of ∂A. Let Γ_B denote the component of ∂B which meets the unbounded component of $\overline{\mathbb{C}} \setminus B$. Let $J \in \mathcal{J}$ be such that $\Gamma \subset J$. Let $\epsilon = \min\{dist(\Gamma_A, J), dist(\Gamma_B, J)\}$. By Corollary 29 we see that $B(J, \epsilon/2)$ contains $J(g)$ for some $g \in G$ and thus, using Corollary 33, $\Gamma_B > J(g) > \Gamma_A$.

Let $J_A \in \mathcal{J}$ be such that $\Gamma_A \subset J_A$ and set $\epsilon' = dist(J_A, J(g))$. By the Corollary 29 we see that $B(J_A, \epsilon'/2)$ contains $J(f)$ for some $f \in G$ and thus $J(g) > J(f)$. Choose m large enough so that $g^{-m}(J(f)) > \Gamma_A$. By Lemma 36 there exists $k \in G$ such that $g^{-m}(J(f)) < J(k) < g^{-(m+1)}(J(f)) < J(g)$. By choosing $m_1, m_2 \in \mathbb{N}$ large (as in Example 16), we will find that $\alpha_1 = k^{m_1}$ and $\alpha_2 = g^{m_2}$ generate a subsemigroup H of G where $J(H)$, by Theorem 3.5 in [12], is a Cantor set of topological circles each of which separate A from B and also satisfy the conclusion of the theorem. **QED**

4 Proof of Theorems 19 and 21

Example 37 *Let $f_1(z) = z^2 + c$ where $c > 0$ is small (thus $J(f_1)$ is a quasi-circle). Let $z_0 \in \mathbb{R}$ denoted the finite attracting fixed point of f_1. Note that $f_1^k(0)$ increases to z_0. Choose $f_2(z) = \frac{(z-z_0)^2}{(c-z_0)} + z_0$ and note that $J(f_2) = C(z_0, |c - z_0|)$. For $m_1, m_2 \in \mathbb{N}$ large $h_1 = f_1^{m_1}$ and $h_2 = f_2^{m_2}$ each map $B(z_0, |c - z_0|)$ into itself and $J(G)$ is disconnected for $G = \langle h_1, h_2 \rangle$. Note that $P^*(G) \subset \overline{B(z_0, |c - z_0|)}$ and so $G \in \mathcal{G}$. We have $H = \langle h_2 \rangle$ is hyperbolic, but since $f_1(0) = c \in J(h_2) \subset J(G)$, G is not hyperbolic.*

By conjugating h_2 by a rotation we may assume that $\{h_2^k(c) : k \in \mathbb{N}\}$ is dense in $J(h_2)$ and therefore we see that H can be hyperbolic and have G fail to even be sub-hyperbolic. However, Theorem 19 implies that G is semi-hyperbolic.

Lemma 38 *Let H_1 be a polynomial semigroup in $\mathcal{G}$. Let $H_2 = \langle H_1, h_1, \ldots, h_n \rangle$ be the semigroup generated by H_1 and $h_1, \cdots, h_n$. Suppose*

(1) $H_2 \in \mathcal{G}_{dis}$, and

(2) $J(h_j) \cap J_{\min}(H_2) = \emptyset$ for $j = 1, \ldots, n$.

Then $int\hat{K}(H_1) = int\hat{K}(H_2)$, which then implies $J_{\min}(H_1)$ meets $J_{\min}(H_2)$ since $\partial\hat{K}(H_1) \subset J_{\min}(H_1)$ and $\partial\hat{K}(H_2) \subset J_{\min}(H_2)$.

Remark 39 *We recall the facts given in [12] that for any $G \in \mathcal{G}$ we have $int\hat{K}(G) = \hat{K}(G) \cap F(G)$. Moreover, for any $G \in \mathcal{G}_{dis}$, if $int\hat{K}(G) \neq \emptyset$, then we have $g(\hat{K}(G)) \subset int\hat{K}(G)$ for any $g \in G$ such that $J(g) \cap J_{\min}(G) = \emptyset$.*

Proof: First note that since $H_1 \subset H_2$ we have $\hat{K}(H_1) \supset \hat{K}(H_2)$. Supposing $int\hat{K}(H_1) = \emptyset$, yields $int\hat{K}(H_2) = \emptyset$ also. Then by Theorem 1.9 in [12] each $g \in H_2$ is of the form $g(z) = a(z - z_0)^m + z_0$ for some $z_0 \in \mathbb{C}$. Thus, since $int\hat{K}(H_1) = int\hat{K}(H_2) = \emptyset$, we see that $\hat{K}(H_1) = \{z_0\} = \hat{K}(H_2)$ and thus the lemma holds.

Now suppose that $int\hat{K}(H_1) \neq \emptyset$ and that the lemma does not hold. Thus there exists a point $\eta \in \partial\hat{K}(H_2) \cap int\hat{K}(H_1)$.

We first consider the case that $int\hat{K}(H_2) \neq \emptyset$. By hypothesis (2) and Remark 39 we see that $h_j(\hat{K}(H_2)) \subset int\hat{K}(H_2)$ for all $j = 1, \ldots, n$. Thus there exists $\epsilon > 0$ such that $w \in \hat{K}(H_2)$ implies $h_j(B(w, \epsilon)) \subset int\hat{K}(H_2)$ for all $j = 1, \ldots, n$. Since $int\hat{K}(H_1) = \hat{K}(H_1) \cap F(H_1)$ by Remark 39, H_1 is normal at η and so there exists $\delta > 0$ such that $diam f(B(\eta, \delta)) < \epsilon$ for all $f \in H_1$. We assume that $\delta < \epsilon$ and $B = B(\eta, \delta) \subset int\hat{K}(H_1)$.

We now show that $g \in H_2$ implies $g(B)$ lies in the bounded set $int\hat{K}(H_1)$ which gives the contradiction that $\eta \in B \subset int\hat{K}(H_2)$. If $g \in H_1$, then $g(B) \subset g(int\hat{K}(H_1)) \subset int\hat{K}(H_1)$. If $g \notin H_1$, then we may write $g = k_2 h_j k_1$ where $k_1 \in H_1 \cup \{id\}, k_2 \in H_2 \cup \{id\}$ and $j \in \{1, \ldots, n\}$. Then $k_1(B) \subset int\hat{K}(H_1)$ with $diam\, k_1(B) < \epsilon$ and $k_1(\eta) \in \hat{K}(H_2)$ (since $\eta \in \hat{K}(H_2)$). Then $h_j(k_1(B)) \subset h_j(B(k_1(\eta), \epsilon)) \subset int\hat{K}(H_2)$ and so $g(B) = k_2(h_j(k_1(B))) \subset k_2(int\hat{K}(H_2)) \subset int\hat{K}(H_2)$. This concludes the proof in the case that $int\hat{K}(H_2) \neq \emptyset$.

If $int\hat{K}(H_2) = \emptyset$, then by Theorem 1.9 in [12] we have $\hat{K}(H_2) = \{z_0\}$ and each $g \in H_2$ is of the form $g(z) = a(z - z_0)^m + z_0$ which implies $\hat{K}(H_1) = \{z_0\}$ or $\hat{K}(H_1) = \overline{B(z_0, \rho)}$ for some $\rho > 0$. If $\hat{K}(H_1) = \{z_0\}$, then we are done. Otherwise, the proof of the lemma then follows by using the above argument with $B = B(z_0, r)$ where $r = \min\{\rho, \min_{j=1,\ldots,n} dist(z_0, J(h_j))\}$.
QED

Definition 40 *Let G be a rational semigroup and let N be a positive integer. We define $SH_N(G)$ to be the set of all $z \in \overline{\mathbb{C}}$ such that there exists a neighborhood U of z such that for all $g \in G$ we have $\deg(g : V \to U) \leq N$ for each connected component V of $g^{-1}(U)$.*

Definition 41 *Let G be a rational semigroup. We define $UH(G) = \overline{\mathbb{C}} \setminus \cup_{N=1}^{\infty} SH_N(G)$.*

Remark 42 *For a rational semigroup G we note that each $SH_N(G)$ is open and thus $UH(G)$ is closed.*

Remark 43 *For a rational semigroup G we see that $UH(G) \subset P(G)$. This holds since for $z \notin P(G)$ and $U = B(z, \delta)$ such that $U \cap P(G) = \emptyset$ it must be the case (by an application of the Riemann Hurwitz relation) that $\deg(g : V \to U) = 1$ for each connected component V of $g^{-1}(U)$.*

Remark 44 *We note from Lemma 1.14 in [10] that, the attracting cycles of g, parabolic cycles of g, and the boundary of every Siegel disk of g are contained in $UH(\langle g \rangle)$, for any polynomial g with $\deg(g) \geq 2$. Hence we may conclude that such points are also in $UH(G)$ for any G containing g.*

Theorem 45 *Let $H \in \mathcal{G}$ and let $G = \langle H, h_1, \ldots, h_m \rangle$ be a polynomial semigroup generated by H and $h_1, \cdots, h_m$. Suppose*

(1) $G \in \mathcal{G}_{dis}$,

(2) $J(h_j) \cap J_{\min}(G) = \emptyset$ for each $j = 1, \cdots, m$, and

(3) $\mathbb{C} \cap J(H) \cap UH(H) = \emptyset$.

Then, $\mathbb{C} \cap J(G) \cap UH(G) = \emptyset$.

Remark 46 *This theorem does not require that H or G be finitely generated.*

Proof: Assume the conditions stated in the hypotheses. Since $UH(G) \subset P(G)$ and $P^*(G) \cap J(G) \subset J_{\min}(G)$, we have only to show $J_{\min}(G) \subset \mathbb{C} \setminus UH(G)$.

If $\#J_{\min}(G) = 1$, then Theorem 1.9 in [12] implies that there exists $z_0 \in \mathbb{C}$ such that each map in G is of the form $a(z - z_0)^n + z_0$ and $J_{\min}(G) = \{z_0\}$. We also then see that $UH(H) = P(H) = \{z_0, \infty\}$. Letting $r = \min\{dist(J(H), z_0), \min_{j=1,\dots,m} dist(J(h_j), z_0)\} > 0$, one can easily show that $B(z_0, r)$ is forward invariant under each map in G and thus $z_0 \notin J(G)$. This contradiction shows that $\#J_{\min}(G) > 1$. This, together with hypothesis (2), implies $h_j^{-1}(J_{\min}(G)) \cap J_{\min}(G) = \emptyset$ for $j = 1, \dots, m$ (else $\overline{\cup_{n=1}^{\infty} h_j^{-1}(J_{\min}(G))}$ would be a connected set in $J(G)$ meeting both $J_{\min}(G)$ and $J(h_j)$ contradicting hypothesis (3)), which in turn implies

$$h_j^{-1}(J(G)) \cap A = \emptyset \tag{I}$$

where $A = PH(J_{\min})$.

Let $d = \min_{j=1,\dots,m} dist(h_j^{-1}(J(G)), A) > 0$. By (I) there exists $d_1 > 0$ such that for all $j = 1, \dots, m$, for all $z \in J(G)$, and all components U of $h_j^{-1}(B(z, d_1))$ we have

$$U \cap B(A, d/2) = \emptyset. \tag{II}$$

Now by Lemma 38 and by hypothesis (3) we have $UH(H) \cap \mathbb{C} \subset P^*(H) \cap F(H) \subset \hat{K}(H) \cap F(H) = int\hat{K}(H) = int\hat{K}(G) \subset F(G)$ and so, taking compliments, $J_{\min}(G) \subset \mathbb{C} \setminus UH(H)$.

Claim: There exists $b \in UH(H) \cap int\hat{K}(H)$.

Proof of claim: Note that $int\hat{K}(H) = int\hat{K}(G) \neq \emptyset$. Let $g_0 \in H$ and consider the iterates $\{g_0^n\}$ at any $w \in int\hat{K}(H) \subset F(H)$. Hypothesis (3) implies $UH(H) \cap \mathbb{C} \subset F(H)$ which implies that g_0 cannot have a cycle of Siegel disks nor a parabolic cycle (see Remark 44). Thus by Sullivan's No Wandering Domains Theorem the orbit $\{g_0^n(w)\}$ must be drawn toward an attracting cycle. By replacing, if necessary, g_0 by an iterate we may assume that $g_0^n(w)$ approaches a fixed point b of g_0. Thus $b \in UH(H) \cap \mathbb{C} \subset P^*(H) \cap F(H) \subset \hat{K}(H) \cap F(H) = int\hat{K}(H)$ which completes the proof of the claim.

Now let $z \in J_{\min}(G) \subset \mathbb{C} \setminus UH(H)$. Then there exists $\delta > 0$ such that $B(z, 2\delta) \subset \mathbb{C} \setminus UH(H)$. Since $g(UH(H)) \subset UH(H)$ for each $g \in H$, we must have $g(b) \notin B(z, 2\delta)$. Since H is normal at b, there exists $\epsilon_1 > 0$ such that $g \in H$ gives $g(B(b, \epsilon_1)) \cap B(z, \delta) = \emptyset$, which implies $g^{-1}(B(z, \delta)) \cap B(b, \epsilon_1) = \emptyset$. Since $z \in \mathbb{C} \setminus UH(H)$ there exists $\delta_1 < \delta$ and $N \in \mathbb{N}$ such that for all $h \in H$ and for all components V of $h^{-1}(B(z, \delta_1))$ we have $\deg(h : V \to B(z, \delta_1)) \leq N$.

Fix $h \in H$ and consider a component V of $h^{-1}(B(z, \delta_1))$ and note that the maximum principle implies that V is simply connected. Let $\phi_{V,h} : B(0, 1) \to V$ be the Riemann map chosen such that $h \circ \phi_{V,h}(0) = z$. By applying the distortion Theorem 1.10 in [10], there exists $0 < \delta_2 < \delta_1$ such that any component W of $(h \circ \phi_{V,h})^{-1}(B(z, \delta_2))$ is such that $diamW \leq c$ where $c > 0$ is a small number independent of h, to be specified later.

The family $\{\phi_{V,h}\}$ is normal on $B(0, 1)$ since $\phi_{V,h}(B(0, 1)) = V$ does not meet $B(b, \epsilon_1)$. Thus

$$diam\phi_{V,h}(W) < d_1/10 \tag{III}$$

when c is sufficiently small.

Let $g \in G$. If $g \in H$, then $\deg(g : V \to B(z, \delta_2)) \leq N$ where V is any component of $g^{-1}(B(z, \delta_2))$. If $g \notin H$, then we write $g = hh_j g_1$ where $g_1 \in G \cup \{id\}, h \in H \cup \{id\}$ and $j \in \{1, \ldots, m\}$. Let V_0 be a component of $h_j^{-1}h^{-1}(B(z, \delta_2))$. Thus we have $\deg(hh_j : V_0 \to B(z, \delta_2)) \leq NM$ where $M = \max_{j=1,\ldots,m}\{\deg h_j\}$. By (III) we have $diamh_j(V_0) < d_1/10$. By the definition of d_1 we have $V_0 \cap B(A, d/2) = \emptyset$ and thus $V_0 \cap P(G) = \emptyset$. Using the maximum principle applied to the polynomial hh_j implies V_0 is simply connected and hence each branch of g_1^{-1} is well defined on V_0. So for all components V_1 of $g^{-1}(B(z, \delta_2))$ we have $\deg(g : V_1 \to B(z, \delta_2)) \leq NM$.

In the above, N depends on z, but what we have shown is that $z \in J_{\min}(G)$ implies $z \in J_{\min}(G) \cap SH_N(H)$ for some N, which in turn implies $z \in J_{\min}(G) \cap SH_{NM}(G)$, thus giving $z \notin UH(G)$. **QED**

Proof of Theorem 19: Assume the hypotheses hold. Since H is semi-hyperbolic, the point ∞, which is an attracting fixed point for every map in H and therefore in $UH(H)$, must lie in $F(H)$. From this it follows easily that $\infty \in F(G)$ since there must then be a neighborhood of ∞ which is forward invariant under the finite number of maps h_j as well as each map in H. Applying Theorem 45 now gives the desired conclusion. **QED**

Proof of Theorem 21: The proof follows the same line at the proof of Theorem 45. We note that the usual Koebe Distortion Theorem applies, and on the domains of interest in the proof each h_j is one-to-one by hypothesis (4) and $h \in H$ is one-to-one by hypothesis (3). We omit the details. **QED**

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