COMPLETELY INVARIANT SETS OF NORMALITY FOR RATIONAL SEMIGROUPS

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Abstract. Let G be a semigroup of rational functions of degree at least two where the semigroup operation is composition of functions. We prove that the largest open subset of the Riemann sphere on which the semigroup G is normal and is completely invariant under each element of G, can have only 0, 1, 2, or infinitely many components.

1. Introduction

It is well known in iteration theory that the set of normality of a rational function can have only 0, 1, 2, or infinitely many components (see [1], p. 94). In this paper we generalize this result by showing that the completely invariant set of normality of a rational semigroup can have only 0, 1, 2, or infinitely many components. The proof not only generalizes the iteration result, but it also provides an alternative proof for it.

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2. Definitions and basic facts

In what follows all notions of convergence will be with respect to the spherical metric on the Riemann sphere $\overline{\mathbb{C}}$.

A rational semigroup G is a semigroup of rational functions of degree greater than or equal to two defined on the Riemann sphere $\overline{\mathbb{C}}$ with the semigroup operation being functional composition. When a semigroup G is generated by the functions $\{f_1, f_2, \ldots, f_n, \ldots\}$, we write this as

$$G = \langle f_1, f_2, \ldots, f_n, \ldots \rangle.$$

In [3], p. 360 the definitions of the set of normality, often called the Fatou set, and the Julia set of a rational semigroup are as follows:

Definition 1. For a rational semigroup G we define the set of normality of G, $N(G)$, by

$$N(G) = \{z \in \overline{\mathbb{C}} : \text{there is a neighborhood of } z \text{ on which } G \text{ is a normal family}\}$$

and define the Julia set of G, $J(G)$, by

$$J(G) = \overline{\mathbb{C}} \setminus N(G).$$

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Clearly from these definitions we see that $N(G)$ is an open set and therefore its complement $J(G)$ is a compact set. These definitions generalize the case of iteration of a single rational function and we write $N(\langle h \rangle) = N_h$ and $J(\langle h \rangle) = J_h$.

Note that $J(G)$ contains the Julia set of each element of G. In fact, we have $J(G) = \overline{\cup_{f \in G} J_f}$ (see [3], p. 365).

Definition 2. If h is a map of a set Y into itself, a subset X of Y is:

$$\begin{aligned} &i)\, \textit{forward invariant} \text{ under } h \text{ if } h(X) \subset X; \\ &ii)\, \textit{backward invariant} \text{ under } h \text{ if } h^{-1}(X) \subset X; \\ &iii)\, \textit{completely invariant} \text{ under } h \text{ if } h(X) \subset X \text{ and } h^{-1}(X) \subset X. \end{aligned}$$

It is well known that the set of normality of h and the Julia set of h are completely invariant under h (see [1], p. 54), in fact,

$$h(N_h) = N_h = h^{-1}(N_h) \text{ and } h(J_h) = J_h = h^{-1}(J_h). \tag{2.1}$$

Further, we have the following result.

Property 1. *The set J_h is the smallest closed completely invariant (under h) set which contains three or more points (see* [1]*, p. 67).*

In fact, this may be chosen as an alternate definition of J_h, equivalent to that given in Definition 1.

From Definition 1, we get that $N(G)$ is forward invariant under each element of G and $J(G)$ is backward invariant under each element of G (see [3], p. 360). The sets $N(G)$ and $J(G)$ are, however, not necessarily completely invariant under the elements of G. This is in contrast to the case of single function dynamics as noted in (2.1). The question then arises, what if we required the Julia set of the semigroup G to be completely invariant under each element of G? We consider in this paper the consequences of such an extension, given in the following definition.

Definition 3. For a rational semigroup G we define the completely invariant Julia set of G, $E = E(G)$, by

$$E(G) = \bigcap \{S : S \text{ is closed, completely invariant under each } g \in G, \#(S) \geq 3\}$$

where $\#(S)$ denotes the cardinality of S.

We note that $E(G)$ exists, is closed, is completely invariant under each element of G and contains the Julia set of each element of G by Property 1.

Definition 4. For a rational semigroup G we define the completely invariant set of normality of G, $W = W(G)$, to be the complement of $E(G)$, i.e.,

$$W(G) = \overline{\mathbb{C}} \setminus E(G).$$

Note that $W(G)$ is open and it is also completely invariant under each element of G. The main result of this paper is the following theorem.

Theorem 1. *For a rational semigroup G the set $W(G)$ can have only 0, 1, 2, or infinitely many components.*

3. Proof of the main result

Property 2. *If $E(G)$ has nonempty interior, then $E(G) = \overline{\mathbb{C}}$.*

Proof. For a collection of sets $\mathcal{A}$, and a function h, we denote new collections of sets by $h(\mathcal{A}) = \{h(A) : A \in \mathcal{A}\}$ and $h^{-1}(\mathcal{A}) = \{h^{-1}(A) : A \in \mathcal{A}\}$.

Choose $g \in G$. Let us define the following countable collections of sets:

$$\mathcal{E}_0 = \{J_g\},$$
$$\mathcal{E}_1 = \bigcup_{f\in G} f^{-1}(\mathcal{E}_0) \cup \bigcup_{f\in G} f(\mathcal{E}_0),$$
$$\mathcal{E}_{n+1} = \bigcup_{f\in G} f^{-1}(\mathcal{E}_n) \cup \bigcup_{f\in G} f(\mathcal{E}_n),$$

and $\mathcal{E} = \bigcup_{n=0}^{\infty} \mathcal{E}_n$.

In the same manner as in the proof of Lemma 1 in [7] we conclude $E(G) = \overline{\bigcup_{A\in\mathcal{E}} A}$. Using this result we then finish the proof in the same manner as done in the proof of Lemma 2 in [7]. $\square$

The remainder of this section will be devoted to the proof of Theorem 1.

Lemma 1. *If W_0 is a component of W, then $f(W_0)$ is also a component of W for any $f \in G$.*

Proof. Let W_1 be the component of W that contains $f(W_0)$. We show that $f(W_0) = W_1$. Suppose to the contrary that $z \in W_1 \setminus f(W_0)$. Since f is continuous on the compact set $\overline{W_0}$ and an open map on W_0, we have $\partial f(W_0) \subset f(\partial W_0) \subset f(E) \subset E$. Let γ be a path in W_1 connecting z to a point $w \in f(W_0)$. Hence γ must cross $\partial f(W_0) \subset E$. This contradicts the fact that $\gamma \subset W_1$ and so we conclude that $f(W_0) = W_1$. $\square$

Since the remainder of this section will be devoted to the proof of Theorem 1, we will assume that W has L components where $2 \le L < +\infty$. We remark here that the strategy will be to show that each of the L components of W is simply connected and then the result will follow by an application of the Riemann-Hurwitz relation.

Definition 5. Let W have components W_j for $j = 0, \ldots, L-1$.

Remark 1. We see by Lemma 1 that each $f \in G$ (and hence each f^{-1} as well) permutes the W_j for $j = 0, \ldots, L-1$ since f is a continuous map of W onto W.

We may assume that $\infty \in W_0$, else we may impose this condition by conjugating each $f \in G$ by the same rotation of the sphere.

Definition 6. For $j = 1, \ldots, L-1$, we define

$K_j = \{z \notin W_j :$ there exists a simple closed curve $\gamma \subset W_j$ such that $Ind_\gamma(z) = 1\}$

where the winding number is given by $Ind_\gamma(z) = (1/2\pi i)\int_\gamma 1/(w-z)\, dw$. If $z \in K_j$ and the simple closed curve $\gamma \subset W_j$ is such that $Ind_\gamma(z) = 1$, then we say that γ works for $z \in K_j$.

In order to properly define K_0 we first need to move W_0 so that it no longer contains ∞. Let ϕ be a rotation of the sphere so that $\infty \in \phi(W_1)$ and denote $\widetilde{W_j} = \phi(W_j)$ for $j = 0, \ldots, L-1$.

Definition 7. We define

$$\widetilde{K_0} = \{z \notin \widetilde{W_0} : \text{there exists a simple closed curve } \gamma \subset \widetilde{W_0} \text{ such that } Ind_\gamma(z) = 1\}$$

and

$$K_0 = \phi^{-1}(\widetilde{K_0}).$$

If $z \in K_0$ and simple closed curve $\gamma \subset \widetilde{W_0}$ is such that $Ind_\gamma(\phi(z)) = 1$, then we say that the simple closed curve $\phi^{-1}(\gamma)$ works for $z \in K_0$.

Remark 2. Note that saying $\phi^{-1}(\gamma)$ works for $z \in K_0$ does not necessarily imply that $Ind_{\phi^{-1}(\gamma)}(z) = 1$, since it may be the case that $Ind_\gamma(\phi(\infty)) = 1$ and hence $Ind_{\phi^{-1}(\gamma)}(z) = 0$ since z lies in the unbounded component of $\overline{\mathbb{C}} \setminus \phi^{-1}(\gamma)$.

Definition 8. We define

$$K = \bigcup_{j=0}^{L-1} K_j.$$

Definition 9. We define

$$W_j' = W_j \cup K_j.$$

Lemma 2. *For $j = 0, \ldots, L-1$, the set W_j' is open, connected and simply connected. Thus each K_j is the union of the "holes" in W_j.*

Proof. Suppose that $1 \le j \le L-1$, so that W_j is a bounded domain in the complex plane. Define A to be the unbounded component of $\overline{\mathbb{C}} \setminus W_j$. Hence $B = \overline{\mathbb{C}} \setminus A$ is open, connected and simply connected.

Let F be a bounded component of $\overline{\mathbb{C}} \setminus W_j$. Since A and F are each components of the closed set $\overline{\mathbb{C}} \setminus W_j$, there exists a simple polygon $\gamma \subset W_j$ which separates A from F (see [5], p. 134). Hence we see that $F \subset K_j$. Since F was an arbitrary bounded component of $\overline{\mathbb{C}} \setminus W_j$, we conclude that K_j contains all the bounded components of $\overline{\mathbb{C}} \setminus W_j$, i.e., the "holes" of W_j. Hence $W_j' \supset B$. Clearly K_j cannot contain any points of A since any simple closed path $\gamma \subset W_j$ which would wind around such a point would have to necessarily wind around every point of A (since A is a component of the complement of W_j) including ∞ which cannot happen. Hence we conclude $W_j' = B$ and is therefore open, connected and simply connected.

We show that $\phi(W_0')$ is open, connected and simply connected using the same argument as above, and this implies that W_0' is open, connected and simply connected. $\square$

Definition 10. We define

$$W' = \bigcup_{j=0}^{L-1} W_j'.$$

Note that we have $W' = W \cup K$.

Lemma 3. *If for some distinct $r, s \in \{0, \ldots, L-1\}$, we have $W_r' \cap W_s' \neq \emptyset$, then either $\overline{W_r'} \subset W_s'$ or $\overline{W_s'} \subset W_r'$. In particular, if $W_r \cap W_s' \neq \emptyset$ for some distinct $r, s \in \{0, \ldots, L-1\}$, then $\overline{W_r'} \subset W_s'$.*

Proof. Let $z \in W_r' \cap W_s'$. Since $W_r \cap W_s = \emptyset$, we may assume that $z \in K_s$, say. Let γ_s work for $z \in K_s$. Let I_{γ_s} be the component of $\overline{\mathbb{C}} \setminus \gamma_s$ which contains z. Note that $I_{\gamma_s} \setminus W_s = \{z : \gamma_s \text{ works for } z\}$ whether or not $s = 0$ (see Definitions 6 and 7 and Remark 2). Since $z \in W_r'$, we have two cases, either $z \in K_r$ or $z \in W_r$.

Suppose that $z \in K_r$ and let γ_r work for $z \in K_r$. As $\gamma_s \cap \gamma_r = \emptyset$ (since $W_r \cap W_s = \emptyset$) we see that either $\gamma_r \subset I_{\gamma_s}$ or $\gamma_s \subset I_{\gamma_r}$, where I_{γ_r} is the component of $\overline{\mathbb{C}} \setminus \gamma_r$ which contains z. By switching the roles of r and s, if necessary, we assume $\gamma_r \subset I_{\gamma_s}$ and we note that this can be done since $z \in K_r \cap K_s$. In particular, $W_r \cap I_{\gamma_s} \neq \emptyset$.

If $z \in W_r$, then we still get $W_r \cap I_{\gamma_s} \neq \emptyset$ since $z \in I_{\gamma_s}$.

Since $W_r \cap I_{\gamma_s} \neq \emptyset$, $W_r \cap W_s = \emptyset$, $\overline{W_r}$ is connected, and $\gamma_s \subset W_s$, we conclude that $\overline{W_r} \subset I_{\gamma_s}$. Hence $\overline{W_r} \subset W_s'$ since γ_s then works for every $z \in \overline{W_r}$. Since W_s' is simply connected we see that $\overline{W_r'} \subset W_s'$. □

Lemma 4. *The boundary of W_0' is a nondegenerate continuum and as such contains more than three points.*

Proof. We will first show that $W_0' \cap W_1' = \emptyset$. The set W_1' cannot contain W_0' as $\infty \in W_0'$ and W_1' is a bounded subset of $\mathbb{C}$ (since W_1 is a bounded subset of $\mathbb{C}$). The same argument also shows that $\phi(W_0')$ cannot contain $\phi(W_1')$ where ϕ is as in Definition 7, and so we conclude that W_0' cannot contain W_1'. By Lemma 3 we conclude that $W_0' \cap W_1' = \emptyset$.

Since W_0' is simply connected, $\partial W_0'$ contains a nondegenerate continuum unless $\partial W_0'$ consists of just a single point. If $\partial W_0'$ consists of just a single point, then $W_0' \cup \partial W_0' = \overline{\mathbb{C}}$, but this contradicts the fact that $W_0' \cap W_1' = \emptyset$. □

Lemma 5. *For each $j = 0, \ldots, L-1$, we have $J_f \subset \partial W_j$ for each $f \in G$. Since $J(G) = \overline{\cup_{f \in G} J_f}$, we have $J(G) \subset \partial W_j$ for each $j = 0, \ldots, L-1$.*

Proof. Since f permutes the W_j by Remark 3, we may select a positive integer n so that $f^n(W_j) = W_j = f^{-n}(W_j)$ for each $j = 0, \ldots, L-1$. Then we have $\overline{\bigcup_{k=1}^{\infty} f^{-kn}(W_j)} \supset J_{f^n} = J_f$ (see [1], p.71 and p.51). But since $\overline{\bigcup_{k=1}^{\infty} f^{-kn}(W_j)} = \overline{W_j}$ we see that $\partial W_j \supset J_f$, since $W_j \cap J_f = \emptyset$. □

Lemma 6. *We have $W_r \not\subseteq W_s'$ for distinct $r, s \in \{0, \ldots, L-1\}$, and therefore, by Lemma 3, the W_j' are disjoint for $j = 0, \ldots, L-1$.*

Proof. If $L = 2$, then the proof of Lemma 4 shows that $W_0' \cap W_1' = \emptyset$.

We assume now that $L \geq 3$. We will first show that no bounded W_s' can contain any W_r with $r \neq s$. Suppose that this does occur. Then there exists a simple closed curve $\gamma_s \subset W_s$ such that $\overline{W_r} \subset I_{\gamma_s}$ where I_{γ_s} is the component of $\overline{\mathbb{C}} \setminus \gamma_s$ which contains the points z such that $Ind_{\gamma_s}(z) = 1$. Hence, by Lemma 5, $J(G) \subset \partial W_r \subset \overline{W_r} \subset I_{\gamma_s}$. But since $\overline{W_0} \subset \overline{\mathbb{C}} \setminus I_{\gamma_s}$ we see that $J(G) \subset \partial W_0 \subset \overline{W_0} \subset \overline{\mathbb{C}} \setminus I_{\gamma_s}$. This contradiction implies no bounded W_s' can contain any W_r.

We see that W_0' cannot contain any W_r with $r \geq 1$ by the following similar argument. If $\overline{W_r} \subset W_0'$, then there exists a simple closed curve $\gamma \subset \widetilde{W}_0$ such that $Ind_\gamma(z) = 1$ for every $z \in \overline{\widetilde{W}_r}$. Let I_γ be the component of $\overline{\mathbb{C}} \setminus \gamma$ which contains $\overline{\widetilde{W}_r}$. So $\phi(J(G)) \subset \phi(\partial W_r) = \partial\phi(W_r) = \partial\widetilde{W}_r \subset I_\gamma$. Since $\overline{\widetilde{W}_1} \subset \overline{\mathbb{C}} \setminus I_\gamma$ (recall $\infty \in \widetilde{W}_1$), we see that $\phi(J(G)) \subset \phi(\partial W_1) = \partial\phi(W_1) = \partial\widetilde{W}_1 \subset \overline{\mathbb{C}} \setminus I_\gamma$. This contradiction implies W_0' cannot contain any W_r with $r \geq 1$. □

Corollary 1. *The set K has no interior and therefore each $K_j \subset \partial W_j$.*

Proof. By Lemma 6 we see that each $K_j \subset E$ and hence $K \subset E$. The Corollary then follows from Property 2. $\square$

Corollary 2. *We have $\partial W_j = K_j \cup \partial W'_j$.*

Proof. By Corollary 1 we get $K_j \cup \partial W'_j \subset \partial W_j$. We also have $\partial W_j = \overline{W_j} \setminus W_j \subset \overline{W'_j} \setminus W_j = (W'_j \cup \partial W'_j) \setminus W_j = (W_j \cup K_j \cup \partial W'_j) \setminus W_j = K_j \cup \partial W'_j$. $\square$

Lemma 7. *We have $f(K) \subset K$ for all $f \in G$.*

Proof. Let $z \in K_j$ be such that $\gamma \subset W_j$ works for z.

Suppose that $W_l = f(W_j) \neq W_0$. So W'_j contains no poles of f, else such a pole would be in W_j (by the complete invariance of W under f since $\infty \in W_0 \subset W$ and Lemma 6) and hence $f(W_j) = W_0$. By the argument principle, $f(\gamma) \subset W_l$ winds around $f(z)$, thus $f(z) \in K_l$ as $f(z) \notin W_l$ by the complete invariance of W under the map f. Note that $f(\gamma)$ might not work for $f(z) \in K_l$ since it might not be simple, but $f(z) \in K_l$ since it cannot be in the unbounded component of $\overline{\mathbb{C}} \setminus W_l$ and have a curve in W_l, namely $f(\gamma)$, wind around it.

Now suppose that $f(W_j) = W_0$. So $(\phi \circ f)(W_j) = \widetilde{W_0}$ is bounded and W'_j contains no poles of $\phi \circ f$ (else $f(W_j) = W_1$). So $(\phi \circ f)(\gamma)$ winds around $(\phi \circ f)(z)$ and hence $(\phi \circ f)(z) \in \widetilde{K_0}$, i.e., $f(z) \in K_0$.

So $f(K_j) \subset K$ and hence we conclude $f(K) \subset K$. $\square$

Lemma 8. *We have for all $f \in G$, $f(W') \cap \partial W'_0 = \emptyset$. Also $W' \subset N(G)$ and in particular $K \cap J(G) = \emptyset$.*

Proof. We have $f(W') = f(W \cup K) = f(W) \cup f(K) \subset W \cup K = W'$. Since $W' \cap \partial W'_0 = \emptyset$ (since W' is open), Lemma 4 and Montel's Theorem finish the proof. $\square$

Corollary 3. *We have $J(G) \subset \partial W'_j$ for each $j = 0, \ldots, L-1$.*

Proof. This follows immediately from Lemma 5, Corollary 2 and Lemma 8. $\square$

Remark 3. It is of interest to note that for any positive integer n there exist disjoint simply connected domains $D_1, \ldots, D_n$ in $\overline{\mathbb{C}}$ with $\partial D_1 = \partial D_2 = \cdots = \partial D_n$ (see [4], p. 143). Thus Corollary 3 does not imply that $L < 3$ from a purely topological perspective.

Lemma 9. *We have $f^{-1}(K) \subset K$ for all $f \in G$. Hence by Lemma 7, K is completely invariant under each $f \in G$.*

Proof. Let $z \in K_j \subset \partial W_j$ and say $f(w) = z$. Define $W_k = f^{-1}(W_j)$ by Remark 3. We obtain sequences $z_n \in W_j$ such that $z_n \to z$, and $w_n \in W_k$ such that $w_n \to w$ and $f(w_n) = z_n$. Hence we see that $w \in \partial W_k$, else $w \in W_k$ and $z = f(w) \in W_j$. If $w \notin K_k$, then $w \in \partial W'_k$ by Corollary 2. Let Γ be the component of ∂W_j that contains $f(\partial W'_k)$. Since $z \in \Gamma$, the set Γ must be one of the components of K_j. By Corollary 3 we see that there exists a $\zeta \in \partial W'_k \cap J_f$. Hence $f(\zeta) \in K_j \cap J_f$ which is a contradiction since we know by Lemma 8 that K is disjoint from $J(G) \supset J_f$. This contradiction implies $w \in K_k$ and hence $f^{-1}(K) \subset K$. $\square$

Lemma 10. *If W has L components where $2 \leq L < +\infty$, then each is simply connected.*

Proof. Since K and W are each completely invariant under each $f \in G$, so is $W' = W \cup K$. By Lemma 8 we see that $\overline{\mathbb{C}} \setminus W'$ is completely invariant under each $f \in G$, closed, and contains $J(G)$. Hence $E \subset \overline{\mathbb{C}} \setminus W'$. This implies that $W = W'$ and hence each component of W is then simply connected. □

We are now able to present the proof of Theorem 1.

Proof of Theorem 1. If W has L components where $2 \leq L < +\infty$, then each is simply connected by Lemma 10. Select a map $f \in G$. Letting $n \geq 1$ be selected so that each of the components W_j of W is completely invariant under f^n, we get by the Riemann-Hurwitz relation (see [8], p. 7)

$$\delta_{f^n}(W_j) = \deg(f^n) - 1$$

where we write $\delta_g(B) = \sum_{z \in B}[v_g(z) - 1]$ and $v_g(z)$ is the valency of the map g at the point z.

Hence we obtain

$$L(\deg(f^n) - 1) = \sum_{j=0}^{L-1} \delta_{f^n}(W_j) \leq \delta_{f^n}(\overline{\mathbb{C}}) = 2(\deg(f^n) - 1)$$

and so $L \leq 2$. The last equality follows from Theorem 2.7.1 in [1]. □

Remark 4. Note that if $L = 2$, then each component of W is necessarily simply connected.

4. Conclusions

We know from iteration theory that each of the four possibilities $(0, 1, 2, \infty)$ for the number of components of the set of normality can be achieved. So by constructing semigroups G such that all the elements have the same Julia set we know that the only four possibilities for the number of components of the completely invariant set of normality of G can also be achieved. However, it does not seem possible that all four possibilities can be achieved if we restrict ourselves to the cases where two elements of the semigroup G have nonequal Julia sets. For example, if G contains two polynomials with nonequal Julia sets then the completely invariant set of normality is necessarily empty (see [7], Theorem 1).

We do have the following examples however.

Example 1 Let $f(z) = 2z - z^{-1}$ and $g(z) = (z^2 - 1)/2z$. In this case J_f is a Cantor subset of the interval $[-1, 1]$ and $J_g = \overline{\mathbb{R}}$, the extended real line. It is shown in [6] that $J(\langle f, g\rangle) = \overline{\mathbb{R}} = E(\langle f, g\rangle)$.

Example 2 Let $f(z) = 2z - z^{-1}$ and $g(z) = f(z - 1) + 1$. In this case J_f is a Cantor subset of the interval $[-1, 1]$ and $J_g = J_f + 1$. It is shown in [6] that $J(\langle f, g\rangle) = [-1, 2]$ and $E(\langle f, g\rangle) = \overline{\mathbb{R}}$.

So we see that it is possible for a completely invariant set of normality of a semigroup G which contains two elements with nonequal Julia sets, to have 0 or exactly 2 components. We feel that the interplay between functions with nonequal Julia sets and the fact that if $E(G)$ has interior then $E(G) = \overline{\mathbb{C}}$ demands that only under special circumstances can we have $W(G)$ be nonempty, when two elements of the semigroup G have nonequal Julia sets.

We state the following conjectures which are due to Aimo Hinkkanen and Gaven Martin.

Conjecture 1. *If G is a rational semigroup which contains two maps f and g such that $J_f \neq J_g$ and $E(G) \neq \overline{\mathbb{C}}$, then $W(G)$ has exactly two components, each of which is simply connected, and $E(G)$ is equal to the boundary of each of these components.*

Conjecture 2. *If G is a rational semigroup which contains two maps f and g such that $J_f \neq J_g$ and $E(G) \neq \overline{\mathbb{C}}$, then $E(G)$ is a simple closed curve in $\overline{\mathbb{C}}$.*

Of course Conjecture 1 would follow from Conjecture 2.

We finish by including some comments on the number of components of the set of normality $N(G)$ of a rational semigroup G. It is not known if the set $N(G)$ must have only 0, 1, 2, or infinitely many components when G is a finitely generated rational semigroup. However, for each positive integer n, an example of an infinitely generated polynomial semigroup G can be constructed with the property that $N(G)$ has exactly n components. These examples were constructed by David Boyd in [2].

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