Journal of Difference Equations and Applications
Vol. 00, No. 00, January 2008, 1–8

RESEARCH ARTICLE

Density of repelling fixed points in the Julia set of a Rational or Entire Semigroup

Rich Stankewitz*

Department of Mathematical Sciences, Ball State University, Muncie, IN 47306, USA

(*v3.5 released October 2008*)

We briefly survey several methods of proof that the Julia set of a rational or entire function is the closure of the repelling cycles, in particular, focusing on those methods which can be extended to the case of semigroups. We then present an elementary proof that the Julia set of either a non-elementary rational or entire semigroup is the closure of the set of repelling fixed points.

Keywords: Complex dynamics, Julia sets, Random Dynamics, Dynamics of Semigroups.

AMS Subject Classification: 37F10, 37F50, 30D05.

1. Introduction

It is with great pleasure that I submit this paper in honour of Bob Devaney and the celebration of his 60th birthday. I also wish to express my deep appreciation for the mathematical contributions, great inspiration, and the friendly support and guidance he has given to me and many others.

In this paper we concern ourselves with the dynamics of semigroups, a natural generalisation of the study of the dynamics of iteration of a complex analytic map. Instead of repeatedly iterating the same map over and over again, semigroup dynamics allows for a more flexible setting where the map may be changed at each point of the orbit, exactly as in a random walk. Starting with a family of maps $\{h_\lambda : \lambda \in \Lambda\}$, we consider the dynamics of any iteratively defined *composition sequence* of maps, that is, any sequence $h_{\lambda_n} \circ \cdots \circ h_{\lambda_1}$ where each $\lambda_k \in \Lambda$. Assigning probabilities to the choice of map at each stage is the setting for research of random dynamics (see [6–9, 12, 27, 28]). One may consider questions of dynamic stability both along a single previously chosen composition sequence or along all possible composition sequences. Restricting one's attention to the case where all h_λ are rational (respectively, entire), one is lead to study the dynamics of rational (respectively, entire) semigroups.

A *rational (respectively, entire) semigroup* is a semigroup generated by non-constant rational (respectively, entire) maps on the Riemann sphere $\overline{\mathbb{C}}$ (respectively, complex plane $\mathbb{C}$) with the semigroup operation being the composition of maps. We denote by $\langle h_\lambda : \lambda \in \Lambda\rangle$ the semigroup generated by the family of maps $\{h_\lambda : \lambda \in \Lambda\}$. Thus $\langle h_\lambda : \lambda \in \Lambda\rangle$ denotes the family of all maps which can be created through composition of any finite number of maps h_λ. Research on the dynamics of rational

*Email: rstankewitz@bsu.edu

ISSN: 1023-6198 print/ISSN 1563-5120 online

DOI: 10.1080/1023619YYxxxxxxxx
http://www.informaworld.com

semigroups was initiated by Hinkkanen and Martin in [14], where each rational semigroup was always taken to have at least one element of degree at least two — a restriction we do not impose here. Also, Ren, Gong, and Zhou studied such rational semigroups from the perspective of random dynamical systems (see [13, 30]). Later, Kriete and Sumi in [16] studied semigroups of entire maps.

Of primary concern is the set where the dynamics of a semigroup is stable and the set where the dynamics is chaotic. We thus follow [14] in saying that for a rational (respectively, entire) semigroup G the *Fatou set* $F(G)$ is the set of points in $\overline{\mathbb{C}}$ (respectively, $\mathbb{C}$) which have a neighbourhood on which G is normal, and its complement in $\overline{\mathbb{C}}$ (respectively, $\mathbb{C}$) is called the *Julia set* $J(G)$. The more classical Fatou set and Julia set of the cyclic semigroup $\langle g \rangle$ generated by a single map (i.e., the collection of iterates $\{g^n : n \geq 1\}$) is denoted by $F(g)$ and $J(g)$, respectively.

We quote the following results from [14], noting that although they were stated only for rational semigroups, the proofs given for these basic results work as well for entire semigroups. The Fatou set $F(G)$ is *forward invariant* under each element of G, i.e., $g(F(G)) \subset F(G)$ for all $g \in G$, and thus $J(G)$ is *backward invariant* under each element of G, i.e., $g^{-1}(J(G)) \subset J(G)$ for all $g \in G$.

We should take a moment to note that the sets $F(G)$ and $J(G)$ are, however, not necessarily completely invariant under the elements of G. This is in contrast to the case of *iteration* dynamics, i.e., the dynamics of *cyclic* semigroups generated by a single function. For a treatment of alternatively defined *completely* invariant Julia sets of rational semigroups the reader is referred to [23–26].

This paper is concerned with the connection between repelling cycles (which are just repelling fixed points for a higher iterate) and the Julia set. We first give a definition.

Definition 1.1. A point $w \in \overline{\mathbb{C}}$ is called *periodic with period* p for the map f if $f^p(w) = w$ and $w, f(w), \dots, f^{p-1}(w)$ are distinct points. In this case we call the set $\{w, f(w), \dots, f^{p-1}(w)\}$ a *p-cycle* for the map f.

Periodic points are then classified according to their local dynamic behaviour which can be analysed easily by considering the multiplier of the cycle.

Definition 1.2. Suppose the set $\{w_0, \dots, w_{p-1}\}$ forms a p-cycle for the map f with $w_0 \neq \infty$. We define the *multiplier* λ of this cycle (also called the multiplier of each point $w_0, \dots, w_{p-1}$ of period p) to be the derivative of the p-th iterate f^p at its fixed point w_0. Then the p-cycle $\{w_0, \dots, w_{p-1}\}$ of the map f is called
a) *attracting* if $|\lambda| < 1$
b) *repelling* if $|\lambda| > 1$
c) *indifferent* if $|\lambda| = 1$.
Note that the standard accommodations are made to extend the notion of multiplier when $w_0 = \infty$.

It is elementary to prove that a repelling cycle of a map f must lie in $J(f)$. What may be surprising is that the set of all such cycles for f is dense in $J(f)$. In this paper we will consider several proofs of this fact in the iteration (i.e., cyclic semigroup) setting. Then we will look at possible extensions of these methods to the more general semigroup setting. An important ingredient used in many of these proofs will be to show that the Julia set is perfect, i.e., is closed and contains no isolated points. Thus we begin by stating and proving this in the more general setting of semigroups after first making an important definition.

Definition 1.3. Let G be a rational or entire semigroup. We call G *non-elementary* when $J(G)$ contains three or more points.

It is well known that if a semigroup G contains either a transcendental entire map or a rational map of degree two or more, then it is non-elementary (see [18], p. 69). However, not all semigroups are non-elementary. For example, the Julia set of the rational semigroup $\langle 2z, z/2\rangle$ is $\{0,\infty\}$. The number three in the definition of non-elementary is special because of the role it plays in Montel's theorem which we present here in a strengthened form.

THEOREM 1.4 (Montel, see [10], p. 203) *Suppose $\mathcal{F}$ is a family of analytic functions defined on a domain $U \subset \overline{\mathbb{C}}$ or punctured domain $U \setminus \{p\}$ and mapping into $\overline{\mathbb{C}} \setminus \{a, b, c\}$ where $a, b, c \in \overline{\mathbb{C}}$ are three distinct points. Then $\mathcal{F}$ is a normal family on U.*

With this one can now prove that for a non-elementary rational (respectively, entire) semigroup G, the set $J(G)$ is the smallest closed subset of $\overline{\mathbb{C}}$ (respectively, $\mathbb{C}$) which contains three or more points and is backward invariant. Letting the *backward orbit* of z be denoted by $G^{-1}(z) = \cup_{g\in G} g^{-1}(\{z\})$, we then have that $J(G) = \overline{G^{-1}(z)}$ for any $z \in J(G)$ whose backward orbit contains three or more points.

Also using the strengthened Montel's theorem one can prove the following lemma which was presented for rational semigroups as Lemma 3.1 of [14]. For completeness we reproduce their proof below, noting that it also applies to entire semigroups.

LEMMA 1.5. *For a rational or entire semigroup G, the set $J(G)$ is perfect when G is non-elementary.*

Proof. Let $\{a, b, c\} \subset J(G)$. Suppose U is an open neighbourhood of $z \in J(G)$ such that $U \setminus \{z\}$ does not meet $J(G)$, i.e., $U \setminus \{z\} \subset F(G)$. Since $g(F(G)) \subset F(G) = \overline{\mathbb{C}} \setminus J(G)$ for each map $g \in G$, we see that each $g \in G$ omits $\{a, b, c\}$ on $U \setminus \{z\}$. Hence, by Theorem 1.4 we must have that G is normal on all of U, which contradicts the assumption that $z \in J(G)$. □

We now present the following two well known main results from iteration theory which we wish to generalise.

THEOREM 1.6 (Fatou, Julia) *For a rational map f of degree at least two, the set of repelling cycles of f is dense in $J(f)$.*

THEOREM 1.7 (Baker) *For a transcendental entire map f, the set of repelling cycles of f is dense in $J(f)$.*

Theorems 1.6 and 1.7 have been proven by several people using various techniques, some of which we now discuss.

1. *Julia's proof of Theorem 1.6.* The idea behind Julia's proof in [15] (see also [17], p. 156 for a nice exposition) is to first show the existence of a fixed point $z_0 \in J(f)$ which is either repelling or has multiplier exactly equal to 1. Assuming U is an open set which meets $J(f)$, the local behaviour of f near such a fixed point is analysed in order to construct a special homoclinic orbit (sequence z_j such that $\cdots \overset{f}{\mapsto} z_2 \overset{f}{\mapsto} z_1 \overset{f}{\mapsto} z_0$ and $\lim z_j = z_0$) which meets U. The properties of this homoclinic orbit are then used to show the existence of a repelling cycle which also meets U, thus completing the proof.

A critical obstruction to extending this proof to the more general entire semigroup case (and in fact just the case of iteration of a transcendental map) is the reliance on the fact that rational maps always have such a fixed point as z_0 above, a point which a transcendental map (e.g., $e^z + z$) need not have. We do note, however, that when a semigroup G is rational containing a map of degree at least two, the method can be properly adjusted. It can also be adjusted for the case of an entire semigroup G which contains a transcendental map. However, in this case one must first show that such a map has a repelling cycle which is itself a complicated task. We do note though that this method, when applicable, does have the appeal that it mainly relies on dynamical methods, and not on either value distribution theory or Ahlfors covering theory. We, however, wish to seek a method which will more easily apply to all of the semigroups under consideration.

2. *Fatou's proof of Theorem 1.6.* The idea behind Fatou's proof in [11] is to show (a) $J(f)$ lies in the closure of the set of *all* cycles of f, and (b) the number of non-repelling cycles is finite. The result then follows since $J(f)$ is a perfect set as shown in Lemma 1.5.

We make a few comments on the approaches to prove (a) and (b). In considering (b) we let C_A be the number of attracting cycles and C_I be the number of indifferent cycles. Since the immediate attracting basin for each attracting cycle contains at least one critical value and there are at most $2\deg(f)-2$ such critical values, we have $C_A \leq 2\deg(f)-2$. Fatou was able to strengthen this to the relation $C_A + C_I/2 \leq 2\deg(f)-2$ by showing that a sufficiently small *analytic* perturbation of f will produce a map of the same degree where at least half of the indifferent cycles of f become attracting cycles for the perturbed map, while the attracting cycles remain attracting. Shishikura [22] later strengthened this even further showing $C_A+C_I \leq 2\deg(f)-2$ by *quasiconformally* perturbing the map in a process called quasiconformal surgery.

Part (a) follows from a quick application of Montel's Theorem, even when a transcendental map f is considered (see [18], p. 70), but relies heavily on the fact that we are only considering the iterates of f, i.e., a cyclic semigroup. Furthermore, part (b) does not hold in general for entire maps or even rational semigroups. For example, we claim that the rational semigroup $G = \langle z^2/3, (z^2+2)/3\rangle$ has an infinite number of maps, each with a different *finite* attracting fixed point in $[0,1]$. One way to see this is to note that for fixed $n \in N$, the 2^n disjoint images of the interval $[0,1]$ under the each of the maps made as an n-fold composition of the two generating functions are proper sub-intervals of $[0,1]$. Thus the contraction mapping principle implies that each such sub-interval contains a fixed point of the corresponding map. Since this happens for all $n \in N$, the claim follows.

Due to these restrictions outlined above, we must then continue to consider other methods which might apply to semigroups.

3. *Baker's proof of Theorem 1.7.* The method used by Baker in [1] employs both Marty's Criterion (see [19], p. 75) and the beautiful and deep Five Island Theorem from Ahlfors' theory of covering surfaces (see [19], p. 177). This method also turns out to be flexible enough to be adapted to the more general setting of semigroups and was thus employed to do so by Hinkkanen and Martin in [14] (shown only for rational semigroups containing at least one map of degree two or more, though the method applies also for both non-elementary rational and entire semigroups).

In [4] Bergweiler asked if there is a more elementary proof of Theorem 1.7 which does not rely on Ahlfors' deep theory of covering surfaces. [1] In [21] Schwick did exactly that providing a proof which employs a simple consequence of Nevanlinna's second fundamental theorem along with a result of Zalcman to describe non-normality. The main result of this paper is to show that Schwick's ideas can be implemented in both the rational semigroup and entire semigroup settings. Since the proof does not rely on such things as the degree of the map, existence of special fixed points, or Ahlfors deep theory, it offers a new perspective on the issue which is flexible enough to be applied to semigroups thus getting to what some may regard

[1] It is important to note, however, that in [5] Bergweiler does provide elementary proofs of the special cases of the key results of the Ahlfors theory which are used in complex dynamics.

as more of the heart of the matter. The following two theorems will be proved.

THEOREM 1.8. *The Julia set of a semigroup of entire functions which contains at least one transcendental element is the closure of the set of repelling fixed points.*

If an entire semigroup does not contain a transcendental element (thus it is a semigroup of polynomials acting on $\mathbb{C}$ instead of a **rational** semigroup acting on $\overline{\mathbb{C}}$), we can use the following result if it is also non-elementary **as a rational semigroup**.

THEOREM 1.9. *The Julia set of non-elementary rational semigroup is the closure of the set of repelling fixed points.*

We can use the above results to obtain some important corollaries. It is immediate from the definition that $J(G)$ contains the Julia set of each element of G and thus also $J(G) \supset \overline{\bigcup_{f\in G} J(f)}$. However, the previous two results show that more can be said. In particular, we present the following Corollary 1.10 (first stated for rational semigroups by Hinkkanen and Martin in [14]) and Corollary 1.11 (first stated for rational semigroups by Zhou and Ren in [30]).

COROLLARY 1.10. *For a rational or entire semigroup G which is non-elementary, we have*

$$J(G) = \overline{\bigcup_{f\in G} J(f)}.$$

COROLLARY 1.11. *The Fatou set of a non-elementary rational (respectively, entire) semigroup $G = \langle h_\lambda : \lambda \in \Lambda \rangle$ is precisely the set of $z \in \overline{\mathbb{C}}$ (respectively, $z \in \mathbb{C}$) which has a neighbourhood on which every composition sequence generated by $\{h_\lambda : \lambda \in \Lambda\}$ is normal.*

2. Proof of main results

The following important result known as Zalcman's lemma provides the key perspective on the non-normality condition to be employed later (see [29] for the original statement and see [20] for the slightly modified statement which we adopt here).

THEOREM 2.1. *A family $\mathcal{F}$ of meromorphic functions on $\{z : |z| < 1\}$ is not normal at 0 if and only if there exists a sequence $f_j \in \mathcal{F}$, a sequence $z_j \to 0$, a sequence of positive real numbers $r_j \to 0$ and a nonconstant meromorphic function f on $\mathbb{C}$ such that*

$$f_j(z_j + r_j z) \to f(z)$$

spherically uniformly on all compact subsets of $\mathbb{C}$.

The following is a consequence of Nevanlinna's second fundamental theorem. The details of the proof are notationally cumbersome and so we refer the reader to p. 61 of [18].

LEMMA 2.2 (Consequences of Nevanlinna's second fundamental theorem)
a) For a transcendental entire function f there are at most two values $w \in \mathbb{C}$ for which the equation $f(z) = w$ does not have an infinite number of simple solutions, i.e., $z_0 \in \mathbb{C}$ such that $f(z_0) = w$ and $f'(z_0) \neq 0$.

b) For a transcendental meromorphic function f there are at most four values $w \in \overline{\mathbb{C}}$ for which the equation $f(z) = w$ does not have an infinite number of simple solutions.

The last result we will need to prove the main theorems involves totally ramified points, i.e., values $w \in \overline{\mathbb{C}}$ such that each preimage z of w under f maps to w with valency (local degree), denoted here and below by $v_f(z)$, strictly greater than one.

LEMMA 2.3. *A non-constant rational map f has at most three totally ramified points and a polynomial g has at most one finite totally ramified point.*

Proof. Let $z_1, \ldots, z_k$ be totally ramified for the rational function f whose degree we denote by d. Suppose z_1 has preimages $a_1, \ldots, a_m$. Then $d = \sum_{j=1}^{m} v_f(a_j) \geq 2m$ since each $v_f(a_j) \geq 2$. Hence we have

$$d/2 \leq d - m = \sum_{z \in f^{-1}(\{z_1\})} [v_f(z) - 1].$$

Repeating this for each of the k points z_i and summing gives

$$kd/2 \leq \sum_{z \in f^{-1}(\{z_1,\ldots,z_k\})} [v_f(z) - 1] \leq \sum_{z \in \overline{\mathbb{C}}} [v_f(z) - 1] = 2d - 2$$

where the Riemann-Hurwitz relation (see [3], p. 43) was used in the last step. Thus $k \leq 4 - 4/d$ and since k is a positive integer we see that $k \leq 3$.

Similar reasoning for the polynomial case will work by using the fact that for a polynomial g we have $v_g(\infty) = \deg(g)$. □

Proof. [of Theorem 1.8] Select a transcendental entire function $g \in G$. Since $J(g) \subset J(G)$ we immediately get that G is non-elementary since it is known that $J(g)$ contains more than three points. By Lemma 2.2(a) there is a set A_g consisting of at most two points such that for any $w \notin A_g$ the equation $g(z) = w$ has an infinite set of simple solutions.

We will show that the repelling fixed points of the semigroup G are dense in $J(G) \setminus A_g$. The conclusion then follows from the fact that $J(G)$ is perfect (see Lemma 1.5).

For $w_0 \in J(G) \setminus A_g$ we apply Theorem 2.1 (formally to the family of maps $\mathcal{F} = \{f(z + w_0) : f \in G\}$) to see that there exists a sequence $f_k \in G$, complex numbers $z_k \to w_0$, $r_k \searrow 0$ and an *entire* nonconstant function h such that

$$f_k(z_k + r_k z) \to h(z) \tag{1}$$

uniformly on compact subset of $\mathbb{C}$. Note that h is entire since each f_k is entire. Thus we see that

$$(g \circ f_k)(z_k + r_k z) \to (g \circ h)(z) \tag{2}$$

uniformly on compact subset of $\mathbb{C}$.

Let ζ_j for $j \in \mathbb{N}$ be distinct simple solutions to the equation $g(z) = w_0$, i.e., $g(\zeta_j) = w_0$ and $g'(\zeta_j) \neq 0$. If h is transcendental, then by Lemma 2.2(a) there must be a simple solution to one of the equations $h(z) = \zeta_j$ for $j = 1, 2, 3$. Hence there exists a point z_0 such that $h(z_0) = \zeta_1$, say, with $h'(z_0) \neq 0$. If h is a polynomial, then h can have at most 1 totally ramified point by Lemma 2.3. Hence there exists a point z_0 such that $h(z_0) = \zeta_1$, say, with $h'(z_0) \neq 0$. In either case (h transcendental or polynomial), we have $(g \circ h)(z_0) = w_0$, and $(g \circ h)'(z_0) \neq 0$.

Thus

$$(g \circ f_k)(z_k + r_k z) - (z_k + r_k z) \to (g \circ h)(z) - w_0.$$

The fact that the equation $(g \circ h)(z) = w_0$ has a simple solution z_0 (or using the fact that neither g nor h is constant) shows that the limit function $(g \circ h)(z) - w_0$ is not constant and so Hurwitz's theorem then implies that the equation

$$(g \circ f_k)(z_k + r_k z) = (z_k + r_k z)$$

has a solution $z = \tilde{z}_k$ for all large k, and $\tilde{z}_k \to z_0$. We see that the fixed points $z_k + r_k \tilde{z}_k$ of $g \circ f_k \in G$ tend to w_0 and for large k these are repelling fixed points since

$$r_k (g \circ f_k)'(z_k + r_k \tilde{z}_k) = \frac{d}{dz} \left[(g \circ f_k)(z_k + r_k z) \right] |_{z = \tilde{z}_k} \to (g \circ h)'(z_0) \neq 0.$$

$\square$

Proof. [of Theorem 1.9] We will only slightly modify the above proof in the following two cases.

Case 1. Suppose that G contains a non Möbius map f. Set $g = f^3$ and note that $\deg(g)$ is greater than five. Let $A = \{w \in \mathbb{C} : w \text{ is a critical value of } g\} \cup \{\infty, g(\infty)\}$. For $w_0 \in J(G) \setminus A$ we obtain (1), where h is meromorphic on the plane, and the convergence is locally uniform in the spherical metric. Then (2) also holds as g is uniformly continuous on the sphere. The result will follow if we are able to find a point z_0 as above, i.e., such that $(g \circ h)(z_0) = w_0$ and $v_{g \circ h}(z_0) = 1$. By the choice of w_0 the equation $g(z) = w_0$ has at least five simple solutions $\zeta_1, \ldots, \zeta_5 \in \mathbb{C}$. If h is transcendental, then Lemma 2.2(b) implies that there exists a simple solution z_0 to $h(z) = \zeta_j$ for at least one of the ζ_j. If h is rational, then Lemma 2.3 implies that at least one of $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ is not totally ramified under h. In either case (h transcendental or rational), for at least one of the ζ_j there is a simple solution z_0 to the equation $h(z_0) = \zeta_j$. Thus we have found the point z_0 as desired.

Case 2. Suppose that G consists of only Möbius maps. Select any $g \in G$. Let $A = \{\infty, g(\infty)\}$ and select $w_0 \in J(G) \setminus A$. In this case the function h in (1) is meromorphic on $\mathbb{C}$, but also one-to-one (since it is the nonconstant limit of one-to-one functions). Hence h is a Möbius map. We may now let $z_0 = h^{-1}(g^{-1}(w_0))$ and proceed as above. $\square$

We close with a final note regarding another proof of Theorem 1.7. In [2] Bargmann was able to adapt Schwick's method to prove Theorem 1.7 without using results from Nevanlinna theory. He instead used Picard's little theorem and the fact that the non-periodic recurrent points in the Julia set are dense, thus providing a simpler overall method. However, this approach does not lend itself to further adaptation to the more general semigroup setting as the method relies on the fact that only a cyclic semigroup (of iterates) $\langle f \rangle$ is being considered. Specifically, the iterates $f^{\alpha(n)}$ and f^i appearing in his proof would, in the more general semigroup setting, be replaced by more generic maps k_n and k from the semigroup, and these maps might not be adequately related to the other. Since the map with the sought after repelling fixed point in Bargmann's proof is $f^{\alpha(n)-i}$ for some large n, we would correspondingly look for a map ℓ_n such that $k_n = \ell_n \circ k$ so that the map ℓ_n would have the sought after repelling fixed point. However, we are not guaranteed that k_n and k are related by such a map.

Acknowledgements

The author would like to thank both Hiroki Sumi and the referee for their careful reading and helpful comments regarding this manuscript.

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