SOME COUNTEREXAMPLES IN DYNAMICS OF RATIONAL SEMIGROUPS

RICH STANKEWITZ, TOSHIYUKI SUGAWA, AND HIROKI SUMI

Abstract. We give an example of two rational functions with non-equal Julia sets that generate a rational semigroup whose completely invariant Julia set is a closed line segment. We also give an example of polynomials with unequal Julia sets that generate a non nearly Abelian polynomial semigroup with the property that the Julia set of one generator is equal to the Julia set of the semigroup. These examples show that certain conjectures in the field of dynamics of rational semigroups do not hold as stated and therefore require modification.

1. Introduction

In [3], Hinkkanen and Martin develop a theory of dynamics of rational semigroups as a generalization of the classical theory of the dynamics of the iteration of a rational function defined on the complex sphere $\overline{\mathbb{C}}$. In that paper and in subsequent communications, they put forth several conjectures, some of which will be addressed here. In particular we provide counterexamples to Conjectures 1.1, 1.2 and 1.4. In light of these examples the conjectures are then suitably modified and as such remain open questions. We begin by developing the necessary background to state these questions.

In what follows all notions of convergence will be with respect to the spherical metric on the Riemann sphere $\overline{\mathbb{C}}$. A rational semigroup G is a semigroup of rational functions of degree at least two defined on the Riemann sphere $\overline{\mathbb{C}}$ with the semigroup operation being functional composition. (One may wish to allow some or all of the maps in G to be Möbius, for example, when one is considering Kleinian groups as in [8], but since the examples constructed here all contain maps of degree two or more, we will use our simplified definition to avoid any technical complications which are not pertinent to this paper.) When a semigroup G is generated by the functions $\{f_1, f_2, \dots, f_n, \dots\}$, we write this as

$$G = \langle f_1, f_2, \dots, f_n, \dots \rangle. \tag{1.1}$$

On p. 360 of [3], the definitions of the set of normality, often called the Fatou set, and the Julia set of a rational semigroup are as follows:

The second author was partially supported by the Ministry of Education, Grant-in-Aid for Encouragement of Young Scientists, 11740088.

2000 Mathematics Subject Classification: Primary 37F10, 37F50, 30D05. Key words and phrases. Complex dynamics, Julia sets.

Definition 1.1. For a rational semigroup G we define the set of normality of G, $N(G)$, by

$$N(G) = \{z \in \overline{\mathbb{C}} : \exists \text{ a neighborhood of } z \text{ on which } G \text{ is a normal family}\}$$

and define the Julia set of G, $J(G)$, by

$$J(G) = \overline{\mathbb{C}} \setminus N(G).$$

Clearly from these definitions we see that $N(G)$ is an open set and therefore its complement $J(G)$ is a compact set. These definitions generalize the case of iteration of a single rational function and we write $N(\langle h \rangle) = N_h$ and $J(\langle h \rangle) = J_h$. Note that $J(G)$ contains the Julia set of each element of G. For research on (semi-)hyperbolicity and Hausdorff dimension of Julia sets of rational semigroups, see [9, 10, 11, 12].

Definition 1.2. If h is a map of a set Y into itself, a subset X of Y is:

$i)$ *forward invariant* under h if $h(X) \subset X$;
$ii)$ *backward invariant* under h if $h^{-1}(X) \subset X$;
$iii)$ *completely invariant* under h if $h(X) \subset X$ and $h^{-1}(X) \subset X$.

It is well known that for a rational function h, the set of normality of h and the Julia set of h are completely invariant under h (see [2], p. 54), i.e.,

$$h(N_h) = N_h = h^{-1}(N_h) \text{ and } h(J_h) = J_h = h^{-1}(J_h). \tag{1.2}$$

In fact, the following property holds.

Property 1.1. *For a rational map h of degree at least two the set J_h is the smallest closed completely invariant (under h) set which contains three or more points (see* [2]*, p. 67).*

From Definition 1.1, it follows that $N(G)$ is forward invariant under each element of G and $J(G)$ is backward invariant under each element of G (see [3], p. 360). The sets $N(G)$ and $J(G)$ are, however, not necessarily completely invariant under the elements of G. This is in contrast to the case of single function dynamics as noted in (1.2). However, we could have generalized the classical notion of the Julia set of a single function in such a way as to force the Julia set of a rational semigroup to be completely invariant under each element of the semigroup. Thus we give the following definition.

Definition 1.3. For a rational semigroup G we define the **completely invariant** Julia set of G

$$E(G) = \bigcap \{S : S \text{ is closed, completely invariant under each } g \in G, \#(S) \geq 3\}$$

where $\#(S)$ denotes the cardinality of S.

We note that $E(G)$ exists, is closed, is completely invariant under each element of G and contains the Julia set of each element of G by Property 1.1.

Definition 1.4. For a rational semigroup G we define the **completely invariant** set of normality of G, $W(G)$, to be the complement of $E(G)$, i.e.,

$$W(G) = \overline{\mathbb{C}} \setminus E(G).$$

Note that $W(G)$ is open and it is also completely invariant under each element of G.

We state the following conjectures which are due to A. Hinkkanen and G. Martin (see [7]).

Conjecture 1.1. *If G is a rational semigroup which contains two maps f and g such that $J_f \neq J_g$ and $E(G) \neq \overline{\mathbb{C}}$, then $W(G)$ has exactly two components, each of which is simply connected, and $E(G)$ is equal to the boundary of each of these components.*

Conjecture 1.2. *If G is a rational semigroup which contains two maps f and g such that $J_f \neq J_g$ and $E(G) \neq \overline{\mathbb{C}}$, then $E(G)$ is a simple closed curve in $\overline{\mathbb{C}}$.*

In section 2 we give a method for constructing functions (as well as providing concrete functions) whose Julia sets are unequal, but which generate a semigroup whose completely invariant Julia set is a line segment. Hence the above conjectures do not hold. But since the only completely invariant Julia sets of rational semigroups which are known at this time (when the semigroup contains two maps with unequal Julia sets) are $\overline{\mathbb{C}}$ (see [6] and [7]) or sets which are Möbius equivalent to a line segment or circle, the authors put forth the following conjecture, which is currently unresolved.

Conjecture 1.3. *If G is a rational semigroup which contains two maps f and g such that $J_f \neq J_g$ and $E(G)$ is not the whole Riemann sphere, then $E(G)$ is Möbius equivalent to a line segment or a circle.*

Remark 1.1. We briefly explain some evidence that compels us to pose Conjecture 1.3 the way we did. Our example of a rational semigroup G with $E(G)$ being a line segment is rigid since G contains a Tchebycheff polynomial, which is known to be postcritically finite (and hence, rigid). On the other hand, an example of a rational semigroup G with $E(G)$ being a (unit) circle generated by rational functions $f_1, \ldots, f_n$ with non-equal Julia sets is easily constructed by choosing finite Blaschke products as the f_j's. However, it seems difficult to quasiconformally deform $f_1, \ldots, f_n$ simultaneously so that the completely invariant Julia set of the resulting rational semigroup is not a circle.

In section 3 we provide a counterexample to the following conjecture also due to Hinkkanen and Martin [4].

Conjecture 1.4. *Let G be a polynomial semigroup such that $J_h = J(G)$ for some $h \in G$. Then $J_f = J_g$ for all $f, g \in G$ (and hence G is nearly abelian by Theorem 3.2).*

In our counterexample $J(G)$ is a closed line segment. Since no other types of counterexamples are known, we modify this conjecture as follows and note that it remains unresolved.

Conjecture 1.5. *Let G be a polynomial semigroup such that $J_h = J(G)$ for some $h \in G$ where $J(G)$ is not a line segment. Then $J_f = J_g$ for all $f, g \in G$ (and hence G is nearly abelian by Corollary 3.2).*

2. Counterexamples to Conjectures 1.1 and 1.2

We begin this section with some notation and lemmas. Let $\phi(z) = \frac{z^2-1}{z^2+1}$ and denote the upper half plane as $U = \{z : \Im z > 0\}$. Then ϕ maps U one-to-one onto $\Omega = \overline{\mathbb{C}} \setminus [-1, 1]$ and ϕ maps $\overline{\mathbb{R}}$ two-to-one onto $I = [-1, 1]$. We call a map f *odd* if $f(-z) = -f(z)$ and we call a map f *even* if $f(-z) = f(z)$.

Lemma 2.1. *A function f is an odd rational map such that $f(U) = U$ if and only if it has the form*

$$f(z) = az - \frac{b}{z} - \sum_{j=1}^{N} \frac{B_j z}{z^2 - A_j} \tag{2.1}$$

where $a, b, A_j, B_j \geq 0$.

Proof. Let f be an odd rational map such that $f(U) = U$. Then any preimage of infinity must be real (else there would exist a preimage of infinity in U) and simple (else there would be points in U that map outside of U). Again, since $f(U) = U$, it follows that f must be of the form $az - \frac{b}{z} - \sum_{j=1}^{k} \frac{c_j}{z-a_j}$ where $a, b, c_j \geq 0$ and $a_j \in \mathbb{R}$. Since $f(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$ and $f(-z) = -f(z)$ we conclude that the poles other than the one which might possibly exist at the origin must come in pairs of real numbers symmetric about the origin. Hence $f(z) = az - \frac{b}{z} - \sum_{j=1}^{N} \frac{b_j}{z-a_j} - \sum_{j=1}^{N} \frac{b_j}{z+a_j}$ where $b_j > 0$, which can be algebraically reduced to (2.1).

Let f be a map of the form (2.1). Hence f is odd, rational, maps U into U (since each term in the sum does), and maps $\overline{\mathbb{R}}$ into $\overline{\mathbb{R}}$ (since the coefficients are all real). From this it easily follows that $f(U) = U$. $\square$

Lemma 2.2. *Let f be a rational map. Then $[f(z)]^2$ is even if and only if f is even or odd.*

Proof. Suppose $[f(z)]^2 = [f(-z)]^2$. Then an analytic square root of $[f(-z)]^2$ (defined locally away from the zeroes and poles of f) is either $f(z)$ or $-f(z)$. The identity theorem can then be used to show that $f(-z)$ is either $f(z)$ or $-f(z)$ globally, i.e., f is either even or odd.

The reverse implication is immediate. $\square$

Lemma 2.3. *Let f be a rational map. Then $\phi \circ f$ is even if and only if $[f(z)]^2$ is even.*

Proof. Since $\phi(z) = \psi(z^2)$ for $\psi(z) = \frac{z-1}{z+1}$ we see that $[f(z)]^2 = \psi^{-1} \circ \phi(f(z))$ and $\psi^{-1} \circ \phi(f(-z)) = [f(-z)]^2$. The lemma easily follows. $\square$

Lemma 2.4. *If g is an even rational function, then $g(z) = h(z^2)$ for some rational map h.*

Proof. For $z \neq 0$ or ∞ we define $h(z) = g(\pm\sqrt{z})$ and note that h is well defined since g is even. Since h is analytic on $\mathbb{C} \setminus \{0\}$ and can be extended in the obvious way to be continuous on $\overline{\mathbb{C}}$, h is rational and satisfies $h(z^2) = g(z)$. □

Lemma 2.5. *Let f be a rational map such that $f(U) = U$. Then there exists a rational map $\tilde{f}$ such that $\phi \circ f = \tilde{f} \circ \phi$ if and only if f is odd (and therefore of the form in Lemma 2.1).*

Proof. Let f be odd. Since $f(z) = -f(-z)$ we see that $[f(z)]^2$ is an even rational function and therefore by Lemma 2.4 $[f(z)]^2 = h(z^2)$ for some rational map h. Define $\tilde{f}(z) = \frac{h(\frac{1+z}{1-z})-1}{h(\frac{1+z}{1-z})+1}$ (hence $\tilde{f}$ is a rational map as it is a composition of rational maps). Let $w = \phi(z) = \frac{z^2-1}{z^2+1}$ and note that $z^2 = \frac{1+w}{1-w}$. Hence $(\tilde{f} \circ \phi)(z) = \tilde{f}(w) = \frac{h(\frac{1+w}{1-w})-1}{h(\frac{1+w}{1-w})+1} = \frac{[f(z)]^2-1}{[f(z)]^2+1} = (\phi \circ f)(z)$.

Suppose there exists a rational map $\tilde{f}$ such that $\phi \circ f = \tilde{f} \circ \phi$. Then $\tilde{f} \circ \phi$ is even since ϕ is even. The semi-conjugacy implies $\phi \circ f$ is also even, which by Lemmas 2.3 and 2.2 gives that f is either even or odd. If f were even, then $f(\overline{\mathbb{C}} \setminus U) = f(\overline{U}) = \overline{U}$ and the preimage of the lower half plane would be empty. This contradicts the fact that the image of $\overline{\mathbb{C}}$ under a rational map is always $\overline{\mathbb{C}}$. Hence we conclude that f must be odd. □

Lemma 2.6. *If $\tilde{f}$ is a rational map such that $\Omega = \tilde{f}^{-1}(\Omega)$, then there exists an odd rational map f such that $U = f^{-1}(U)$ and $\phi \circ f = \tilde{f} \circ \phi$.*

Proof. Let h denote the branch of the inverse of ϕ which maps Ω onto U. Then $f = h \circ \tilde{f} \circ \phi$ maps U onto U properly and is therefore a rational map (Blaschke product of the upper half plane). Clearly, $\phi \circ f = \tilde{f} \circ \phi$ on U and so by the identity Theorem this semi-conjugacy holds on all of $\overline{\mathbb{C}}$. By Lemma 2.5 f is odd. □

Remark 2.1. Lemmas 2.5 and 2.6 classify those rational functions that can be semi-conjugated by ϕ.

Lemma 2.7. *For rational semigroups $G = \langle g_j : j \in \mathcal{I} \rangle$ and $H = \langle h_j : j \in \mathcal{I} \rangle$ where there exists a rational function k satisfying the semi-conjugacy relation $k \circ h_j = g_j \circ k$ for each $j \in \mathcal{I}$, we have $J(G) = k(J(H))$ and $N(G) = k(N(H))$.*

Proof. We first note that the semi-conjugacy relation on the generators translates to a semi-conjugacy relation between corresponding elements of the semigroups. More precisely, if $h = h_{j_1} \circ \cdots \circ h_{j_n} \in H$, then for $g = g_{j_1} \circ \cdots \circ g_{j_n}$ we have $k \circ h = g \circ k$ since $k \circ h_{j_1} \circ \cdots \circ h_{j_n} = g_{j_1} \circ k \circ h_{j_2} \circ \cdots \circ h_{j_n} = g_{j_1} \circ g_{j_2} \circ k \circ h_{j_3} \circ \cdots \circ h_{j_n} = ... = g_{j_1} \circ g_{j_2} \circ \cdots \circ g_{j_n} \circ k$.

Let z_0 be a point in $N(H)$ and let Δ be a small open set in $N(H)$ containing z_0 such that $h(\Delta)$ has spherical diameter less than ϵ for all $h \in H$. Denoting the Lipschitz constant of k by C (see [2], p. 32), we see that for any $g = g_{j_1} \circ \cdots \circ g_{j_n} \in G$ the diameter of $g(k(\Delta)) = k(h_{j_1} \circ \cdots \circ h_{j_n}(\Delta)) = k(h(\Delta))$ is less than $C\epsilon$. Hence $k(z_0) \in N(G)$ and so we conclude that $k(N(H)) \subset N(G)$.

Let z_0 be a repelling fixed point for some $h = h_{j_1} \circ \cdots \circ h_{j_n} \in H$, but which is not a critical point of k. Then for $g = g_{j_1} \circ \cdots \circ g_{j_n}$ we have $g \circ k = k \circ h$ and hence g has a fixed point at $k(z_0)$ with the same multiplier as that of h at z_0 (using the chain rule and the fact that $g = k \circ h \circ k^{-1}$ for the branch of k^{-1} which maps $k(z_0)$ to z_0). Hence we have shown that the repelling fixed points of the maps in H, which are not any of the finite number of critical points of k, map under k to repelling fixed points of maps in G. Since the Julia set of a rational semigroup is a perfect set equal to the the closure of the set of repelling fixed points of the elements of the semigroup (see [3], Theorem 3.1 and Corollary 3.1), it then follows that $k(J(H)) \subset J(G)$.

Since $J(H) = \overline{\mathbb{C}} \setminus N(H)$ and $J(G) = \overline{\mathbb{C}} \setminus N(G)$ the lemma now follows from the fact that $k(\overline{\mathbb{C}}) = \overline{\mathbb{C}}$. □

One might expect that a result similar to Lemma 2.7 would hold for completely invariant Julia sets, however, we require an additional hypothesis as noted in the following lemmas.

Lemma 2.8. *Suppose rational functions g, h, k satisfy the semi-conjugacy relation $k \circ h = g \circ k$. If $\tilde{S}$ is completely invariant under g, then $k^{-1}(\tilde{S})$ is completely invariant under h. Also, if S is completely invariant under h and $k^{-1}(k(S)) = S$, then $k(S)$ is completely invariant under g.*

The proof of Lemma 2.8 follows readily from the semi-conjugacy and will therefore be omitted.

Lemma 2.9. *For rational semigroups $G = \langle g_j : j \in \mathcal{I} \rangle$ and $H = \langle h_j : j \in \mathcal{I} \rangle$ where there exists a rational function k satisfying the semi-conjugacy relation $k \circ h_j = g_j \circ k$ for each $j \in \mathcal{I}$, we have $k(E(H)) \subset E(G)$ (and thus $W(G) \subset k(W(H))$). If we also have that $k^{-1}(k(E(H)) = E(H)$, then $k(E(H)) = E(G)$ and $W(G) = k(W(H))$.*

Remark 2.2. The hypothesis $k^{-1}(k(E(H))) = E(H)$ stated above would automatically follow from the other assumptions if, in addition, k is a (branched) Galois covering. We, however, do not require that form of the statement because one can easily check that this hypothesis holds in the situations we consider below.

Proof. Let $h = h_{j_1} \circ \cdots \circ h_{j_n} \in H$ and consider the corresponding $g = g_{j_1} \circ \cdots \circ g_{j_n} \in G$. Since $E(G)$ is completely invariant under g and $k \circ h = g \circ k$, Lemma 2.8 shows that the closed set $k^{-1}(E(G))$ is completely invariant under h. Since $h \in H$ was arbitrary, we conclude that $E(H) \subset k^{-1}(E(G))$. Thus $k(E(H)) \subset E(G)$.

Similarly one can use Lemma 2.8 to show that $k^{-1}(k(E(H)) = E(H)$ implies $E(G) \subset k(E(H))$ and so $E(G) = k(E(H))$. When $k^{-1}(k(E(H)) = E(H)$, k maps $E(H)$ in a deg(k)-to-one fashion onto $k(E(H)) = E(G)$. Since k is a rational map of global degree deg(k), it must then map $W(H) = \overline{\mathbb{C}} \setminus E(H)$ onto $W(G) = \overline{\mathbb{C}} \setminus E(G)$ (also in a deg(k)-to-one fashion). □

Example 2.1 (Counterexamples to Conjectures 1.1 and 1.2). Let f be an odd rational map such that $f(U) = U$. Then by Lemma 2.5 there exists a rational function $\tilde{f}$ satisfying the semi-conjugacy relation $\phi \circ f = \tilde{f} \circ \phi$. Similarly we let g be an odd rational map with $g(U) = U$ and so there exists a rational map $\tilde{g}$ with $\phi \circ g = \tilde{g} \circ \phi$. By choosing f and g such that $J_f \neq \overline{\mathbb{R}}$ and $J_g = \overline{\mathbb{R}}$, we have that $J_{\tilde{f}} \neq I$ and $J_{\tilde{g}} = I$ by Lemma 2.7. Since $\overline{\mathbb{R}}$ is completely invariant under both f and g we have $E(G) \subset \overline{\mathbb{R}}$ where $G = \langle f, g\rangle$. Since $E(G) \supset J_g = \overline{\mathbb{R}}$, we conclude that $E(G) = \overline{\mathbb{R}}$. For $\tilde{G} = \langle \tilde{f}, \tilde{g}\rangle$ we see that since $\phi^{-1}(\phi(E(G))) = \phi^{-1}(\phi(\overline{\mathbb{R}})) = \overline{\mathbb{R}} = E(G)$, we must have $E(\tilde{G}) = \phi(\overline{\mathbb{R}}) = I$. Since $J_{\tilde{f}} \neq J_{\tilde{g}}$, $\tilde{G}$ is a counterexample to Conjectures 1.1 and 1.2.

Specifically we may select $f(z) = 2z - \frac{1}{z}$ and $g(z) = \frac{z^2-1}{2z}$. Hence J_f is a Cantor subset of I (see [2], p.21). Since g is the conjugate of $z \mapsto z^2$ under $z \mapsto i\frac{1+z}{1-z}$ we see that $J_g = \overline{\mathbb{R}}$. In this case one can calculate (via the proof of Lemma 2.5) that $\tilde{f}(z) = \frac{3z+5z^2}{1+3z+4z^2}$ and $\tilde{g}(z) = 2z^2 - 1$.

In the next example, we construct a semigroup G that provides a counterexample to Conjectures 1.1 and 1.2 with the additional property that $J(G) \subsetneq E(G)$.

Example 2.2. Consider $f(z) = 2z - 1/z$ as in Example 2.1. Let $\varphi(z) = 2z$, and set $g(z) = (\varphi \circ f \circ \varphi^{-1})(z) = 2z - 4/z$. Note that $J_g = \varphi(J_f) = 2J_f$ and that $\overline{\mathbb{R}}$ is completely invariant under g. Hence for $G = \langle f, g\rangle$, we have $E(G) \subset \overline{\mathbb{R}}$.

Suppose that $E(G) \neq \overline{\mathbb{R}}$. Since $\overline{\mathbb{R}}$ is completely invariant under both f and g, it follows from Lemma 3.2.5 in [5] that if $E(G)$ contains a non-degenerate interval in the real line, then $E(G) = \overline{\mathbb{R}}$. Hence we may select an open interval $L = (x, y)$ in $\mathbb{R} \setminus E(G)$ with both x, y large. Since the length of the intervals $f^n(L)$ tends to $+\infty$, we may assume that $y - x$ is large. By expanding the interval we may also assume that $x, y \in E(G)$ (note that we used here that ∞ is a non-isolated point in $E(G)$ which follows since $2 \in J_g \subset E(G)$ and $f^n(2) \to \infty$).

Since x is large, we can use the fact that $f(x)$ is slightly greater than $g(x)$ to see that $g^{-1}(\{f(x)\})$ contains a point slightly larger than x (and hence less than y). But by the complete invariance of the set $E(G)$ under f and g, we get $g^{-1}(\{f(x)\}) \subset E(G)$. This is a contradiction since the interval (x, y) does not meet $E(G)$. We conclude that $E(G) = \overline{\mathbb{R}}$.

Since ∞ is an attracting fixed point under both f and g, we see that small neighborhoods of ∞ map inside themselves under each map in G. Hence $\infty \in N(G)$ and so $J(G) \neq \overline{\mathbb{R}}$.

As in Example 2.1 we may semi-conjugate the odd rational maps f and g by ϕ to get maps $\tilde{f}(z) = \frac{3z+5z^2}{1+3z+4z^2}$ and $\tilde{g}(z) = \frac{5z^2+40z-29}{3z^2+40z-27}$. Hence for $\tilde{G} = \langle \tilde{f}, \tilde{g}\rangle$ we have $J(\tilde{G}) = \phi(J(G)) \subsetneq \phi(\overline{\mathbb{R}}) = I$ and $E(\tilde{G}) = \phi(E(G)) = \phi(\overline{\mathbb{R}}) = I$. Since $J_{\tilde{f}} \neq J_{\tilde{g}}$ (otherwise one would have $E(\tilde{G}) = J(\tilde{G}) = J_{\tilde{f}} = J_{\tilde{g}}$), we see that $\tilde{G}$ is a counterexample to Conjectures 1.1 and 1.2.

3. Counterexamples to Conjecture 1.4

In [3], p. 366 Hinkkanen and Martin give the following definition.

Definition 3.1. A rational semigroup G is **nearly abelian** if there is a compact family of Möbius transformations $\Phi = \{\phi\}$ with the following properties:

(i) $\phi(N(G)) = N(G)$ for all $\phi \in \Phi$, and

(ii) for all $f, g \in G$ there is a $\phi \in \Phi$ such that $f \circ g = \phi \circ g \circ f$.

Theorem 3.1 ([3], Theorem 4.1)**.** *Let G be a nearly abelian semigroup. Then for each $g \in G$ we have $J_g = J(G)$.*

A natural question is to what extent does the converse to Theorem 3.1 hold. Using a result of A. Beardon (see [1], Theorem 1) Hinkkanen and Martin have proved the following result for polynomial semigroups.

Theorem 3.2 ([3], Corollary 4.1). *Let $\mathcal{F}$ be a family of polynomials of degree at least 2, and suppose that there is a set J such that $J_g = J$ for all $g \in \mathcal{F}$. Then $G = \langle \mathcal{F} \rangle$ is a nearly abelian semigroup.*

Note that under the hypotheses of Theorem 3.2 we have $J_h = J(G)$ for each generator $h \in \mathcal{F}$. So we see that Conjecture 1.4 is suggesting that if $J_h = J(G)$ for just one $h \in G$, then G is still nearly abelian. However, this is not the case as we see by the following counterexample.

Example 3.1 (Counterexample to Conjecture 1.4). Let $f(z) = z^2 - 2, g(z) = 4z^2 - 2$ and $G = \langle f, g \rangle$. It is well known that f is a conjugate of $2z^2 - 1$ by $z \mapsto 2z$ and so $J_f = [-2, 2]$ (see [2], p. 9). It can easily be seen that g maps $[-1, 1]$ onto $[-2, 2]$ in a two-to-one fashion. Since $g^{-1}([-1, 1]) \subset g^{-1}([-2, 2]) = [-1, 1]$ it follows that $J_g \subset [-1, 1]$. In particular $J_g \subsetneq J_f$. We also note that $\overline{\mathbb{C}} \setminus [-2, 2]$ is forward invariant under both f and g and as such must lie in $N(G)$ by Montel's Theorem. It follows that $J(G) = [-2, 2] = J_f$, yet $J_f \neq J_g$.

We remark that any map g that maps a proper sub-interval of $[-2, 2]$ onto $[-2, 2]$ in a $deg(g)$-to-one fashion would suffice in the above example and such functions can easily be obtained by constructing real polynomials with appropriate graphs. Also, f may be replaced by any Tchebycheff polynomial (see section 1.4 of [2]), normalized so that $J_f = [-2, 2]$.

Acknowledgements: The authors would like to thank Aimo Hinkkanen for his advice given during the preparation of this manuscript.

References

[1] Alan F. Beardon. Symmetries of Julia sets. *Bull. London Math. Soc.*, 22:576–582, 1990.

[2] Alan F. Beardon. *Iterations of Rational Functions.* Springer-Verlag, New York, 1991.

[3] A. Hinkkanen and G.J. Martin. The dynamics of semigroups of rational functions I. *Proc. London Math. Soc.*, 3:358–384, 1996.

[4] A. Hinkkanen and G.J. Martin. Personal communication. 1997.

[5] Rich Stankewitz. *Completely invariant Julia sets of rational semigroups.* PhD thesis, University of Illinois, 1998.

[6] Rich Stankewitz. Completely invariant Julia sets of polynomial semigroups. *Proc. Amer. Math. Soc.*, 127(10):2889–2898, 1999.

[7] Rich Stankewitz. Completely invariant sets of normality for rational semigroups. *Complex Variables Theory Appl.*, 40(3):199–210, 2000.

[8] Rich Stankewitz. Uniformly perfect sets, rational semigroups, Kleinian groups and IFS's. *Proc. Amer. Math. Soc.*, 128(9):2569–2575, 2000.

[9] Hiroki Sumi. On hausdorff dimension of julia sets of hyperbolic rational semigroups. *Kodai. Math.J.*, 21(1):10–28, 1998.

[10] Hiroki Sumi. Skew product maps related to finitely generated rational semigroups. *Nonlinearity*, 13:995–1019, 2000.

[11] Hiroki Sumi. Dynamics of sub-hyperbolic and semi-hyperbolic rational semigroups and skew products. *Ergod.Th.& Dynam. Sys.*, 21:563–603, 2001.

[12] Hiroki Sumi. Semi-hyperbolic fibered rational maps and rational semigroups. preprint.

(Rich Stankewitz) School of Science, Penn State University at Erie, Station Road, Erie, PA 16563, USA
E-mail address: `rls42@psu.edu`

(Toshiyuki Sugawa) Department of Mathematics, Kyoto University, 606-8502 Kyoto, Japan
Current address: Department of Mathematics, University of Helsinki, P.O.Box 4 (Yliopistonkatu 5), FIN-00014, Helsinki, Finland
E-mail address: `sugawa@kusm.kyoto-u.ac.jp`

(Hiroki Sumi) Department of Mathematics, Tokyo Institute of Technology 2-12-1, O-Okayama, Meguro-ku, Tokyo, 152-8551, Japan
E-mail address: `sumi@math.titech.ac.jp`