DYNAMICS OF RATIONAL SEMIGROUPS

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Contents

1. Introduction

These notes are based on a series of lectures given by the authors at Georg-August-Universität in Göttingen in June 22 - July 2, 1998. The authors would like to thank Professors Manfred Denker and Hartje Kriete for their hospitality.

For a treatment of the classical iteration theory one may see [5] and [33]. We use these texts as the basic references for such material. The material in Sections 1−3 on rational semigroup dynamics is largely taken from the papers [15] and [16] by Aimo Hinkkanen and Gaven Martin. The material in Section 4 is taken from the papers [32], [30] and [31] by Rich Stankewitz. The material from Sections 5 and 6 is taken from the papers [8], [7] and [6] by David Boyd.

The study of the dynamics of rational semigroups is a generalization of the study of the dynamics associated with the iteration of a rational function defined on the Riemann sphere. A main focus of this study is to see how far and in what sense does the classical theory of Fatou and Julia extend to this new setting. In particular, it is of interest to understand to what extent such main results as Sullivan's no-wandering-domains theorem and the Classification of fixed components theorem hold in this more general setting. We are also interested in learning what new phenomena can occur and what new insight this might lend to the classical theory.

In what follows all notions of convergence will be with respect to the spherical metric d on the Riemann sphere $\overline{\mathbb{C}}$.

A rational semigroup G is a semigroup of rational functions of degree greater than or equal to two defined on the Riemann sphere $\overline{\mathbb{C}}$ with the semigroup operation being functional composition. When a semigroup G is generated by the functions $\{f_1, f_2, \dots, f_n, \dots\}$, we write this as

$$G = \langle f_1, f_2, \dots, f_n, \dots \rangle. \tag{1.1}$$

In [15], p. 360 the definitions of the set of normality, often called the Fatou set, and the Julia set of a rational semigroup are as follows:

Definition 1.1. For a rational semigroup G we define the set of normality of G, $N(G)$, by

$$N(G) = \{z \in \overline{\mathbb{C}} : \exists \text{ a neighborhood of } z \text{ on which } G \text{ is a normal family}\}$$

and define the Julia set of G, $J(G)$, by

$$J(G) = \overline{\mathbb{C}} \setminus N(G).$$

Clearly from these definitions we see that $N(G)$ is an open set and therefore its complement $J(G)$ is a compact set. These definitions generalize the case of iteration of a single rational function and we write $N(\langle h \rangle) = N(h) = N_h$ and $J(\langle h \rangle) = J(h) = J_h$.

Note that $J(G) \supset J(g)$ and $N(G) \subset N(g)$ for all $g \in G$.

Definition 1.2. If h is a map of a set Y into itself, a subset X of Y is:

i) *forward invariant* under h if $h(X) \subset X$;

ii) *backward invariant* under h if $h^{-1}(X) \subset X$;

iii) *completely invariant* under h if $h(X) \subset X$ and $h^{-1}(X) \subset X$.

It is well known that the set of normality of h and the Julia set of h are completely invariant under h (see [5], p. 54), in fact,

$$h(N(h)) = N(h) = h^{-1}(N(h)) \qquad \text{and} \qquad h(J(h)) = J(h) = h^{-1}(J(h)). \tag{1.2}$$

Theorem 1.1 (Montel's Theorem)**.** *The family of all analytic maps f from a domain Ω to $\overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ is normal in Ω.*

By using Montel's Theorem one can obtain the following result.

Property 1.1. *The set $J(h)$ is the smallest closed completely invariant (under h) set which contains three or more points (see* [5]*, p. 67).*

In fact, this may be chosen as an alternate definition to the definition of $J(h)$ given in Definition 1.1.

Proposition 1.1 ([15], p. 360)**.** *The set $N(G)$ is forward invariant under each element of G and $J(G)$ is backward invariant under each element of G.*

Proof. We make use of the fact that a family of continuous functions defined on a domain of the Riemann sphere is equicontinuous if, and only if, it is a normal family. Let $g \in G$ and $z \in N(G)$. For $\epsilon > 0$ there exists a neighborhood $\triangle$ of z such that $diam f(\triangle) < \epsilon$ for all $f \in G$. Hence $diam\, h(g(\triangle)) = diam(h \circ g)(\triangle) < \epsilon$ for all $h \in G$. Hence G is equicontinuous on $g(\triangle)$, and we conclude that $g(N(G)) \subset N(G)$. $\square$

Remark 1.1. Since $J(G)$ is backward invariant we can characterize $J(G)$ as the smallest closed set that contains three or more points which is backward invariant under each element of G. This follows since the complement of such a set is forward invariant under each element of G and therefore in the set of normality of G by Montel's theorem.

Proposition 1.2 ([34], Lemma 1.1.4)**.** *If* $G = \langle g_1, \dots, g_N \rangle$, *then* $J(G) = \cup_{i=1}^N g_i^{-1}(J(G))$ *and* $N(G) = \cap_{i=1}^N g_i^{-1}(N(G))$.

Proof. By Proposition 1.1 we have

$$N(G) \subset \cap_{i=1}^N g_i^{-1}(N(G)).$$

Take any $z_0 \in \cap_{i=1}^N g_i^{-1}(N(G))$ and set $w_j = g_j(z_0) \in N(G)$. For any $\epsilon > 0$ there is a neighborhood $\triangle_j$ of w_j for each $j = 1, \dots, N$ such that if $f \in G$, then $diam f(\triangle_j) < \epsilon$ for each $j = 1, \dots, N$. Consider the neighborhood $\triangle = \cap_{j=1}^N g_j^{-1}(\triangle_j)$ of z_0 and note that for any $f \in G$ we have $diam(f \circ g_j)(\triangle) = diam f(g_j(\triangle)) \leq diam f(\triangle_j) < \epsilon$ for each $j = 1, \dots, N$. Hence $G \circ g_j = \{h \circ g_j : h \in G\}$ is equicontinuous at z_0. Since $G = (\cup_{i=1}^N G \circ g_j) \cup (\cup_{i=1}^N g_j)$ we see that G is equicontinuous at z_0.

The corresponding statement for $J(G)$ readily follows. □

The sets $N(G)$ and $J(G)$ are, however, not necessarily completely invariant under the elements of G. This is in contrast to the case of single function dynamics as noted in (1.2).

Example 1.1. Let $a \in \mathbb{C}, |a| > 1$ and $G = \langle z^2, z^2/a \rangle$. One can easily show that $J(G) = \{z : 1 \leq |z| \leq |a|\}$ (see [15], p. 360). Note that $J(z^2) = \{z : |z| = 1\}$ and $J(z^2/a) = \{z : |z| = |a|\}$. Clearly in this example $J(G)$ is not completely invariant.

We will study completely invariant Julia sets for rational semigroups in Section 4.

Note also that $J(G)$ has nonempty interior and yet $J(G) \neq \overline{\mathbb{C}}$. This is not possible for the Julia set of a single rational function.

1.1. **The expanding property of the Julia set.**

Let G be a rational semigroup. A point $z \in \overline{\mathbb{C}}$ is called *exceptional* if its backward orbit $O^-(z) = \{w : \exists g \in G \text{ such that } g(w) = z\}$ is finite. The set of exceptional points is denoted by $E(G)$. When $G = \langle f \rangle$, we denote the set of exceptional points by $E(f)$.

For any rational function f of degree at least 2, it is well known that $|E(f)| \leq 2$ where $|E(f)|$ denotes the cardinality of the set $E(f)$ (see [5], Theorem 4.1.2). If $|E(f)| = 1$, then f is conjugate to a polynomial. If $|E(f)| = 2$, then f is conjugate to a map of the form $z \mapsto z^d$ where $d \in \mathbb{Z}$ and $|d| \geq 2$. For general semigroups of rational functions, we have the following proposition.

Proposition 1.3 ([15], Lemma 3.4)**.** *Let* G *be a rational semigroup. Then* $|E(G)| \leq 2$. *If* $|E(G)| = 1$, *then* G *is conjugate to a polynomial semigroup. If* $|E(G)| = 2$, *then* G *is conjugate to a semigroup whose elements are all of the form* az^n *for* $a \in \mathbb{C}$ *and* $n \in \mathbb{Z}$.

Remark 1.2. If G is a finitely generated semigroup, then $E(G) \subset N(G)$. This need not be the case if G is not finitely generated. We leave it to the reader to provide an example of such a semigroup.

Proposition 1.4 ([15], Lemma 3.2). *Let G be rational semigroup and let $a \in \overline{\mathbb{C}} \setminus E(G)$. Then $J(G)$ is a subset of the accumulations points of $O^-(a)$.*

Let G be a rational semigroup and select an element $g \in G$. Note that $J(g) \subset J(G)$. We will now show how $J(G)$ can be "built up" from $J(g)$.

For a collection of sets $\mathcal{A}$, and a function h, we denote new collections of sets by $h(\mathcal{A}) = \{h(A) : A \in \mathcal{A}\}$ and $h^{-1}(\mathcal{A}) = \{h^{-1}(A) : A \in \mathcal{A}\}$.

Consider the countable collection of sets

$$\mathcal{F}_0 = \{J(g)\},$$

$$\mathcal{F}_{n+1} = \bigcup_{f \in G} f^{-1}(\mathcal{F}_n),$$

$$\text{and } \mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{F}_n.$$

Since $J(G)$ is backward invariant under each $f \in G$, closed, and contains $J(g)$, we have $J(G) \supset \overline{\bigcup_{A \in \mathcal{F}} A}$.

Lemma 1.1 ([32], Lemma 3). *We have $J(G) = \overline{\bigcup_{A \in \mathcal{F}} A}$.*

Proof. Since the set on the right is closed, backward invariant under each f in G (since rational functions are continuous open maps) and clearly contains more that three points, it must contain $J(G)$ as the complement is then in the set of normality of G. □

Remark 1.3. In fact, if we had let $\mathcal{F}_0 = \{\{a, b, c\}\}$ where a, b, c are three points known to be in $J(G)$ (for example, if $a, b, c \in J(g)$) and we defined each $\mathcal{F}_n$ and $\mathcal{F}$ as above in terms of this new collection $\mathcal{F}_0$, then we would arrive at the same description of $J(G)$ as given in Lemma 1.1. This is due to the minimality condition for Julia sets as noted in Property 1.1.

Corollary 1.1 ([15], Lemma 3.1). *The set $J(G)$ is perfect.*

Proof. Since $J(g)$ is perfect (see [5], p. 68) and backward and forward images of perfect sets under rational maps are perfect, we see that each set in $\mathcal{E}$ is perfect by a routine inductive argument. The corollary then follows since the closure of a union of perfect sets is perfect. □

The above proof due to Rich Stankewitz (see [32]) is given as an alternative to the original proof found in [15].

Theorem 1.2 ([15], Theorem 3.1 and Corollary 3.1)**.** *If G is a rational semigroup, then the repelling fixed points of the elements of G are dense in $J(G)$. Hence also*

$$J(G) = \overline{\bigcup_{g \in G} J(g)}.$$

Proof. The proof will follow along the lines of that of Baker in [2]. As the repelling fixed points of any element $g \in G$ are in $J(g)$ and each $J(g) \subset J(G)$, we have that the repelling fixed points of the elements of G are in $J(G)$. We will now show that such points are dense in $J(G)$. Pick $z_0 \in J(G)$ and let U be a neighborhood of z_0. We will show that U contains a repelling point of some element of G.

Since $J(G)$ is perfect, we may find disks $B_j = \{z : |z-a_j| < \epsilon\} \subset U \setminus (E(G) \cup \{z_0\})$ with disjoint closures, centered at finite points $a_j \in J(G)$, for $1 \leq j \leq 5$. We denote the spherical derivative of a meromorphic function by $f^{\#}$; thus $f^{\#}(z) = |f'(z)|/(1 + |f(z)|^2)$, with the usual modification if $z = \infty$ or $f(z) = \infty$. Let C be the positive constant associated with the set $\{B_j : 1 \leq j \leq 5\}$ by the Ahlfors Five Island Theorem (see Theorem 7.4 below). Thus C is chosen so that if f is any meromorphic function defined on the unit disk with $f^{\#}(0) > C$, then the unit disk contains a simply connected subdomain that is mapped conformally by f onto some B_j.

If $1 \leq j \leq 5$, then G is not normal in any neighborhood of a_j. Thus by Marty's criterion (see [28], p. 75), there is some $f_j \in G$ and a point $b_j \in D_j = \{z : |z-a_j| < \epsilon/3\}$ such that $f^{\#}(b_j) > 3C/\epsilon$. Write $E_j = \{z : |z-b_j| < \epsilon/3\} \subset B_j$. Then $g_j(z) = f_j(b_j+z)$ is meromorphic in the disc centered at the origin of radius $\epsilon/3$ with $g_j^{\#}(0) = f_j^{\#}(b_j) > 3C/\epsilon$. Hence we deduce that g_j maps some simply connected subdomain of the disc centered at the origin of radius $\epsilon/3$ conformally onto some B_i, where $1 \leq i \leq 5$. Thus the corresponding f_j maps some simply connected subdomain of E_j, and consequently some simply connected relatively compact subdomain of B_j, conformally onto some B_i. Repeating this argument at most five times, we find some k with $1 \leq k \leq 5$, and an element $g \in G$ arising as a composition of the f_j, such that g maps some simply connected relatively compact subdomain of B_k conformally onto B_k. It now follows that some branch of g^{-1} has an attracting fixed point, and hence g has a repelling fixed point in $B_k \subset U$. $\square$

The iterates of a single rational function f expand open sets which meet $J(f)$ as explained in the following proposition.

Proposition 1.5 ([5], Theorem 6.9.4). *Let f be a rational function with $\deg f \geq 2$, let W be a non-empty open set intersecting $J(f)$, and let K be a compact subset of $\overline{\mathbb{C}} \setminus E(f)$. Then there exists an integer N such that $K \subset f^n(W)$ for all $n \geq N$.*

Correspondingly, there is an expanding property for finitely generated semigroups of rational functions. For a rational semigroup $G = \langle f_1, \ldots, f_k \rangle$ we define the *length* of a word $g = f_{i_n} \circ f_{i_{n-1}} \circ \cdots \circ f_{i_1}$ with $i_j \in \{1, \ldots, k\}$ to be $l(g) = n$. We note that it is possible for an element of G to be represented by multiple words.

Proposition 1.6 ([7], Lemma 1). *Let $G = \langle f_1, \ldots, f_k \rangle$ be a finitely generated rational semigroup with $\deg f_j \geq 2$ for $j = 1, \ldots, k$, let W be a non-empty open set intersecting $J(G)$, and let K be a compact subset of $\overline{\mathbb{C}} \setminus E(G)$. Then there exists a positive integer N such that for all $n \geq N$,*

$$K \subset \bigcup_{l(g)=n} g(W)$$

where g ranges over the words of G of length n.

We leave the proof of Proposition 1.6 as an exercise.

2. Uniformly Perfect Sets

In this section we show that $J(G)$ is uniformly perfect when G is finitely generated. Uniformly perfect sets were introduced by A. F. Beardon and Ch. Pommerenke in 1979 in [3]. We begin with some preliminary definitions.

Definition 2.1. A conformal annulus is an open subset $\mathcal{A}$ of $\overline{\mathbb{C}}$ that can be conformally mapped onto the genuine annulus $Ann(0; r_1, r_2) = \{z : 0 \leq r_1 < |z| < r_2 \leq \infty\}$ and the modulus of such a conformal annulus is given by

$$mod(\mathcal{A}) = \frac{1}{2\pi} \log \frac{r_2}{r_1}.$$

We note that $mod(\mathcal{A})$ is a conformal invariant.

Definition 2.2. A conformal annulus $\mathcal{A}$ is said to separate a set F if F intersects both components of $\overline{\mathbb{C}} \setminus \mathcal{A}$ and $F \cap \mathcal{A} = \emptyset$.

Definition 2.3. A closed curve γ is said to separate a set F if F intersects more than one component of $\overline{\mathbb{C}} \setminus \gamma$ and $F \cap \gamma = \emptyset$.

Definition 2.4. A closed curve γ is said to separate the nonempty sets A and B if there does not exist a component of $\overline{\mathbb{C}} \setminus \gamma$ that intersects both A and B and if γ is disjoint from both A and B.

Definition 2.5. ([27], p. 192) We say that a compact subset $F \subset \overline{\mathbb{C}}$ is uniformly perfect if F has at least two points and if the moduli of conformal annuli in $\overline{\mathbb{C}} \setminus F$ which separate F are bounded.

Remark 2.1. Uniformly perfect sets are necessarily perfect (see [27], p. 192).

Remark 2.2. By a scaling and normal families argument one can show that conformal annuli of large modulus contain genuine annuli of large modulus. Thus the compact set E is uniformly perfect if, and only if, there is a $c > 0$ such that for any finite $z_0 \in E$ and $r > 0$ (and $r < r_0$ when $\infty \notin E$), the Euclidean annulus $\{z : cr < |z - z_0| < r\}$ meets E.

Remark 2.3. For a hyperbolic domain $U \subset \mathbb{C}$ it is known (from estimates when $U = \mathbb{C} \setminus \{0, 1\}$) that the hyperbolic density $\lambda_U(z) \to +\infty$ as z tends to any finite point on the boundary of U.

Lemma 2.1. *Thus the boundary of a domain D is uniformly perfect if, and only if, there is a positive constant δ such that every Jordan curve in D separating ∂D has hyperbolic length at least δ, with respect to the hyperbolic metric in D.*

Proof. We observe in the annulus of radii 1 and $R > 1$, the circle of radius $\sqrt{R}$ has hyperbolic length $2\pi^2/\log R$. This can be calculated using the density for the annulus in [23], p. 12. Hence, if ∂D is not uniformly perfect then there exist separating annuli A_n of modulus $\frac{1}{2\pi} \log R_n \to \infty$. The circle centered at the center of A_n of radius $\sqrt{R_n}$ therefore has hyperbolic length less than or equal to $2\pi^2/\log R_n \to 0$. (Note that the hyperbolic density in D is less than the hyperbolic density in A_n.)

Suppose that ∂D is uniformly perfect. We may also assume that $\infty \in \partial D$ since the property of being uniformly perfect is invariant under Möbius maps (see [27], p. 192). Since ∂D is uniformly perfect there exists a $c > 0$ such that

$$\lambda(z) > \frac{c}{\delta(z)}$$

where $\lambda(z)$ denotes the hyperbolic density on D and $\delta(z)$ denotes the (Euclidean) distance from z to ∂D (see [3], p. 476). Let γ be a curve in D that separates ∂D. Let $z \in \partial D$ be a finite point that lies in a bounded component of $\overline{\mathbb{C}} \setminus \gamma$. Letting R denote the maximum distance from z to a point on γ, we see that since γ "winds around" z we must have

$$\int_\gamma |dz| \geq 2R.$$

Hence the hyperbolic length of γ satisfies

$$l(\gamma) = \int_\gamma \lambda(z)|dz| \geq \frac{c}{R}\int_\gamma |dz| \geq 2c.$$

$\square$

Claim 2.1. *Let U be a domain in $\overline{\mathbb{C}}$ such that $\#(\overline{\mathbb{C}} \setminus U) \geq 3$ and let γ_n be (smooth) curves in U. Then if the hyperbolic length of γ_n tends to 0, the spherical lengths of γ_n also tend to 0.*

2.1. **Logarithmic capacity.** In this section we state Pommerenke's criterion in terms of logarithmic capacity for a set to be uniformly perfect. We state a few facts about logarithmic capacity, but for a more thorough treatment see [1].

Definition 2.6. For a measure ν on a compact set F we define the logarithmic potential of ν by

$$p_\nu(z) = -\int_F \log|z-\zeta|\ d\nu(\zeta).$$

Definition 2.7. For a measure ν on a compact set F we define

$$S_\nu = \sup_{z\in F} p_\nu(z).$$

We note that S_ν may be infinite, as is the case when $\nu = \delta_{z_0}$ for some $z_0 \in F$.

¿From all the measures ν with total measure $\nu(F) = 1$, there is one that minimizes S_ν (see [1], p. 25). This measure is called the equilibrium measure.

Definition 2.8. If we call

$$S = \min\{S_\nu : \nu \text{ is a measure of total measure } \nu(F) = 1\},$$

then we define the capacity of F by

$$cap(F) = e^{-S}.$$

If $S = \infty$, i.e., no ν can be chosen such that S_ν is finite, then we say that F is a set of zero capacity.

We note that one can show that $cap(\{w : |w-z| \leq r\}) = r$.

Pommerenke [27] has shown that a set E is uniformly perfect if, and only if, there exists a constant $\delta > 0$ such that

$$cap(E \cap \{w : |w-z| \leq r\}) \geq \delta r \tag{2.1}$$

for all $z \in E$ whenever $0 < r < diam(E)$. We note that if 2.1 holds for all $r < r_0$ then 2.1 holds for $0 < r < diam(E)$ if δ is replaced by $\frac{\delta r_0}{diam(E)}$. This immediately implies the following lemma.

Remark 2.4. Note that this implies that if E is uniformly perfect then each component of $\overline{\mathbb{C}} \setminus E$ is regular for the Dirichlet problem.

Lemma 2.2. *The union of finitely many uniformly perfect sets is uniformly perfect.*

Lemma 2.3. *If A is a uniformly perfect set and B is a compact set which does not contain A, then $A \setminus B$ contains a uniformly perfect subset X.*

Proof. Let z be a finite point in $A \setminus B$ and let $\epsilon > 0$ be chosen such that $\triangle(z,\epsilon) \cap B = \emptyset$. Observe that $\overline{A \cap \triangle(z,\epsilon)}$ is a perfect and closed set. If $\overline{A \cap \triangle(z,\epsilon)}$ is not totally disconnected, then we may select for X any component of $\overline{A \cap \triangle(z,\epsilon)}$ which is not a single point, for it will then be a compact and connected set with more than one point and hence uniformly perfect. If $\overline{A \cap \triangle(z,\epsilon)}$ is totally disconnected, then one may find a simple closed curve in $\triangle(z,\epsilon) \setminus A$ which separates A (see Lemma 2.4). Letting D denote the component of $\overline{\mathbb{C}} \setminus A$ which does not intersect $\overline{\mathbb{C}} \setminus \triangle(z,\epsilon)$ (i.e., the inside component) we let $X = D \cap A$ and note that it is uniformly perfect as can be seen by using Pommerenke's criterion above and the fact that A is uniformly perfect. To see this we use the fact that X is then both open in A and compact and therefore there exists a $r_0 > 0$ such that every point of X is at a distance at least r_0 from $A \setminus X$. $\square$

Lemma 2.4. *If $\overline{A \cap \triangle(z,\epsilon)}$ is (nonempty) totally disconnected, perfect and closed, then there exists a simple closed curve γ in $\triangle(z,\epsilon) \setminus A$ which separates A.*

Proof. We see in [18], p. 100 that $\overline{A \cap \triangle(z,\epsilon)}$ is homeomorphic to the middle third Cantor set C. Let $f : \overline{A \cap \triangle(z,\epsilon)} \to C$ be a homeomorphism. Consider the open set $f(\triangle(z_0,\delta)) \cap A)$ in C where $z_0 \in A$ and δ is small enough so that $\triangle(z_0,\delta)$ is a subset of $\triangle(z,\epsilon)$ and so that A is not contained in $\triangle(z_0,\delta)$. Since C contains infinitely many small copies of itself, we may find such a copy C' in $f(\triangle(z_0,\delta) \cap A)$. Note that C' is open in C. Now $f^{-1}(C')$ is open in $\overline{A \cap \triangle(z,\epsilon)}$ and as such equals the intersection of $\overline{A \cap \triangle(z,\epsilon)}$ with an open subset U of $\triangle(z_0,\delta)$. Since C' is closed in C, $f^{-1}(C')$ is closed in $\overline{A \cap \triangle(z,\epsilon)}$ and hence no points of $f^{-1}(C')$ can approach the boundary of U. Say that all the points of $f^{-1}(C')$ are always a distance ρ from the boundary of U. Using a grid of squares of size $\rho/4$ we can construct a simple polygonal path $\gamma \subset U \setminus A$ that separates A. $\square$

Note that the set $f^{-1}(C')$ in the proof of Lemma 2.4 can be seen to be uniformly perfect when A is uniformly perfect once it was known that $f^{-1}(C')$ is both compact and open in $A \cap \triangle(z, \epsilon)$ by Pommerenke's criterion (without having to find a curve γ).

2.2. **Julia sets of finitely generated rational semigroups are uniformly perfect.** It is known that the Julia set of a rational function is uniformly perfect. Several proofs of this fact have been given, namely by Eremenko [11], Hinkkanen [13], and Mañé and da Rocha [22].

We first point out the following fact.

Claim 2.2. *Let γ be a simple closed curve in $\overline{\mathbb{C}}$ and let f be a rational function. Let D be a component of $\overline{\mathbb{C}} \setminus \gamma$ and C a component of $\overline{\mathbb{C}} \setminus f(\gamma)$. Then if $f(D) \cap C \neq \emptyset$, then $C \subset f(D)$.*

Proof. If C is a proper subset of $f(D)$ then there would exist a point $w \in \partial f(D) \cap C$. This implies that there exists a point $z \in \overline{D}$ with $f(z) = w$. The point z cannot be in D else $w = f(z) \in f(D)$. Hence $z \in \partial D \subset \gamma$ and $w = f(z) \in f(\gamma)$. □

Theorem 2.1 ([16], Theorem 3.1)**.** *Let $G = \langle g_1, g_2, \ldots, g_N \rangle$ be a finitely generated rational semigroup. Then the Julia set $J(G)$ is uniformly perfect.*

Proof. Let $J(G)$ denote the Julia set of G and J_i the Julia set of the generator g_i. Since connected closed sets containing at least two points are uniformly perfect, we shall assume that $J(G)$ is not connected and not uniformly perfect. In particular, then $N(G) \neq \emptyset$. Since the union of finitely many uniformly perfect sets is uniformly perfect, we may assume that $J(G) \neq \cup_{i=1}^N J_i$. By Remark 1.1, there is $h \in G$ such that $h^{-1}(\cup_{i=1}^N J_i) \not\subseteq \cup_{i=1}^N J_i$. Now $h^{-1}(\cup_{i=1}^N J_i)$ is uniformly perfect since each J_i is uniformly perfect and h is rational (to see this in detail one can argue as in the proof of Lemma 2 in [13]). By Lemma 2.3 we choose X to be a uniformly perfect compact subset of

$$h^{-1}(J_j) \setminus (\bigcup_{i=1}^N J_i).$$

Note that $J(G)$ has positive logarithmic capacity. Thus $N(G)$ has a hyperbolic metric (which is defined in each component of $N(G)$ separately). Since $J(G)$ is not uniformly perfect, there is a sequence of simple closed curves $\gamma_k \subset N(G)$ such that each γ_k separates $J(G)$ and such that the hyperbolic length

$$l(\gamma_k) \to 0$$

as $k \to \infty$. Since X is uniformly perfect, we may assume that no γ_k separates X. Thus for each k there exists D_k a component of $\overline{\mathbb{C}} \setminus \gamma_k$ such that $X \cap D_k = \emptyset$ and $J(G) \cap D_k \neq \emptyset$.

For each $k = 1, 2, \ldots$ choose $h_k \in G$ to be an element of shortest word length such that

$$X \subset h_k(D_k).$$

The existence of these maps follows from the density of the repelling fixed points of the elements of G in $J(G)$ and the use of Claim 2.2. (Of course, there may be no uniqueness in the choice for h_k even if the word length is minimal.) Now each h_k can be written in the form

$$h_k = g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_m},$$

where $m = m(k)$ is as small as possible and each $i_\nu \in \{1, 2, \ldots N\}$ (and each integer i_ν depends on k). Passing to a subsequence and, if necessary, relabeling the generators, we may assume that $g_{i_1} = g_1$ for all k. Let us define $f_k = g_{i_2} \circ g_{i_3} \circ \cdots \circ g_{i_m}$.

We first claim that there are only finitely many k for which $h_k(\gamma_k)$ separates X from J_1. To see this, simply note that h_k is an analytic map from $N(G)$ into $N(G)$ and therefore a contraction in the hyperbolic metric. Thus the length of $h_k(\gamma_k)$ is less than the length of γ_k and this is going to zero. But any curve separating X from J_1 has a length which is bounded below by a fixed constant since both these sets have positive diameter (see Claim 2.1). Similarly, there are only finitely many k for which $f_k(\gamma_k)$ separates X from at least one J_i. Thus, after passing to a subsequence, we may assume that neither $h_k(\gamma_k)$ nor $f_k(\gamma_k)$ ever separates X from J_1. By the minimality in the word length of h_k, the set $f_k(D_k)$ does not contain X while $g_1(f_k(D_k))$ does. Now $f_k(\gamma_k)$ separates X for only finitely many k, because any loop that separates X has hyperbolic length (in the hyperbolic metric of $\overline{\mathbb{C}} \setminus X$) bounded below by a fixed positive constant as X is uniformly perfect (and since the hyperbolic metric of $\overline{\mathbb{C}} \setminus X$ is smaller than that of $N(G)$). Thus, after again passing to a subsequence, we may assume that $f_k(\gamma_k)$ never separates X. Similarly, we may assume that $f_k(\gamma_k)$ never separates J_1. Write $\beta_k = f_k(\gamma_k)$. We have arrived at the situation where β_k does not separate X or J_1, nor does β_k separate X from J_1. Thus X and J_1 lie in the same component of $\overline{\mathbb{C}} \setminus \beta_k$. This component does not meet $f_k(D_k)$ and in particular $f_k(D_k)$ does not meet J_1. Now $h_k(D_k) = g_1(f_k(D_k))$ covers X and therefore must meet J_1 as both X and J_1 meet the same component of $\overline{\mathbb{C}} \setminus h_k(\gamma)$. This is a contradiction, as any $z \in f_k(D_k)$ which maps by g_1 to a point in J_1 must itself be in J_1 since $g_1^{-1}(J_1) = J_1$. But this contradicts the fact that $f_k(D_k)$ foes not meet J_1. $\square$

Theorem 2.2 ([16], Theorem 4.1)**.** *Let G be a rational semigroup such that $J(G)$ is uniformly perfect. Suppose that z_0 is a superattracting fixed point of an element $h \in G$. Let A be the union of all the components of $N(h)$ in which the iterates of h tend to z_0. Then either $z_0 \in N(G)$ or $A \subset J(G)$. In particular, either $z_0 \in N(G)$ or z_0 lies in the interior of $J(G)$.*

Proof. Note first that by the forward invariance of $N(G)$ under G, if $N(G) \cap A \neq \emptyset$, then $N(G)$ contains points as close to z_0 as we like. We may assume that $z_0 \in J(G)$ and that $N(G) \cap A \neq \emptyset$, for otherwise there is nothing to prove. Close to z_0 we may conformally conjugate h to $z \mapsto z^d$ for some $d \geq 2$. Let us use the coordinates in which h is equal to $z \mapsto z^d$. In these coordinates, let V be a small disk close to $z_0 = 0$ contained in $N(G)$. Also $h^n(V) \subset N(G)$ for all $n \geq 1$, and since the application of h (that is $z \mapsto z^d$) multiplies the argument of a point in V by d, we see that for all sufficiently large n, the set $h^n(V)$ contains an annulus centered at $z_0 = 0$. Suppose that V_n is the component of $N(G)$ containing $h^n(V)$. Now if, for a certain n, the set V_n contains the annulus $\{z : r_1 < |z| < r_2\}$, then for any $k \geq 1$, the set V_{n+k} contains the annulus $\{z : r_1^{d^k} < |z| < r_2^{d^k}\}$. Since $\{z_0\} \cup J(h) \subset J(G)$. it follows that V_{n+k} separates $J(G)$. The moduli of these annuli are equal to $\log(r_2^{d^k}/r_1^{d^k}) = d^k \log(r_2/r_1) \to \infty$ as $k \to \infty$. This contradicts the assumption that $J(G)$ is uniformly perfect. $\square$

Corollary 2.1 ([16], Corollary 4.1)**.** *If G is a finitely generated rational semigroup and z_0 is a superattracting fixed point of some element of G, then either z_0 lies in (the interior of) the Fatou set of G or in the interior of the Julia set of G.*

Theorem 2.3 ([16], Theorem 5.1)**.** *There exists an infinitely generated rational semigroup G (all of whose elements have degree at least two) with the property that for any positive integer N, the semigroup G contains only finitely many elements of degree at most N, such that $J(G)$ is not uniformly perfect, and such that G contains an element g with a superattracting fixed point α with $\alpha \in \partial J(G) \subset J(G)$.*

3. Nearly Abelian Semigroups

A natural question regarding rational semigroups is how the algebraic structure of the semigroup affects its dynamics. If the algebraic structure is in some way simple, this may provide information about the dynamics.

In this section we discuss the concept of nearly abelian semigroups as introduced in [15].

As a motivating example we consider the following lemma which is due to Julia.

Lemma 3.1. *Let f and g be rational functions of degree at least two that commute, i.e., $f \circ g = g \circ f$. Then $J(f) = J(g)$.*

Proof. Since g is uniformly continuous on $\overline{\mathbb{C}}$ in the spherical metric, the family $\{g \circ f^n : n \geq 1\}$ is normal on $N(f)$. This is the same family as $\{f^n \circ g : n \geq 1\}$ and so $\{f^n : n \geq 1\}$ is normal on the open set $g(N(f))$. Thus $g(N(f)) \subset N(f)$. So all the g^n omit $J(f)$ on $N(f)$. As the degree of f is greater than two, $J(f)$ contains at least three points and so it follows that the family $\{g^n : n \geq 1\}$ is normal on $N(f)$. Hence $N(f) \subset N(g)$. By symmetry we obtain $N(g) \subset N(f)$. This gives $N(f) = N(g)$ and hence $J(f) = J(g)$ as desired. □

In particular, if the rational semigroup G is abelian, then $J(g) = J(G)$ for every $g \in G$ by Theorem 1.2. However, we are able to obtain a similar result for a more general class of rational semigroups.

Definition 3.1. A rational semigroup G is **nearly abelian** if there is a compact family of Möbius transformations $\Phi = \{\phi\}$ with the following properties:

(i) $\phi(N(G)) = N(G)$ for all $\phi \in \Phi$, and

(ii) for all $f, g \in$ there is a $\phi \in \Phi$ such that $f \circ g = \phi \circ g \circ f$.

Note that when G is nearly abelian, the family $\Phi(G)$ of Möbius transformations ϕ for which $f \circ g = \phi \circ g \circ f$ for some $f, g \in G$ by assumption is precompact, i.e., any sequence of elements of $\Phi(G)$ contains a subsequence that converges to a Möbius transformation uniformly on $\overline{\mathbb{C}}$. Hence we may take Φ to be the closure of $\Phi(G)$. We make a couple of observations that apply when $\Phi(G)$ is precompact, although they will not be used in what follows. First, if ϕ_n is any sequence from $\Phi(G)$ and D is any disk, it cannot be the case that ϕ_n converges to a constant function on D. Further, if all the ϕ_n have their poles outside a fixed disk larger than D, this implies a uniform upper and lower bound for $|\phi_n'|$ in D.

Here is our first result about nearly abelian semigroups.

Theorem 3.1 ([15], Theorem 4.1)**.** *Let G be a nearly abelian semigroup. Then for each $g \in G$ we have $J(g) = J(G)$.*

Proof. Let f be a fixed element of G and consider an arbitrary element g of G. Set $J = J(f)$ and $N = \overline{\mathbb{C}} \setminus J$. We will show that $J(g) = J$. Assume for a while that this is true. Recall that $J = J(f) \subset J(G)$. On the other hand, for each $g \in G$, g omits J on N so that G is normal on N. Thus $N \subset N(G)$. It now follows that $J(G) = J$, as claimed. (Or one could reach the same conclusion using the fact that $J(G) = \overline{\bigcup_{g\in G} J(g)}$. See Theorem 1.2.)

We proceed to prove that if $g \in G$, then $J(g) = J$. For each $n \geq 1$, there is $\phi_n \in \Phi(G)$ with $f^n \circ g = \phi_n \circ g \circ f^n$. We begin by showing that $g(N(f)) \subset N(f)$. Choose some point $x \in N(f)$ and a neighborhood U of x such that $\overline{U} \subset N(f)$. Then $g(U)$ is a neighborhood of $g(x)$. Consider a sequence of iterates f^{n_j} on $g(U)$. As $\overline{U} \subset N(f)$ we may pass to a subsequence, say f^{m_j}, in such a manner that $f^{m_j} \to \Psi$ uniformly on U and where Ψ is meromorphic on U. Since g is rational, and hence uniformly continuous on $\overline{\mathbb{C}}$, we have $g \circ f^{m_j} \to g \circ \Psi = \psi$ uniformly on U. Passing to a further subsequence without changing notation, we may assume that $\phi_{m_j} \to \phi$ uniformly on the sphere, where ϕ is a Möbius transformation. Now $f^{m_j} \circ g = \phi_{m_j} \circ g \circ f^{m_j} \to \phi \circ \psi = \chi$ uniformly on U. Hence the family $\{f^n \circ g : n \geq 1\}$ is normal on U and so $\{f^n : n \geq 1\}$ is normal on $g(U)$. Since $N(f)$ is the maximal open set on which $\{f^n : n \geq 1\}$ is normal, we have $g(U) \subset N(f)$. Thus $g(x) \in N(f)$ and so $g(N(f)) \subset N(f)$. Hence every g^n omits $J(f)$ on $N(f)$, and so the iterates of g form a normal family on $N(f)$. This implies that $N(f) \subset N(g)$. By symmetry, we obtain $N(g) \subset N(f)$. Hence $N(g) = N(f)$ and so $J(g) = J(f) = J$. □

We note that (since every element of G has degree at least two) the condition $\phi(N(G)) = N(G)$ for all $\phi \in \Phi$ may be replaced by the condition $J(f) = J(g)$ for all $f, g \in G$. For if this latter property holds there clearly is a set J (of cardinality at least 3) such that $J = J(f)$ for all $f \in G$. Thus each $f \in G$ omits J in $\overline{\mathbb{C}} \setminus J$ and so $\overline{\mathbb{C}} \setminus J \subset N(G)$. Since $J = J(f) \subset J(G)$ we have $J = J(g) = J(G)$ for all $g \in G$. Then applying both sides of the equation $f \circ g = \phi \circ g \circ f$ to J we see that $\phi(J) = J$ and hence $\phi(N(G)) = N(G)$. As $f, g \in G$ are arbitrary, the result holds for all such ϕ.

We remark that in many cases the assumed compactness of Φ may be redundant. It is conjectured that the Möbius symmetry group of the Julia set of a rational function is of finite order, unless the Julia set is $\overline{\mathbb{C}}$ or is Möbius equivalent to a circle or a line segment. There are some partial results towards this conjecture in [8].

A natural question is to what extent does the converse to Theorem 3.1 hold?

A. Beardon has proved the following result.

Theorem 3.2 ([4], Theorem 1)**.** *If f and g are polynomials and if $J(f) = J(g)$, then there is a linear mapping $\phi(z) = az + b$ such that $f \circ g = \phi \circ g \circ f$ and $|a| = 1$.*

Corollary 3.1 ([15], Corollary 4.1)**.** *Let $\mathcal{F}$ be a family of polynomials of degree at least 2, and suppose that there is a set J such that $J(g) = J$ for all $g \in \mathcal{F}$. Then $G = \langle \mathcal{F} \rangle$ is a nearly abelian semigroup.*

Proof. As each $g \in \mathcal{F}$ is a polynomial, J is compact in $\mathbb{C}$. We note that $J(f) = J(g)$ for all $f, g \in G$ since the same is true for any pair of functions in $\mathcal{F}$. One way to see this is as follows. It suffices to show that for any $f, g \in \mathcal{F}$, we have $J(f \circ g) = J$. It is easy to see that $J(f \circ g) \subset J$ since J is backwards invariant under f and g. If $J(f \circ g) \neq J$, then by Remark 4.6 below, there are points of J in the basin of attraction of infinity for $f \circ g$. However, J is also forward invariant under $f \circ g$ as it is forward invariant under f and g individually. This is a contradiction and hence $J(f \circ g) = J$ as claimed.

As $J(f) = J(g)$ for all $f, g \in G$, the polynomials f and g nearly commute by Theorem 3.2. We finish by observing that the family of commutators of the form $z \mapsto az + b$, where $|a| = 1$, has compact closure since the numbers b must also be bounded. □

The following conjecture is the strongest converse to Theorem 3.1 that we can reasonably expect.

Conjecture 3.1 ([15], Conjecture 4.1)**.** *Let G be a rational semigroup and suppose that for some $g \in G$ we have $J(g) = J(G)$ and that $J(G)$ is not the image under a Möbius transformation of a circle, line segment or the Riemann sphere. Then G is nearly abelian.*

We remark here that the set of all rational functions that share the same Julia set J where J is Möbius equivalent to the sphere a circle or a line segment will not even nearly commute, i.e., given any two such functions f and g there need not be a Möbius transformation ϕ such that $f \circ g = \phi \circ g \circ f$. Further, if we restrict ourselves to such rational functions that do nearly commute, the set of commutators $\Phi(G)$, being a subset of the symmetries of J may be so large as to not be precompact. It is also relatively easy to construct examples of rational semigroups such that $J(G) = \{z : |z| = 1\}$ such that there is some $g \in G$ with $J(g) = J(G)$ yet theer is another element $h \in G$ such that $J(h) \subsetneq J(G)$.

3.1. **Wandering domains.** One of the major differences that appear when passing from the classical iteration theory to the dynamics of rational semigroups is the existence of wandering domains. Sullivan's result precludes wandering domains in the Fatou set of a rational function. We first need to establish what a wandering domain would mean for a rational semigroup.

Definition 3.2. Let G be a rational semigroup. Given a component U of $N(G)$ and an element $g \in G$, we let U_g denote the component of $N(G)$ containing $g(U)$. The component U is called a **wandering domain** if there are infinitely many distinct components in $\{U_g : g \in G\}$. We remark that $g(U)$ is usually a proper subset of U_g, and further there can be infinitely many distinct elements h_j of G such that the sets $h_j(U)$ are all contained in the same component of $N(G)$, yet the sets $h_j(U)$ might still be mutually disjoint.

Hinkkanen and Martin provide an example of an infinitely generated polynomial semigroup (of finite type) that has a wandering domain. See [15], §5. They further provide an example where the wandering domain returns to the same component infinitely often. They have made the following conjecture.

Conjecture 3.2 ([15], Conjecture 5.1)**.** *Let G be a finitely generated rational semigroup. Then G has no wandering domains.*

Some partial results have been made in this direction. One such result has to do with hyperbolic rational semigroups. A hyperbolic rational semigroup G satisfies the property that $J(G)$ is disjoint from the closed post-critical set of G. This generalization of hyperbolic rational maps was established independently by Hiroki Sumi in [34] and by Hinkkanen and Martin in [14]. Both proved that hyperbolic rational semigroups have no wandering domains. Note that these hyberbolic rational semigroups need not be finitely generated. Sumi has also established a no-wandering-domains theorem for sub- and semi-hyperbolic rational semigroups. Another no-wandering-domains type result is the following theorem.

Theorem 3.3 ([15], Theorem 5.1)**.** *Let G be a nearly abelian rational semigroup. Then G has no wandering domains.*

Proof. Let $f \in G$ be a rational map of degree at least two. Then as G is nearly abelian, we have $J(f) = J(G)$. Let $\Phi(G)$ be the precompact family associated with the commutative properties of G as in Definition 3.1. For a single rational function f, the Fatou set $N(f)$ has a finite number of components that are periodic under f. (See [10], §III, Theorem 2.7 and §VI, Theorem 4.1. The sharp bounds on the number of non-repelling cycles and periodic components was found by Shishikura, see [29].) Further, by Sullivan's no-wandering-domains theorem, every component of $N(f)$ maps under some iterate onto a periodic component of $N(f)$. Thus we may replace f by a suitable iterate of f to assume that if U is a periodic component of f, then U is fixed, that is, $f(U) = U$. Let $\mathcal{U}$ be the collection of all fixed components of f and let $\mathcal{A}$ be the set of all components of $N(G)$ of the form $\phi(U)$, where $\phi \in \Phi(G)$ and $U \in \mathcal{U}$. It is easy to verify from the precompactness of the family $\Phi(G)$, that $\mathcal{A}$ consists of a finite number of components of $N(G)$. (For instance, we may normalize so that $\infty \in N(G)$ and then observe that there are only finitely many components whose area is larger than any given constant. Since $\Phi(G)$ is precompact, there is a uniform bound on the amount by which any element of $\Phi(G)$ can decrease the area of any $U \in \mathcal{U}$.) We now observe that if $g \in G$ and $U \in \mathcal{U}$, then $g(U) \in \mathcal{A}$. To see this, simply observe that for every integer m, we have

$$g(U) = g(f^m(U)) = \phi_m(f^m(g(U))),$$

and if m is taken sufficiently large, then $f^m(g(U)) \in \mathcal{U}$. Next let V be any component of $N(G)$ and suppose that V is wandering. Choose an infinite sequence $g_i \in G$ such that the sets $g_i(V) = V_i$ are disjoint. Choose an integer n such that $f^n(V) = U \in \mathcal{U}$. As f^n has finite degree, the collection $\{f^n(V_i)\}_{i=1}^{\infty}$ must contain an infinite number of components of $N(G)$. However, for each i we see that

$$f^n(V_i) = f^n(g_i(V)) = (\phi_i \circ g_i \circ f^n)(V) = \phi_i(g_i(U)) \in \phi_i(\mathcal{A}),$$

for some $\phi_i \in \Phi(G)$. However, it is again easy to see from the precompactness of the set $\Phi(G)$ that in fact the set $\{\phi(\mathcal{A}) : \phi \in \Phi(G)\}$ is a finite collection of components of $N(G)$, which yields the desired contradiction. $\square$

3.2. **Stable Domains.** Sullivan's no-wandering-domains theorem together with the classification of the periodic components of the Fatou set of a rational function describe the stable dynamics of an arbitrary rational function of degree at least two. In this section we present a partial classification of the dynamics of a rational semigroup on a stable domain. See [15], pp. 362, 374–379.

Definition 3.3. Recall that we write U_g for the component of $N(G)$ containing $g(U)$. We define the **stabilizer** of U to be

$$G_U = \{g \in G : U_g = U\}.$$

If G_U contains an element of degree two or more, we shall say that U is a stable basin for G.

Clearly, G_U is a subsemigroup of G. In particular, $N(G) \subset N(G_U)$, where the containment may be strict.

Definition 3.4. Given a stable basin U for G we say that it is

(i) **attracting** if U is a subdomain of an attracting basin of each $g \in G_U$ with $\deg g \geq 2$;

(ii) **superattracting** if U is a subdomain of a superattracting basin of each $g \in G_U$ with $\deg g \geq 2$;
(iii) **parabolic** if U is a subdomain of a parabolic basin of each $g \in G_U$ with $\deg g \geq 2$;
(iv) **Siegel** if U is a subdomain of a Siegel disk of each $g \in G_U$ with $\deg g \geq 2$;
(v) **Herman** if U is a subdomain of a Herman ring of each $g \in G_U$ with $\deg g \geq 2$.

Remark 3.1. This classification is not exhaustive. See Example 3.4 below. However, we will show in Theorem 3.6 that Definition 3.4 is a complete classification for nearly abelian semigroups.

Before we discuss these definitions, we introduce a concept which in part generalizes to semigroups the relationship between the dynamics of a rational function and the dynamics of an iterate of the function.

Definition 3.5. A subsemigroup H of a semigroup G is said to be of **finite index** if there is a finite collection of elements $\{g_1, g_2, \ldots, g_n\}$ of $G \cup \{\mathrm{Id}\}$ such that

$$G = g_1 \circ H \cup g_2 \circ H \cup \cdots \cup g_n \circ H.$$

If n is chosen to be as small as possible, we say that H has index n in G.

For instance the subsemigroup H of a finitely generated semigroup G consisting of all words of length some multiple of an integer n has finite index in G. (As, for instance, the words of even length in G.) Thus $\langle f^2, g^2, fg, gf\rangle$ has index 3 in $\langle f, g\rangle$: we may take $g_1 = \mathrm{Id}, g_2 = f, g_3 = g$.

Definition 3.6. We say that a subsemigroup H of G has **cofinite index** or **finite coindex** if there is a finite collection of elements $g_1, g_2, \ldots, g_n$ of $G \cup \{\mathrm{Id}\}$ such that for every $g \in G$ there is $j \in \{1, 2, \ldots, n\}$ such that

$$g_j \circ g \in H.$$

The coindex of of H in G is the smallest such number n.

If the semigroup were a group the two notions would coincide. In the example above, the subsemigroup $\langle f^2, g^2, fg, gf\rangle$ has coindex 2 as well as index 3. For the coindex, note that we may take $g_1 = f$ and $g_2 = \mathrm{Id}$.

Theorem 3.4 ([15], Theorem 2.4)**.** *If H is a finite index or finite coindex subsemigroup of G, then $N(H) = N(G)$ and $J(H) = J(G)$.*

Proof. It suffices to show that $N(H) = N(G)$, for then it immediately follows that also $J(H) = J(G)$. Since H is a subsemigroup of G, we have $N(G) \subset N(H)$. It remains to be proved that $N(H) \subset N(G)$.

Suppose that H is a finite index subsemigroup of G. If f_j is a sequence of elements of G, we may pass to a subsequence without changing notation and assume that each f_j can be written as $f_j = g \circ h_j$ where $h_j \in H$ and $g \in \{g_1, \ldots, g_n\}$, where g is independent of j and where the set $\{g_1, \ldots, g_n\}$ is as in Definition 3.5. If U is a domain with $\overline{U} \subset N(H)$ then we may pass to a further subsequence and assume that $h_j \to \phi$ uniformly in U. Hence $f_j \to g \circ \phi$ uniformly in U. It follows that $U \subset N(G)$ and hence $N(H) \subset N(G)$.

Suppose now that H is a cofinite index subsemigroup of G, and let $\{g_1, \ldots, g_n\}$ be as in Definition 3.6. If f_j is a sequence of elements of G, we may pass to a subsequence without changing notation and assume that $h_j = g \circ f_j \in H$ where g is fixed with $g \in \{g_1, \ldots, g_n\}$. Suppose that $z_0 \in N(H)$. Let U be a spherical disk with center z_0 and with $\overline{U} \subset N(H)$. We may pass to a further subsequence and assume that $h_j \to \phi$ uniformly in U. Shrink U, if necessary, so that $\phi(V)$ omits some non-empty open disk $D(U)$ in $\overline{\mathbb{C}}$. Let $D'(U)$ be a non-empty open disk whose closure is contained in $D(U)$. Then the functions f_j eventually omit the set $g^{-1}(D'(U))$ in U, so the functions f_j form a normal family in U. It follows that G is normal in U, and hence $N(H) \subset N(G)$, as desired. $\square$

Remark 3.2. This theorem generalizes the well known fact that for any rational function f of degree at least two and for any integer $n \geq 1$, we have that $J(f) = J(f^n)$.

Example 3.1. Let h be a polynomial of degree at least two with distinct components A and B of $N(h)$ such that $h(A) = h(B) = A$ and A contains the (super)attracting fixed point α of h. Let g be a polynomial of degree at least two with distinct components U and V of $N(g)$ such that $g(U) = g(V) = U$, $\overline{U} \subset B$, $\overline{V} \subset A$, and $\alpha \in V$. There is an integer $m \geq 1$ such that $h^m(V) \subset V$ and $h^m(U) \subset V$. Set $f = h^m$ and $G = \langle f, g \rangle$. Hence U and V are components of $N(G)$. It is easy to see that

$$G_V = \{f \circ F : F \in G\}.$$

Thus G_V is of coindex 1 in G, while G_V is not of finite index in G since $g^n \circ f \circ F \in G$ for all $F \in G$ and $n \geq 1$. Furthermore, G_V is not finitely generated even if G is. For if $G_V = \langle g_1, \ldots, g_k \rangle$, then $g_i \in G$ for all i so that $g_i = f \circ F_i$ where $F_i \in G$. But $f \circ g^n \in G_V$ for all $n \geq 1$, and not every $f \circ g^n$ can lie in $\langle g_1, \ldots g_n \rangle$.

Theorem 3.5. *Let G be a rational semigroup with no wandering domains. Let U be any component of the Fatou set. Then the forward orbit of U under G, that is, $\{U_g : g \in G\}$, contains a stable basin of cofinite index, i.e., a stable basin W such that G_W has cofinite index in G.*

Proof. Let G and U be described as above. Since U is not a wandering domain, the forward orbit of U is finite, where we always include the domain itself in its forward orbit even if $\mathrm{Id} \notin G$. Label the components of the forward orbit $U_1, U_2, \ldots, U_m$, with $U_1 = U$. If for every j there is a $g_j \in G$ such that $g_j(U_j) \subset U_1$, then G_{U_1} is easily seen to have cofinite index in G. (Namely, if $g \in G$ and $g(U_1) \subset U_j$, then $(g_j \circ g)(U_1) \subset U_1$ and hence $g_j \circ g \in G_{U_1}$.) Otherwise choose $k \geq 2$ such that U_1 does not lie in the forward orbit of $V = U_k$. The forward orbit of V is then contained in $\{U_2, \ldots, U_m\}$, so that the number of components in the forward orbit of V is strictly less than that of U. Proceeding by the obvious induction we find a component W whose forward orbit has fewest components, and then $W = U_i$ for some i with $1 \leq i \leq m$. Then for every $h \in G$, for the component W_h of the forward orbit of W there is a function $\tilde{g}$ belonging to a fixed finite subset of G, such that $\tilde{g}(W_h) \subset W$. Thus $W_{\tilde{g} \circ h} = W$ so that $\tilde{g} \circ h \in G_W$ and it follows that G_W has cofinite index in G. $\square$

Corollary 3.2. *Let G be a nearly abelian rational semigroup. Let U be any component of the Fatou set. Then the forward orbit of U under G, $\{U_g : g \in G\}$, contains a stable basin of cofinite index.*

3.3. **Some properties of Stable Basins.** We next discuss a few simple features of some stable basins for rational semigroups.

First we point out that a stable basin can be attracting for a semigroup G, and yet, there need not be a common attracting cycle fixed by each $g \in G$. For instance let $f(z) = z^2 + c$ and $g(z) = z^2 + d$ where $c, d \in \mathbb{C}$. If $|c|, |d|$ are sufficiently small, then the disk $D(1/2)$ of radius $1/2$ centered at 0 is mapped into the disk $D(1/4)$ by f^n, g^n for some large n. Thus $G = \langle f^n, g^n \rangle$ is a polynomial semigroup which contains $\{z : |z| < 1/2\}$ in its Fatou set. This disk contains the attracting cycles for f and g (and hence for f^n and g^n) and these are different if $c \neq d$. Every $h \in G$ maps $D(1/2)$ into $D(1/4)$ and thus contains a (super)attracting fixed point for h. (Question: Is is possible to show that no $h \in G$ is superattracting?)

In the case when G is nearly abelian, we have the following theorem, whose proof can be found in [15], pp. 376–378.

Theorem 3.6 ([15], Theorem 6.2)**.** *Let G be a nearly abelian rational semigroup and U a stable component of $N(G)$. Then U is either attracting, superattracting, parabolic, Siegel or Herman (in the sense of Definition 3.4). In the Siegel case, the basin U contains a single cycle fixed by each element of G_U. If U is of Siegel or Herman type, then G_U is abelian.*

3.4. **Examples.**

Example 3.2 (Common parabolic basins). Set $f(z) = z^2 - 3/4$ and $g = -f$. Then f has a parabolic cycle at $z = -1/2$. Note that if $\phi(z) = -z$, then $f \circ \phi = f$ and $\phi^2(z) = z$, so that g is actually a conjugate of f and so has a parabolic cycle at $1/2$. The semigroup $G = \langle f, g\rangle$ is nearly abelian since

$$f \circ g = f \circ \phi \circ f = f^2 = \phi^2 \circ f^2 = \phi \circ \phi \circ f \circ f = \phi \circ g \circ f.$$

(This also follows from Corollary 3.1.) Thus a nearly abelian semigroup can have different parabolic cycles in the same stable basin. More precisely, there is a component U of $N(G)$ containing the origin such that each of f^2 and g^2 maps U onto itself and has a parabolic fixed point on ∂U, the fixed point being $-1/2$ for f^2 and $1/2$ for g^2.

Example 3.3 (Common superattracting basins). Set $f(z) = (z^2-c^2)^2+c$ and $g = -c - (z^2 - c^2)^2$, i.e., if $\phi(z) = -z$ then $g = \phi \circ f$. Then c is a superattracting fixed point for f and $-c$ is a superattracting fixed point for g. As before, we can see that $\langle f, g\rangle$ is a nearly abelian polynomial semigroup. If $|c|$ is small enough, then both f and g map the disk $\{z : |z| < 1/2\}$ into itself, and thus f and g have a common superattracting basin.

Example 3.4 (Mixed Basin). Set $f(z) = z/(1+z-z^2)$ and $g(z) = \lambda z+z^2$ where $0 < \lambda < 1$. Then $J(g)$ is a Jordan curve (see [5], Theorem 9.9.3) while $J(f)$ is a Cantor subset of the real line (can be shown using the fact that $1/f(1/z) = z+1-1/z$). The mapping f has a parabolic fixed point at 0, $N(f)$ is connected, each of the upper and lower half planes is completely invariant under f, and there is $\epsilon > 0$ such that the interval $(0, \epsilon) \subset N(f) \cap N(g)$ because $f((0, \epsilon)) \subset (0, \epsilon)$ and $g((0, \epsilon)) \subset (0, \epsilon)$. Let $G = \langle f, g\rangle$. (Note that G is not nearly abelian!) Then each $h \in G$ has an attracting or parabolic fixed point at 0. If ϵ is small enough and we set $B = \{z : |z - \epsilon| < \epsilon\}$, then $f(B) \subset B$ (to see this note that $1/f(1/z) = z + 1 - 1/z$). We claim that $g(B) \subset B$. We will leave the details as an exercise, but we will remark that it suffices to choose any ϵ such that $0 < \epsilon < (1 - \lambda)/2$. It follows that $B \subset N(G) \neq \emptyset$ and therefore that $0 \in \partial N(G)$. Thus the stable basin for G containing B is contained in a parabolic basin for f and contained in an attracting basin for g and with the parabolic/attracting fixed point in its boundary.

4. Completely invariant Julia sets

The material from this section is taken from [32] and [30].

We have seen earlier that the Julia set $J(G)$ of a rational semigroup G need not be completely invariant under all the elements of G (see Example 1.1). This is in contrast to the classical situation where $J(f)$ is completely invariant under each iterate f^n.

The question then arises, what if we required the Julia set of the semigroup G to be completely invariant under each element of G? That is, what if we extended the definition of a Julia set given in Property 1.1? We will consider in this section some of the consequences of such an extension which is given in the following definition.

Definition 4.1. For a rational semigroup G we define the completely invariant Julia set

$$I = I(G) = \bigcap \{S : S \text{ is closed, completely invariant under each } g \in G, \#(S) \geq 3\}$$

where $\#(S)$ denotes the cardinality of S.

We note that $I(G)$ exists, is closed, is completely invariant under each element of G and contains the Julia set of each element of G by Property 1.1.

Definition 4.2. For a rational semigroup G we define the completely invariant set of normality of G, $W = W(G)$, to be the complement of $I(G)$, i.e.,

$$W(G) = \overline{\mathbb{C}} \setminus I(G).$$

Note that $W(G)$ is open and it is also completely invariant under each element of G.

So we see that we that in the effort to generalize the dynamics associated with the iteration of a rational function to the more general dynamics of rational semigroups, we are able to extend certain key notions in more than one way. In particular, we can define our Julia set in terms of normality, as we did in defining $J(G)$ or in terms of complete invariance, as we did in defining $I(G)$. It is of interest to pursue a greater understanding of how these two extensions differ, and to learn which is better for studying certain phenomena.

One key difference in the theory is that when studying the action of the elements of the semigroup, one finds that components of the set of normality $N(G)$ only map into other components and not onto as in the action of the elements on the components of the completely invariant set of normality $W(G)$ (see Lemma 4.4). This, of course, has a large impact on how one works to extend, and even define, the concepts involved in two cornerstone theorems of iteration theory, the classification of the fixed components and Sullivan's no-wandering-domains theorem.

We will see below that the extension of the Julia set given by $J(G)$ is better if one wishes to study the dynamics on the extension of the set of normality. This is seen in Theorem 4.2 which states that if a semigroup G is generated by two polynomials with unequal Julia sets, then $I(G)$ must necessarily be the entire Riemann sphere $\overline{\mathbb{C}}$. Hence , in such a case, the completely invariant set of normality is empty and so there are no dynamics on its components of which to study. In this case, however, $J(G)$ is a compact subset of the plane $\mathbb{C}$ and hence there are dynamics on the components of $N(G)$ to be studied. (If the Julia sets of the two generators are equal, then both $J(G)$ and $I(G)$ are equal to this common Julia set.) We note that if one is studying dynamics from the point of view that complete invariance is required, then, of course, the extension given by $I(G)$ is better.

We now compare the sets $I(G)$ and $J(G)$.

Example 4.1. Suppose that $G = \langle f, g\rangle$ and $J(f) = J(g)$. Then $I = J(f) = J(g)$ since $J(f)$ is completely invariant under f and $J(g)$ is completely invariant under g. It is easily verified that if $J(f) = J(g)$, then $J(G) = J(f) = J(g)$.

We will see in the following example, however, that it is not always the case that $J(G) = I(G)$.

Example 4.2. Let $a \in \overline{\mathbb{C}}, |a| > 1$ and $G = \langle z^2, z^2/a\rangle$. One can easily show that $J(G) = \{z : 1 \le |z| \le |a|\}$ (see [15], p. 360) while $I(G) = \overline{\mathbb{C}}$. Note that $J(z^2) = \{z : |z| = 1\}$ and $J(z^2/a) = \{z : |z| = |a|\}$.

Lemma 4.1 ([32], Corollary 2)**.** *For a rational semigroup G, we have $J(G) \subset I(G)$.*

Proof. Since the $W(G)$ is forward invariant under each element of G with complement $I(G)$ which has more than 3 points, it must lie in the set of normality of G. □

Let G be a rational semigroup and select an element $g \in G$. Note that $J(g) \subset I(G)$. We will now show how $I(G)$ can be "built up" from $J(g)$.

For a collection of sets $\mathcal{A}$, and a function h, we denote new collections of sets by $h(\mathcal{A}) = \{h(A) : A \in \mathcal{A}\}$ and $h^{-1}(\mathcal{A}) = \{h^{-1}(A) : A \in \mathcal{A}\}$.

Choose $g \in G$. Let us define the following countable collections of sets:

$$\begin{aligned} \mathcal{E}_0 &= \{J(g)\}, \\ \mathcal{E}_1 &= \bigcup_{f\in G} f^{-1}(\mathcal{E}_0) \cup \bigcup_{f\in G} f(\mathcal{E}_0), \\ \mathcal{E}_{n+1} &= \bigcup_{f\in G} f^{-1}(\mathcal{E}_n) \cup \bigcup_{f\in G} f(\mathcal{E}_n), \\ \text{and } \mathcal{E} &= \bigcup_{n=0}^{\infty} \mathcal{E}_n. \end{aligned}$$

Since I is completely invariant under each $f \in G$ and contains $J(g)$, we have $I \supset \bigcup_{A\in\mathcal{E}} A$. Since I is also closed, we have

$$I \supset \overline{\bigcup_{A\in\mathcal{E}} A}. \tag{4.1}$$

The following lemma shows that these two sets are actually equal.

Lemma 4.2 ([31], Lemma 3.2.1)**.** *We have*

$$I = \overline{\bigcup_{A\in\mathcal{E}} A}.$$

Proof. We only have $I \subset \overline{\bigcup_{A\in\mathcal{E}} A}$ yet to establish. Since the set on the right is closed and contains $J(g)$ (and therefore more than three points), it remains only to show that it is also completely invariant under each $f \in G$.

We will use the fact that for a non-constant rational function h and a subset B of $\overline{\mathbb{C}}$ we have $h^{-1}(\overline{B}) = \overline{h^{-1}(B)}$ since h is a continuous open map.

Using this fact we see that

$$f^{-1}(\overline{\bigcup_{A\in\mathcal{E}} A}) = \overline{f^{-1}(\bigcup_{A\in\mathcal{E}} A)} = \overline{\bigcup_{A\in\mathcal{E}} f^{-1}(A)} \subset \overline{\bigcup_{A\in\mathcal{E}} A}.$$

Also, by the continuity of f, we have

$$f(\overline{\bigcup_{A\in\mathcal{E}} A}) \subset \overline{f(\bigcup_{A\in\mathcal{E}} A)} = \overline{\bigcup_{A\in\mathcal{E}} f(A)} \subset \overline{\bigcup_{A\in\mathcal{E}} A}.$$

So we conclude that $I \subset \overline{\bigcup_{A\in\mathcal{E}} A}$. □

Remark 4.1. In fact, if we had let $\mathcal{E}_0 = \{\{a, b, c\}\}$ where a, b, c are three points known to be in $I(G)$ (for example, if $a, b, c \in J(g)$) and we defined each $\mathcal{E}_n$ and $\mathcal{E}$ as above in terms of this new collection $\mathcal{E}_0$, then we would arrive at the same description of $I(G)$ as given in

Lemma 4.2. This is due to the minimality condition for Julia sets as noted in Property 1.1. For technical reasons we will, however, use the previous description of I obtained from letting $\mathcal{E}_0 = \{J(g)\}$.

Corollary 4.1 ([31], Corollary 3.2.3)**.** *The set $I(G)$ has no isolated points; i.e., $I(G)$ is perfect.*

Proof. Since $J(g)$ is perfect (see [5], p. 68) and backward and forward images of perfect sets under rational maps are perfect, we see that each set in $\mathcal{E}$ is perfect by a routine inductive argument. The corollary then follows since the closure of a union of perfect sets is perfect. □

Recall the definition of uniformly perfect sets given in Definition 2.5. It is known that Julia sets of rational functions (see [22], [13], and [11]) and Julia sets of finitely generated rational semigroups (see [16]) are uniformly perfect. We put forth the following conjecture due to Aimo Hinkkanen.

Conjecture 4.1. *The set $I(G)$ is uniformly perfect when G is finitely generated.*

Lemma 4.3 ([31], Lemma 3.2.5)**.** *Let B be a set which is completely invariant under each $f \in G$. If $I \cap B$ has nonempty interior relative to B, then $I \supset \overline{B \setminus \{\text{at most two points}\}}$.*

Proof. We will use the following elementary fact:

For any sets D and C and any function h we have

$$D \cap h(C) \neq \emptyset \text{ if and only if } h^{-1}(D) \cap C \neq \emptyset. \tag{4.2}$$

By hypothesis we select an open disc $\triangle$ such that $\triangle \cap B \subset I$ and $\triangle \cap B \neq \emptyset$. By Lemma 4.2 we see then that there exists a set A in $\mathcal{E}_n$, say, such that $\triangle \cap A \neq \emptyset$. Since $A \in \mathcal{E}_n$, it can be expressed as $A = h_n \cdots h_1(J(g))$, where each $h_j \in \{f : f \in G\} \cup \{f^{-1} : f \in G\}$. Considering each h_j as a map on subsets of $\overline{\mathbb{C}}$, as opposed to a map on points of $\overline{\mathbb{C}}$, we can define the "inverse" maps h_j^* accordingly, i.e., $h_1 = f$ implies $h_1^* = f^{-1}$ and $h_2 = f^{-1}$ implies $h_2^* = f$. The h_j^* are not true inverses since $f^{-1}(f(A))$ may properly contain A.

The fact (4.2) does imply, however, that

$$A \cap \triangle \neq \emptyset \implies h_n \cdots h_1(J(g)) \cap \triangle \neq \emptyset \tag{4.3}$$
$$\implies h_{n-1} \cdots h_1(J(g)) \cap h_n^*(\triangle) \neq \emptyset \tag{4.4}$$
$$\vdots \tag{4.5}$$
$$\implies h_1(J(g)) \cap h_2^* \cdots h_n^*(\triangle) \neq \emptyset \tag{4.6}$$
$$\implies J(g) \cap h_1^* \cdots h_n^*(\triangle) \neq \emptyset. \tag{4.7}$$

Since each h_j^* maps open sets to open sets (as each f, f^{-1} do) we see that $U = h_1^* \cdots h_n^*(\triangle)$ is open. We observe that by the expanding property of Julia sets (see [5], p.69) that we have $\bigcup_{n=1}^{\infty} g^n(U) = \overline{\mathbb{C}} \setminus E(g)$, where $E(g)$ is the set of (at most two) exceptional points of g. Since the complete invariance of B and I under each of the maps $f \in G$ implies that $U \cap B \subset I$, we have

$$B \setminus E(g) \subset B \cap \bigcup_{n=1}^{\infty} g^n(U) \subset \bigcup_{n=1}^{\infty} B \cap g^n(U) = \bigcup_{n=1}^{\infty} g^n(B \cap U) \subset E.$$

The result then follows since I is closed. $\square$

Property 4.1 ([31], Corollary 3.2.6)**.** *If $I(G)$ has nonempty interior, then $I(G) = \overline{\mathbb{C}}$.*

Proof. Letting $B = \overline{\mathbb{C}}$ in Lemma 4.3 gives the result. $\square$

Corollary 4.2 ([31], Corollary 3.2.10)**.** *If $J(G)$ has nonempty interior, then $I = \overline{\mathbb{C}}$.*

4.1. **Components of W(G).** It is well known in iteration theory that the set of normality of a rational function can have only 0, 1, 2, or infinitely many components (see [5], p. 94). In this section we generalize this result by showing that the completely invariant set of normality of a rational semigroup can have only 0, 1, 2, or infinitely many components. The proof not only generalizes the iteration result, but it also provides an alternative proof for it. The material in this section is taken entirely from [30].

Theorem 4.1 ([30], Theorem 1)**.** *For a rational semigroup G the set $W(G)$ can have only 0, 1, 2, or infinitely many components.*

Lemma 4.4 ([30], Lemma 1)**.** *If W_0 is a component of W, then $f(W_0)$ is also a component of W for any $f \in G$.*

Proof. Let W_1 be the component of W that contains $f(W_0)$. We show that $f(W_0) = W_1$. Suppose to the contrary that $z \in W_1 \setminus f(W_0)$. Since f is continuous on the compact set $\overline{W_0}$ and an open map on W_0, we have $\partial f(W_0) \subset f(\partial W_0) \subset f(I) \subset I$. Let γ be a path in W_1 connecting z to a point $w \in f(W_0)$. Hence γ must cross $\partial f(W_0) \subset I$. This contradicts the fact that $\gamma \subset W_1$ and so we conclude that $f(W_0) = W_1$. $\square$

Since the remainder of this section will be devoted to the proof of Theorem 4.1, we will assume that W has L components where $2 \leq L < +\infty$. We remark here that the strategy will be to show that each of the L components of W is simply connected and then the result will follow by an application of the Riemann-Hurwitz relation.

Definition 4.3. Let W have components W_j for $j = 0, \dots, L-1$.

Remark 4.2. We see by Lemma 4.4 that each $f \in G$ (and hence each f^{-1} as well) permutes the W_j for $j = 0, \dots, L-1$ since f is a continuous map of W onto W.

We may assume that $\infty \in W_0$, else we may impose this condition by conjugating each $f \in G$ by the same rotation of the sphere.

Definition 4.4. For $j = 1, \dots, L-1$, we define

$$K_j = \{z \notin W_j : \text{there exists a simple closed curve } \gamma \subset W_j \text{ such that } Ind_\gamma(z) = 1\}$$

where the winding number is given by $Ind_\gamma(z) = (1/2\pi i) \int_\gamma 1/(w - z)\, dw$. If $z \in K_j$ and the simple closed curve $\gamma \subset W_j$ is such that $Ind_\gamma(z) = 1$, then we say that γ works for $z \in K_j$.

In order to properly define K_0 we first need to move W_0 so that it no longer contains ∞. Let ϕ be a rotation of the sphere so that $\infty \in \phi(W_1)$ and denote $\widetilde{W_j} = \phi(W_j)$ for $j = 0, \dots, L-1$.

Definition 4.5. We define

$$\widetilde{K_0} = \{z \notin \widetilde{W_0} : \text{there exists a simple closed curve } \gamma \subset \widetilde{W_0} \text{ such that } Ind_\gamma(z) = 1\}$$

and

$$K_0 = \phi^{-1}(\widetilde{K_0}).$$

If $z \in K_0$ and simple closed curve $\gamma \subset \widetilde{W_0}$ is such that $Ind_\gamma(\phi(z)) = 1$, then we say that the simple closed curve $\phi^{-1}(\gamma)$ works for $z \in K_0$.

Remark 4.3. Note that saying $\phi^{-1}(\gamma)$ works for $z \in K_0$ does not necessarily imply that $Ind_{\phi^{-1}(\gamma)}(z) = 1$, since it may be the case that $Ind_\gamma(\phi(\infty)) = 1$ and hence $Ind_{\phi^{-1}(\gamma)}(z) = 0$ since z lies in the unbounded component of $\overline{\mathbb{C}} \setminus \phi^{-1}(\gamma)$.

Definition 4.6. We define

$$K = \bigcup_{j=0}^{L-1} K_j.$$

Definition 4.7. We define

$$W_j' = W_j \cup K_j.$$

Lemma 4.5 ([30], Lemma 2)**.** *For $j = 0, \dots, L-1$, the set W_j' is open, connected and simply connected. Thus each K_j is the union of the "holes" in W_j.*

Proof. Suppose that $1 \leq j \leq L-1$, so that W_j is a bounded domain in the complex plane. Define A to be the unbounded component of $\overline{\mathbb{C}} \setminus W_j$. Hence $B = \overline{\mathbb{C}} \setminus A$ is open, connected and simply connected.

Let F be a bounded component of $\overline{\mathbb{C}} \setminus W_j$. Since A and F are each components of the closed set $\overline{\mathbb{C}} \setminus W_j$, there exists a simple polygon $\gamma \subset W_j$ which separates A from F (see [25], p. 134). Hence we see that $F \subset K_j$. Since F was an arbitrary bounded component of $\overline{\mathbb{C}} \setminus W_j$, we conclude that K_j contains all the bounded components of $\overline{\mathbb{C}} \setminus W_j$, i.e., the "holes" of W_j. Hence $W'_j \supset B$. Clearly K_j cannot contain any points of A since any simple closed path $\gamma \subset W_j$ which would wind around such a point would have to necessarily wind around every point of A (since A is a component of the complement of W_j) including ∞ which cannot happen. Hence we conclude $W'_j = B$ and is therefore open, connected and simply connected.

We show that $\phi(W'_0)$ is open, connected and simply connected using the same argument as above, and this implies that W'_0 is open, connected and simply connected. $\square$

Definition 4.8. We define

$$W' = \bigcup_{j=0}^{L-1} W'_j.$$

Note that we have $W' = W \cup K$.

Lemma 4.6 ([30], Lemma 3)**.** *If for some distinct $r, s \in \{0, \ldots, L-1\}$, we have $W'_r \cap W'_s \neq \emptyset$, then either $\overline{W'_r} \subset W'_s$ or $\overline{W'_s} \subset W'_r$. In particular, if $W_r \cap W'_s \neq \emptyset$ for some distinct $r, s \in \{0, \ldots, L-1\}$, then $\overline{W'_r} \subset W'_s$.*

Proof. Let $z \in W'_r \cap W'_s$. Since $W_r \cap W_s = \emptyset$, we may assume that $z \in K_s$, say. Let γ_s work for $z \in K_s$. Let I_{γ_s} be the component of $\overline{\mathbb{C}} \setminus \gamma_s$ which contains z. Note that $I_{\gamma_s} \setminus W_s = \{z : \gamma_s \text{ works for } z\}$ whether or not $s = 0$ (see Definitions 4.4 and 4.5 and Remark 4.3). Since $z \in W'_r$, we have two cases, either $z \in K_r$ or $z \in W_r$.

Suppose that $z \in K_r$ and let γ_r work for $z \in K_r$. As $\gamma_s \cap \gamma_r = \emptyset$ (since $W_r \cap W_s = \emptyset$) we see that either $\gamma_r \subset I_{\gamma_s}$ or $\gamma_s \subset I_{\gamma_r}$, where I_{γ_r} is the component of $\overline{\mathbb{C}} \setminus \gamma_r$ which contains z. By switching the roles of r and s, if necessary, we assume $\gamma_r \subset I_{\gamma_s}$ and we note that this can be done since $z \in K_r \cap K_s$. In particular, $W_r \cap I_{\gamma_s} \neq \emptyset$.

If $z \in W_r$, then we still get $W_r \cap I_{\gamma_s} \neq \emptyset$ since $z \in I_{\gamma_s}$.

Since $W_r \cap I_{\gamma_s} \neq \emptyset$, $W_r \cap W_s = \emptyset$, $\overline{W_r}$ is connected, and $\gamma_s \subset W_s$, we conclude that $\overline{W_r} \subset I_{\gamma_s}$. Hence $\overline{W_r} \subset W'_s$ since γ_s then works for every $z \in \overline{W_r}$. Since W'_s is simply connected we see that $\overline{W'_r} \subset W'_s$. $\square$

Lemma 4.7 ([30], Lemma 4)**.** *The boundary of W'_0 is a nondegenerate continuum and as such contains more than three points.*

Proof. We will first show that $W_0' \cap W_1' = \emptyset$. The set W_1' cannot contain W_0' as $\infty \in W_0'$ and W_1' is a bounded subset of $\mathbb{C}$ (since W_1 is a bounded subset of $\mathbb{C}$). The same argument also shows that $\phi(W_0')$ cannot contain $\phi(W_1')$ where ϕ is as in Definition 4.5, and so we conclude that W_0' cannot contain W_1'. By Lemma 4.6 we conclude that $W_0' \cap W_1' = \emptyset$.

Since W_0' is simply connected, $\partial W_0'$ contains a nondegenerate continuum unless $\partial W_0'$ consists of just a single point. If $\partial W_0'$ consists of just a single point, then $W_0' \cup \partial W_0' = \overline{\mathbb{C}}$, but this contradicts the fact that $W_0' \cap W_1' = \emptyset$. □

Lemma 4.8 ([30], Lemma 5). *For each $j = 0, \ldots, L-1$, we have $J(f) \subset \partial W_j$ for each $f \in G$. Since $J(G) = \overline{\cup_{f\in G} J(f)}$, we have $J(G) \subset \partial W_j$ for each $j = 0, \ldots, L-1$.*

Proof. Since f permutes the W_j by Remark 4.2, we may select a positive integer n so that $f^n(W_j) = W_j = f^{-n}(W_j)$ for each $j = 0, \ldots, L-1$. Then we have $\overline{\bigcup_{k=1}^{\infty} f^{-kn}(W_j)} \supset J(f^n) = J(f)$ (see [5], p. 71 and p. 51). But since $\overline{\bigcup_{k=1}^{\infty} f^{-kn}(W_j)} = \overline{W_j}$ we see that $\partial W_j \supset J(f)$, since $W_j \cap J(f) = \emptyset$. □

Lemma 4.9 ([30], Lemma 6). *We have $W_r \not\subseteq W_s'$ for distinct $r, s \in \{0, \ldots, L-1\}$, and therefore, by Lemma 4.6, the W_j' are disjoint for $j = 0, \ldots, L-1$.*

Proof. If $L = 2$, then the proof of Lemma 4.7 shows that $W_0' \cap W_1' = \emptyset$.

We assume now that $L \geq 3$. We will first show that no bounded W_s' can contain any W_r with $r \neq s$. Suppose that this does occur. Then there exists a simple closed curve $\gamma_s \subset W_s$ such that $\overline{W_r} \subset I_{\gamma_s}$ where I_{γ_s} is the component of $\overline{\mathbb{C}} \setminus \gamma_s$ which contains the points z such that $Ind_{\gamma_s}(z) = 1$. Hence, by Lemma 4.8, $J(G) \subset \partial W_r \subset \overline{W_r} \subset I_{\gamma_s}$. But since $\overline{W_0} \subset \overline{\mathbb{C}} \setminus I_{\gamma_s}$ we see that $J(G) \subset \partial W_0 \subset \overline{W_0} \subset \overline{\mathbb{C}} \setminus I_{\gamma_s}$. This contradiction implies no bounded W_s' can contain any W_r.

We see that W_0' cannot contain any W_r with $r \geq 1$ by the following similar argument. If $\overline{W_r} \subset W_0'$, then there exists a simple closed curve $\gamma \subset \widetilde{W_0}$ such that $Ind_\gamma(z) = 1$ for every $z \in \overline{\widetilde{W_r}}$. Let I_γ be the component of $\overline{\mathbb{C}} \setminus \gamma$ which contains $\overline{\widetilde{W_r}}$. So $\phi(J(G)) \subset \phi(\partial W_r) = \partial\phi(W_r) = \partial\widetilde{W_r} \subset I_\gamma$. Since $\overline{\widetilde{W_1}} \subset \overline{\mathbb{C}} \setminus I_\gamma$ (recall $\infty \in \widetilde{W_1}$), we see that $\phi(J(G)) \subset \phi(\partial W_1) = \partial\phi(W_1) = \partial\widetilde{W_1} \subset \overline{\mathbb{C}} \setminus I_\gamma$. This contradiction implies W_0' cannot contain any W_r with $r \geq 1$. □

Corollary 4.3 ([30], Corollary 1). *The set K has no interior and therefore each $K_j \subset \partial W_j$.*

Proof. By Lemma 4.9 we see that each $K_j \subset I$ and hence $K \subset I$. The Corollary then follows from Property 4.1. $\square$

Corollary 4.4 ([30], Corollary 2). *We have $\partial W_j = K_j \cup \partial W_j'$.*

Proof. By Corollary 4.3 we get $K_j \cup \partial W_j' \subset \partial W_j$. We also have $\partial W_j = \overline{W_j} \setminus W_j \subset \overline{W_j'} \setminus W_j = (W_j' \cup \partial W_j') \setminus W_j = (W_j \cup K_j \cup \partial W_j') \setminus W_j = K_j \cup \partial W_j'$. $\square$

Lemma 4.10 ([30], Lemma 7). *We have $f(K) \subset K$ for all $f \in G$.*

Proof. Let $z \in K_j$ be such that $\gamma \subset W_j$ works for z.

Suppose that $W_l = f(W_j) \neq W_0$. So W_j' contains no poles of f, else such a pole would be in W_j (by the complete invariance of W under f since $\infty \in W_0 \subset W$ and Lemma 4.9) and hence $f(W_j) = W_0$. By the argument principle, $f(\gamma) \subset W_l$ winds around $f(z)$, thus $f(z) \in K_l$ as $f(z) \notin W_l$ by the complete invariance of W under the map f. Note that $f(\gamma)$ might not work for $f(z) \in K_l$ since it might not be simple, but $f(z) \in K_l$ since it cannot be in the unbounded component of $\overline{\mathbb{C}} \setminus W_l$ and have a curve in W_l, namely $f(\gamma)$, wind around it.

Now suppose that $f(W_j) = W_0$. So $(\phi \circ f)(W_j) = \widetilde{W_0}$ is bounded and W_j' contains no poles of $\phi \circ f$ (else $f(W_j) = W_1$). So $(\phi \circ f)(\gamma)$ winds around $(\phi \circ f)(z)$ and hence $(\phi \circ f)(z) \in \widetilde{K_0}$, i.e., $f(z) \in K_0$.

So $f(K_j) \subset K$ and hence we conclude $f(K) \subset K$. $\square$

Lemma 4.11 ([30], Lemma 8). *We have for all $f \in G$, $f(W') \cap \partial W_0' = \emptyset$. Also $W' \subset N(G)$ and in particular $K \cap J(G) = \emptyset$.*

Proof. We have $f(W') = f(W \cup K) = f(W) \cup f(K) \subset W \cup K = W'$. Since $W' \cap \partial W_0' = \emptyset$ (since W' is open), Lemma 4.7 and Montel's Theorem finish the proof. $\square$

Corollary 4.5 ([30], Corollary 3). *We have $J(G) \subset \partial W_j'$ for each $j = 0, \dots, L-1$.*

Proof. This follows immediately from Lemma 4.8, Corollary 4.4 and Lemma 4.11. $\square$

Remark 4.4. It is of interest to note that for any positive integer n there exist disjoint simply connected domains $D_1, \dots, D_n$ in $\overline{\mathbb{C}}$ with $\partial D_1 = \partial D_2 = \cdots = \partial D_n$ (see [18], p. 143). Thus Corollary 4.5 does not imply that $L < 3$ from a purely topological perspective.

Lemma 4.12 ([30], Lemma 9). *We have $f^{-1}(K) \subset K$ for all $f \in G$. Hence by Lemma 4.10, K is completely invariant under each $f \in G$.*

Proof. Let $z \in K_j \subset \partial W_j$ and say $f(w) = z$. Define $W_k = f^{-1}(W_j)$ by Remark 4.2. We obtain sequences $z_n \in W_j$ such that $z_n \to z$, and $w_n \in W_k$ such that $w_n \to w$ and $f(w_n) = z_n$. Hence we see that $w \in \partial W_k$, else $w \in W_k$ and $z = f(w) \in W_j$. If $w \notin K_k$, then $w \in \partial W'_k$ by Corollary 4.4. Let Γ be the component of ∂W_j that contains $f(\partial W'_k)$. Since $z \in \Gamma$, the set Γ must be one of the components of K_j. By Corollary 4.5 we see that there exists a $\zeta \in \partial W'_k \cap J(f)$. Hence $f(\zeta) \in K_j \cap J(f)$ which is a contradiction since we know by Lemma 4.11 that K is disjoint from $J(G) \supset J(f)$. This contradiction implies $w \in K_k$ and hence $f^{-1}(K) \subset K$. $\square$

Lemma 4.13 ([30], Lemma 10)**.** *If W has L components where $2 \leq L < +\infty$, then each is simply connected.*

Proof. Since K and W are each completely invariant under each $f \in G$, so is $W' = W \cup K$. By Lemma 4.11 we see that $\overline{\mathbb{C}} \setminus W'$ is completely invariant under each $f \in G$, closed, and contains $J(G)$. Hence $I \subset \overline{\mathbb{C}} \setminus W'$. This implies that $W = W'$ and hence each component of W is then simply connected. $\square$

We are now able to present the proof of Theorem 4.1.

Proof of Theorem 4.1. If W has L components where $2 \leq L < +\infty$, then each is simply connected by Lemma 4.13. Select a map $f \in G$. Letting $n \geq 1$ be selected so that each of the components W_j of W is completely invariant under f^n, we get by the Riemann-Hurwitz relation (see [33], p. 7)

$$\delta_{f^n}(W_j) = \deg(f^n) - 1$$

where we write $\delta_g(B) = \sum_{z \in B}[v_g(z) - 1]$ and $v_g(z)$ is the valency of the map g at the point z.

Hence we obtain

$$L(\deg(f^n) - 1) = \sum_{j=0}^{L-1} \delta_{f^n}(W_j) \leq \delta_{f^n}(\overline{\mathbb{C}}) = 2(\deg(f^n) - 1)$$

and so $L \leq 2$. The last equality follows from Theorem 2.7.1 in [5]. $\square$

Remark 4.5. Note that if $L = 2$, then each component of W is necessarily simply connected.

We know from iteration theory that each of the four possibilities $(0, 1, 2, \infty)$ for the number of components of the set of normality can be achieved. So by constructing semigroups G such that all the elements have the same Julia set we know that the only four possibilities for the number of components of the completely invariant set of normality of

G can also be achieved. However, it does not seem possible that all four possibilities can be achieved if we restrict ourselves to the cases where two elements of the semigroup G have nonequal Julia sets. For example, if G contains two polynomials with nonequal Julia sets then the completely invariant set of normality is necessarily empty (see [32], Theorem 1).

We do have the following examples however.

Example 4.3. Consider $f(z) = 2z - \frac{1}{z}$. One can easily show that the extended real line $\overline{\mathbb{R}}$ is completely invariant under f and that $J(f)$ is a Cantor subset of the interval $[-1, 1]$ (see [5], p. 21). Let $\phi(z) = i\frac{1+z}{1-z}$, $h(z) = z^2$, and set $g(z) = (\phi \circ h \circ \phi^{-1})(z) = \frac{z^2-1}{2z}$. Hence $J(g) = \phi(J(h)) = \overline{\mathbb{R}}$, (see [5], p. 50). So we see that $I(\langle f, g\rangle) = \overline{\mathbb{R}} \neq \overline{\mathbb{C}}$, but $J(f) \subsetneq J(g)$. Note that $J(f) \subsetneq J(g) = I$ in this example. We also point out that $J(\langle f, g\rangle) = \overline{\mathbb{R}} = I$.

Example 4.4. Consider $f(z) = 2z - \frac{1}{z}$ as in Example 4.3. Let $\phi(z) = z + 1$, and set $g(z) = (\phi \circ f \circ \phi^{-1})(z) = 2z - \frac{1}{z-1} - 1$.

Claim 4.1. *In Example 4.4 we have* $J(\langle f, g\rangle) = [-1, 2]$ *and* $I(G) = \overline{\mathbb{R}}$.

Proof. Define $A = [-1, 2]$. Since f is a strictly increasing map of each of the intervals

$$A_1 = \left[-1, \frac{1-\sqrt{3}}{2}\right] \subset A \qquad \text{and} \qquad A_2 = \left[\frac{1}{2}, \frac{1+\sqrt{3}}{2}\right] \subset A$$

onto the interval A, we can define two branches, say f_1 and f_2, of f^{-1} on A by $f_1(A) = A_1$ and $f_2(A) = A_2$. As $|f'(z)| > 2$ on A_1 and A_2, we see that f_1 and f_2 are contractions on A.

Since g is a strictly increasing map of each of the intervals

$$A_3 = \left[\frac{1-\sqrt{3}}{2}, \frac{1}{2}\right] \subset A \qquad \text{and} \qquad A_4 = \left[\frac{1+\sqrt{3}}{2}, 2\right] \subset A$$

onto the interval $A = [-1, 2]$, we can define two branches, say g_1 and g_2, of g^{-1} on A by $g_1(A) = A_3$ and $g_2(A) = A_4$. As $|g'(z)| > 2$ on A_3 and A_4, we see that g_1 and g_2 are contractions on A.

We note that A is backward invariant under both f and g since $A_j \subset A$ for $1 \leq j \leq 4$, and so $J(\langle f, g\rangle) \subset A = [-1, 2]$.

We next note that we can define an iterated function system on A using the functions f_1, f_2, g_1, and g_2. Let $W(X) = f_1(X) \cup f_2(X) \cup g_1(X) \cup g_2(X)$ for any compact subset $X \subset A$. We note that $W(A) = A$ and so by Iterated Function Systems (IFS) theory, A is the unique attractor set for this IFS. Let $B = J(f) \cup J(g)$ and note that by the backward invariance of $J(\langle f, g\rangle)$ we get $W^n(B) \subset J(\langle f, g\rangle)$ for all n.

Since $J(\langle f, g\rangle)$ is closed and $W^n(B) \to A$ in the Hausdorff metric, we see that $A \subset J(\langle f, g\rangle)$.

Since $[-1, 2] = J(G) \subset I(G)$ and $\overline{\mathbb{R}}$ is completely invariant under both f and g, we see by Lemma 4.3 that $I(G) = \overline{\mathbb{R}}$. $\square$

So we see that it is possible for a completely invariant set of normality of a semigroup G which contains two elements with nonequal Julia sets, to have 0 or exactly 2 components. We feel that the interplay between functions with nonequal Julia sets and the fact that if $I(G)$ has interior then $I(G) = \overline{\mathbb{C}}$ demands that only under special circumstances can we have $W(G)$ be nonempty, when two elements of the semigroup G have nonequal Julia sets.

We state the following conjectures which are due to Aimo Hinkkanen and Gaven Martin.

Conjecture 4.2. *If G is a rational semigroup which contains two maps f and g such that $J(f) \neq J(g)$ and $I(G) \neq \overline{\mathbb{C}}$, then $W(G)$ has exactly two components, each of which is simply connected, and $I(G)$ is equal to the boundary of each of these components.*

Conjecture 4.3. *If G is a rational semigroup which contains two maps f and g such that $J(f) \neq J(g)$ and $I(G) \neq \overline{\mathbb{C}}$, then $I(G)$ is a simple closed curve in $\overline{\mathbb{C}}$.*

Of course Conjecture 4.2 would follow from Conjecture 4.3.

We finish by including some comments on the number of components of the set of normality $N(G)$ of a rational semigroup G. It is not known if the set $N(G)$ must have only 0, 1, 2, or infinitely many components when G is a finitely generated rational semigroup. However, for each positive integer n, an example of an infinitely generated polynomial semigroup G can be constructed with the property that $N(G)$ has exactly n components. These examples were constructed by David Boyd in [6].

4.2. **Polynomial semigroups.** The material from this section comes entirely from [32]. When the semigroup G contains only elements with the same Julia set J, then we have seen that $I(G) = J = J(G)$. If, however, there are two functions with nonequal Julia sets, then we do not expect that $J(G)$ should necessarily equal $I(G)$, see Example 1.1. For example, if the functions with nonequal Julia sets are polynomials, then we will show that $I(G)$ must coincide with the entire Riemann sphere. Specifically, we prove the following theorems.

Theorem 4.2 ([32], Theorem 1)**.** *For polynomials f and g of degree greater than or equal to two, $J(f) \neq J(g)$ implies $I(G) = \overline{\mathbb{C}}$ where $G = \langle f, g\rangle$.*

The following theorem follows immediately.

Theorem 4.3 ([32], Theorem 2)**.** *For a rational semigroup G' which contains two polynomials f and g of degree greater than or equal to two, $J(f) \neq J(g)$ implies $I(G') = \overline{\mathbb{C}}$.*

We first establish the necessary lemmas.

Lemma 4.14 ([32], Lemma 4)**.** *If f and g are polynomials of degree greater than or equal to two and $J(f) \neq J(g)$, then $\infty \in I$.*

Proof. Denoting the unbounded components of the respective Fatou sets of f and g by F_∞ and G_∞, we recall (see [5], p. 54 and p. 82) that $J(f) = \partial F_\infty$ and $J(g) = \partial G_\infty$.

Since F_∞ and G_∞ are domains with nonempty intersection and $\partial F_\infty \neq \partial G_\infty$, we have $J(f) \cap G_\infty \neq \emptyset$ or $J(g) \cap F_\infty \neq \emptyset$.

Hence we may select $z \in J(g) \cap F_\infty$, say. Denoting the nth iterate of f by f^n, we see that $f^n(z) \to \infty$, and by the forward invariance under the map f of the set I we get that each $f^n(z) \in I$. Since I is closed we see then that $\infty \in I$. $\square$

Remark 4.6. Since it will be necessary later, we make special note of the fact used in the above proof that $J(f) \neq J(g)$ implies $J(f) \cap G_\infty \neq \emptyset$ or $J(g) \cap F_\infty \neq \emptyset$.

Remark 4.7. Note that the proof above shows also that ∞ is not an isolated point of I when $J(f) \neq J(g)$. This, of course, also follows from Corollary 4.1 and Lemma 4.14.

The disc centered at the point z with radius r will be denoted $\triangle(z, r)$.

Lemma 4.15 ([32], Lemma 5)**.** *Suppose that $\triangle(0, r^*) = A \cup B$ where A is open, A and B are disjoint, and both A and B are nonempty. If both A and B are completely invariant under the map $L(z) = z^j$ defined on $\triangle(0, r^*)$ where $0 < r^* < 1$ and $j \geq 2$, then the set A is a union of open annuli centered at the origin and hence B is a union of circles centered at the origin. Furthermore, each of A and B contains a sequence of circles tending to zero.*

Proof. Let $z_0 = re^{i\theta} \in A$. Since A is open we may choose $\delta > 0$ such that the arc $\alpha_{z_0} = \{re^{i\omega} : |\theta - \omega| \leq \frac{\delta}{2}\} \subset A$.

Fix a positive integer n such that $j^n \delta > 2\pi$. Since $L^n(z) = z^{j^n}$ we get

$$L^n(\alpha_{z_0}) = C(0, r^{j^n})$$

where $C(z, r) = \{\zeta : |\zeta - z| = r\}$.

By the forward invariance of A under L, we see that $C(0, r^{j^n}) \subset A$. But now by the backward invariance of A, we get

$$C(0, r) = L^{-n}(C(0, r^{j^n})) \subset A.$$

Thus for any $re^{i\theta} \in A$, we have $C(0, r) \subset A$. Hence A, being open, must be a union of open annuli centered at the origin and therefore B, being the complement of A in $\triangle(0, r^*)$, must be a union of circles centered at the origin.

We also note that if $C(0, r) \subset A$, then $C(0, r^{j^n}) \subset A$ is a sequence of circles tending to zero. Similarly we obtain a sequence of circles in B tending to zero. □

Lemma 4.16 ([32], Lemma 6). *Let $L : \triangle(0, r^*) \to \triangle(0, r^*)$, where $0 < r^* < 1$, be an analytic function such that $L(0) = 0$. Let B be a set with empty interior which is a union of circles centered at the origin and which contains a sequence of circles tending to zero. If B is forward invariant under the map L, then L is of the form*

$$L(z) = az^j$$

for some non-zero complex number a and some positive integer j.

Proof. Since $L(0) = 0$, we have, near $z = 0$,

$$\begin{aligned} L(z) &= az^j + a_1 z^{j+1} + \cdots \\ &= az^j(1 + \frac{a_1}{a} z + \cdots) \end{aligned}$$

for some non-zero complex number a and some positive integer j.

Let $h(z) = L(z)/az^j$ and note that $h(z)$ is analytic and tends to 1 as z tends to 0. We shall prove that $h(z) \equiv 1$ and the lemma then follows.

Let $C_n = C(0, r_n)$ be sequence of circles contained in B with $r_n \to 0$. We claim that each $L(C_n)$ is contained in another circle centered at the origin of, say, radius r_n'. If not, then the connected set $L(C_n)$ would contain points of all moduli between, say, r' and r''. This, however, would imply that B would contain the annulus between the circles $C(0, r')$ and $C(0, r'')$. Thus we have $L(C_n) \subset C(0, r_n')$.

So we see then that $h(C_n) \subset C(0, r_n'/|a|r_n^j)$.

But for large n we see that if h were non-constant, then $h(C_n)$ would be a path which stays near $h(0) = 1$ and winds around $h(0) = 1$. Since $h(C_n)$ is contained in a circle centered at the origin, this cannot happen. We thus conclude that h is constant. □

Lemma 4.17 ([32], Lemma 7). *If $B \subset \triangle(0, r^*)$ for $0 < r^* < 1$ is a nonempty relatively closed set which is completely invariant under the maps $H : z \mapsto z^j$ and $K : z \mapsto az^m$ defined on $\triangle(0, r^*)$ where a is a nonzero complex number and j, m are integers with $j, m \geq 2$, then $B = \triangle(0, r^*)$ or $|a| = 1$.*

Proof. We may assume that $|a| \leq 1$ by the following reasoning. Suppose that $|a| \geq 1$. Let b be a complex number such that $b^{m-1} = a$ and define $\psi(z) = bz$. Since $\psi \circ H \circ \psi^{-1}(z) = z^j/b^{j-1}$ and $\psi \circ K \circ \psi^{-1}(z) = z^m$, we see that the lemma would then imply that $\psi(B) = \triangle(0, |b|r^*)$ or $|b| = 1$. Since we know that $\psi(B) = \triangle(0, |b|r^*)$ exactly when $B = \triangle(0, r^*)$, and $|b| = 1$ exactly when $|a| = 1$, we may then assume that $|a| \leq 1$.

We will assume that $|a| < 1$ and show that this then implies that $B = \triangle(0, r^*)$.

We first note that by Lemma 4.15, B is a union of circles centered at the origin and B contains a sequence of circles tending to zero. If $C(0, \rho) \subset B$, then by the forward invariance of B under H, we see that $C(0, \rho^j) \subset B$. Also we get that if $C(0, \rho) \subset B$, then by the forward invariance of B under K, we have $C(0, |a|\rho^m) \subset B$. Using a change of coordinates $r = \log \rho$ this invariance can be stated in terms of the new functions

$$t(r) = jr \text{ and } s(r) = mr + c \tag{4.8}$$

where $c = \log |a| < 0$.

So the action of H and K on $\triangle(0, r^*)$ is replaced by the action of t and s on $I = [-\infty, \log r^*)$, respectively. In particular, we define

$$B' = \{\log \rho : C(0, \rho) \subset B\} \cup \{-\infty\}$$

keeping in mind that B is a union of circles centered at the origin. Then

$$s(B') \subset B', \tag{4.9}$$

$$s^{-1}(B') \cap I \subset B', \tag{4.10}$$

$$t(B') \subset B', \tag{4.11}$$

$$t^{-1}(B') \cap I \subset B', \tag{4.12}$$

$$B' \text{ is closed in the relative topology on } I. \tag{4.13}$$

In order to make calculations a bit easier we rewrite $s(r) = r_0 + m(r - r_0)$ where $r_0 = -c/(m-1) > 0$.

Hence

$$\begin{aligned}
s^n(r) &= r_0 + m^n(r - r_0), \\
s^{-n}(r) &= r_0 + m^{-n}(r - r_0), \\
t^n(r) &= j^n r, \\
t^{-n}(r) &= j^{-n} r.
\end{aligned}$$

Consider
$$(t^{-n} \circ s^{-n} \circ t^n \circ s^n)(r) = r - r_0 + \frac{r_0}{j^n} + \frac{r_0}{m^n} - \frac{r_0}{m^n j^n}.$$
Let
$$d_n = \frac{r_0}{j^n} + \frac{r_0}{m^n} - \frac{r_0}{m^n j^n} = r_0 \frac{m^n + j^n - 1}{m^n j^n}$$
and note that $0 < d_n \leq r_0$ with $d_n \to 0$ as $n \to \infty$.

We also note that $(t^{-n} \circ s^{-n} \circ t^n \circ s^n)(r) = r - r_0 + d_n$ implies that $(s^{-n} \circ t^{-n} \circ s^n \circ t^n)(r) = r + r_0 - d_n$ since these two functions are inverses of each other.

We claim that $(-\infty, \log r^* - r_0] \subset B'$.

Let us suppose that this is not the case, and suppose that $(r', \tilde{r})$ is an interval disjoint from B' with $-\infty < r' < \tilde{r} \leq \log r^* - r_0$. Since B' is a closed subset of $[-\infty, \log r^* - r_0]$, we may assume that this interval is expanded so that $r' \in B'$. Note that here we used the fact that B contains a sequence of circles going to 0, hence B' contains a sequence of points going to $-\infty$.

Let $r'_n = (t^{-n} \circ s^{-n} \circ t^n \circ s^n)(r') = r' - r_0 + d_n$. We claim that each r'_n is in B'. This is almost obvious from the invariance of B' under s and t in (4.9) through (4.12), but some care needs to be taken to insure that each application of s, t, s^{-1}, and t^{-1} takes points to the right domain. By (4.9) we see that $s(r'), s^2(r'), \ldots, s^n(r') \in B'$. Hence by (4.11) we get $(t \circ s^n)(r'), (t^2 \circ s^n)(r'), \ldots, (t^n \circ s^n)(r') \in B'$.

Since $s^{-1}(r) > r$ for $r \in (-\infty, r_0)$ we see that because $(s^{-n} \circ t^n \circ s^n)(r')$ is clearly less than r' (as $t(r) < r$ for $r \in (-\infty, 0)$), also each of $(s^{-1} \circ t^n \circ s^n)(r'), \ldots, (s^{-n} \circ t^n \circ s^n)(r')$ must be less than $r' < \log r^*$. Hence by (4.10) we see that each of these points lies in B'.

Similarly, since $t^{-1}(r) > r$ for $r \in (-\infty, 0)$ and $(t^{-n} \circ s^{-n} \circ t^n \circ s^n)(r') = r' - r_0 + d_n \leq r' < \log r^* < 0$, also each of $(t^{-1} \circ s^{-n} \circ t^n \circ s^n)(r'), \ldots, (t^{-n} \circ s^{-n} \circ t^n \circ s^n)(r')$ lies in $I = [-\infty, \log r^*)$. Hence by (4.12) each of these points is in B' and so each $r'_n \in B'$.

Hence we conclude that $r' - r_0 \in B'$ since B' is relatively closed in I and $r'_n \to r' - r_0 \in I$. Note also that $r'_n \searrow r' - r_0$.

Now we claim that for any $r'' \in B' \cap (-\infty, \log r^* - r_0)$, we have $r'' + r_0 \in B'$. Let $r''_n = (s^{-n} \circ t^{-n} \circ s^n \circ t^n)(r'') = r'' + r_0 - d_n$. Noting that each $r''_n < r'' + r_0 < \log r^*$ we may again use the invariance of B' under s and t in (4.9) through (4.12) in a similar fashion as above to obtain that each $r''_n \in B'$. Thus also the limit $r'' + r_0 \in B'$.

Consider again $r'_n \searrow r' - r_0$. By applying the above claim to each $r'_n \leq r' < \log r^* - r_0$, we get that each $r'_n + r_0 \in B'$. Since $r'_n + r_0 \searrow r'$ we then see that we have contradicted the statement that $(r', \tilde{r})$ is disjoint from B'.

So we conclude that $(-\infty, \log r^* - r_0] \subset B'$. Clearly then by the partial backward invariance of B' under the map t we get $[-\infty, \log r^*) \subset B'$. Hence we conclude that $\triangle(0, r^*) = B$. $\square$

In order to avoid some technical difficulties we will make use of the following well known result.

Theorem 4.1. *A polynomial f of degree k is conjugate near ∞ to the map $z \mapsto z^k$ near the origin. More specifically, there exists a neighborhood U of ∞ such that we have a univalent*

$\phi : U \to \triangle(0, r^*)$ *for* $0 < r^* < 1$ *with* $\phi(\infty) = 0$ *and* $\phi \circ f \circ \phi^{-1}(z) = z^k$.

Proof. After conjugating f by $z \mapsto 1/z$ we may apply Theorem 6.10.1 in [5], p. 150 to obtain the desired result. $\square$

We will denote the conjugate function of f by F, i.e.,

$$F(z) = \phi \circ f \circ \phi^{-1}(z) = z^k.$$

In order to further simplify some of the following proofs we will assume that $\phi(U) = D = \triangle(0, r^*)$. Note that U is forward invariant under f since $D = \triangle(0, r^*)$ is forward invariant under F. We may and will also assume that U is forward invariant under g as well.

We now define a corresponding function for g using the same conjugating map as we did for f. Let G be the function defined on $D = \triangle(0, r^*)$ given by

$$G = \phi \circ g \circ \phi^{-1}.$$

Note that $G(D) \subset D$.

Via this change of coordinates, we will use the mappings F and G to obtain information about the mappings f and g. In transferring to this simpler coordinate system we make the following definitions.

Let W' denote the image of W under ϕ, i.e., $W' = \phi(U \cap W)$. Let I' denote the image of I under ϕ, i.e., $I' = \phi(U \cap I)$. Thus W' is open and I' is closed in the relative topology of D. Note that W' and I' are disjoint since W and I are disjoint and ϕ is univalent. Also since $W \cup I = \overline{\mathbb{C}}$ it easily follows that $W' \cup I' = \phi(U) = D$.

By the forward invariance of $W \cap U$ under f we see that

$$F(W') = F \circ \phi(W \cap U) = \phi \circ f(W \cap U) \subset \phi(W \cap U) = W'. \tag{4.14}$$

Similarly we get

$$F(I') \subset I'. \tag{4.15}$$

Since I' and W' are disjoint and forward invariant under F, and since $I' \cup W' = D$, we see that

$$F^{-1}(I') \cap D \subset I', \tag{4.16}$$

$$F^{-1}(W') \cap D \subset W'. \tag{4.17}$$

Note that in the same way as we obtained the results for F we get

$$G(W') \subset W', \tag{4.18}$$

$$G(I') \subset I', \tag{4.19}$$

$$G^{-1}(I') \cap D \subset I', \tag{4.20}$$

$$G^{-1}(W') \cap D \subset W'. \tag{4.21}$$

Lemma 4.18 ([32], Lemma 8)**.** *If $G(z) = az^l$ with $|a| = 1$, then $J(f) = J(g)$.*

Proof. The proof relies on the use of Green's functions. It is well known that the unbounded components F_∞ and G_∞ support Green's functions with pole at ∞ which we will denote by G_f and G_g respectively. It is also well known that on U we have

$$G_f(z) = -\log|\phi(z)|$$

since ϕ is a map which conjugates f to $z \mapsto z^k$ (see [5], p. 206).

Since for $\psi(z) = bz$ where $b^{l-1} = a$, the function $\psi \circ \phi$ conjugates g in U to $z \mapsto z^l$, we get in U,

$$G_g(z) = -\log|\psi \circ \phi(z)| = -\log|b\phi(z)| = -\log|\phi(z)|$$

where the last equality uses the fact that $|a| = 1$, and so $|b| = 1$.

Hence $G_f = G_g$ in U. Since G_f and G_g are each harmonic away from ∞ we get that $G_f = G_g$ on the unbounded component C of $F_\infty \cap G_\infty$.

We claim that this implies that $J(f) = J(g)$. Assuming that $J(f) \neq J(g)$, we see by Remark 4.6 that there exists a point which lies in the Julia set of one function, yet in the unbounded component of the Fatou set of the other function. Let us therefore suppose that $z_0' \in J(g) \cap F_\infty$. Let γ be a path in F_∞ connecting z_0' to ∞. We see that γ must intersect ∂C somewhere, say at z_0. Since $z_0 \in F_\infty \cap \partial C$ we get $z_0 \in \partial G_\infty = J(g)$.

We may select a sequence $z_n \in C$ such that $z_n \to z_0$. Since z_0 lies on the boundary of the domain of the Green's function G_g, i.e., $z_0 \in J(g)$, we have $G_g(z_n) \to 0$ (see [5], p.207). Since z_0 lies in the domain of the Green's function G_f we see that $G_f(z_n) \to G_f(z_0) > 0$. We cannot have both of these happen since $G_f(z_n) = G_g(z_n)$ and so we conclude that $J(g) \cap F_\infty = \emptyset$. Hence we conclude that $J(f) = J(g)$. $\square$

We now are able to prove Theorem 4.2.

Proof of Theorem 4.2. Consider whether or not I has nonempty interior. If $I^o \neq \emptyset$, then by Lemma 4.1 we get $I = \overline{\mathbb{C}}$.

If $I^o = \emptyset$, then Lemma 4.15 implies that the set W' is a union of open annuli centered at the origin and hence I' is a union of circles centered at the origin. Since $I^o = \emptyset$, the set I' has empty interior.

Since we know by Remark 4.7 that there exists a sequence of points in I tending to infinity when $J(f) \neq J(g)$, also I' must contain a corresponding sequence of circles tending to zero. By Lemma 4.16 we see that the function G is of the form

$$G(z) = az^l$$

for some non-zero complex number a.

By considering the set I', we see that Lemma 4.17 implies that $|a| = 1$. We see that Lemma 4.18 then implies $J(f) = J(g)$. □

5. An Invariant measure for finitely generated rational semigroups

As stated in Proposition 1.4, the Julia set of a rational semigroup is contained in the set of accumulation points of the backward orbit of any non-exceptional point a. When $a \in J(G)$, we have in fact that $J(G) = \overline{O^-(a)}$. When G is finitely generated, this serves as the basis for a computer algorithm for making an approximate picture of the Julia set. See [26], pp. 35–38 and [24], Appendix E for a discussion of the single generator case. Many are familiar with these pictures in the classical cyclic semigroup case. Experimental evidence indicates that while this procedure often does yield a believable picture, there are certain phenomena which prevent this finite process from giving a complete picture.

Consider the following construction. Let f be a rational function of with $\deg f = d \geq 2$ and let $a \in \overline{\mathbb{C}} \setminus E(f)$. Then for $n \geq 1$ define

$$\mu_a^n = \frac{1}{d^n} \sum_{f^n(z)=a} \delta_z$$

where δ_z is the unit point mass measure at z and the sum is taken over all solutions to $f^n(z) = a$ counted according to multiplicity. As there are exactly d^n such solutions, the measure μ_a^n is a probability measure. Thus μ_n^a is the probability measure evenly distributed (up to multiplicity) over the preimages of a under f^n. The following result was established by Lyubich in [20].

Theorem 5.1 (Lyubich)**.** *The measures μ_n^a converge weakly to a unique regular Borel probability measure $\mu = \mu_f$ independently and locally uniformly in $a \in \overline{\mathbb{C}} \setminus E(f)$. The closed support of μ is $J(f)$. Further, μ satisfies the following properties. For any Borel set E, we have*

$$\mu(E) = \mu\left(f^{-1}(E)\right).$$

We further have

$$\mu(E) \geq \frac{1}{d}\mu\left(f(E)\right)$$

where equality holds if f is injective on E.

Recall that a sequence of measures μ_n on a space X converges weakly to the measure μ if $\int \phi \, d\mu_n \to \int \phi \, d\mu$ for every continuous function ϕ on X.

Roughly, the measure μ has the largest concentration of its support on the part of the Julia set that is best approximated by the above mentioned computer scheme. There is much known about this measure μ. We list a few facts here. For a polynomial f, Brolin showed in [9] that the measure μ is the harmonic measure of $J(f)$ as seen from infinity. Lyubich showed that μ is the measure of maximal entropy for the function f.

Results similar to those of Lyubich were established independently by A. Freire, A. Lopes, and R. Mañé in [19] and [21].

For a finitely generated rational semigroup G, a similar computer scheme can be implemented to create an approximate picture of $J(G)$. As in the cyclic case there are observable instances where no reasonable number of iterations in the computer algorithm will fill in large areas known to belong to the Julia set. For example, consider $G = \langle f, g\rangle$ where $f(z) = z^2 + 2z$ and $g(z) = z^2 + z/2$. It is easy to check that -1 is a superattracting fixed point for f and that $g(-1) = 1/2$ is a repelling fixed point for g. Hence the full component of $N(f)$ containing -1, which in this case is the disk of radius 1 centered at -1, is contained in $J(G)$. See Corollary 2.1. However Figure 1 indicates a large gap in the picture near -1, and the same gap appears in very high numbers of iterations in the program.

A natural question is to what extent does Lyubich's result generalize to rational semigroups? It turns out that the results substantially go through but with some important differences.

The discussion below is influenced by Steinmetz's presentation of Lyubich's result in [33].

Let $G = \langle f_1, \dots, f_k\rangle$ be a finitely generated rational semigroup with $\deg f_j = d_j \geq 2$. We remark that in this setting the semigroup has a "best" generating set. We say that a generating set is **minimal** if no

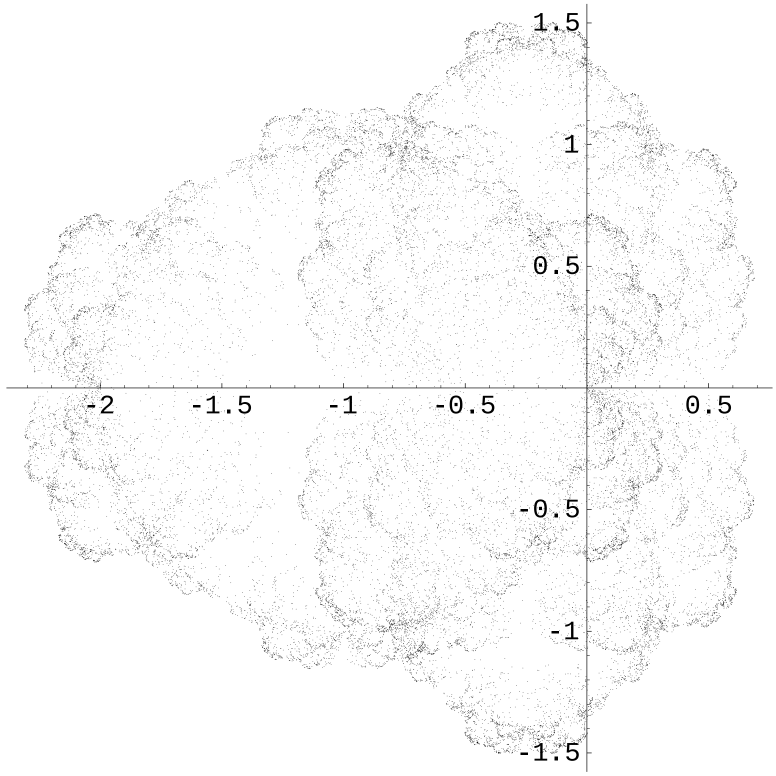

FIGURE 1. Julia set of $\langle z^2 + 2z, z^2 + z/2\rangle$

generator can be expressed as a word in the remaining generators. We have the following result, whose proof is left as an exercise.

Lemma 5.1 ([7], Lemma 2)**.** *Every finitely generated rational semigroup G such that $\deg g \geq 2$ for all $g \in G$ has a unique minimal generating set.*

While the statements of the following results hold for an arbitrary generating set, the conditions of some are likely only to be satisfied by the minimal generating set, and hence from now on, when we refer to the generating set for a finitely generated rational semigroup, we will assume that it is the minimal generating set.

Let $a \in \overline{\mathbb{C}} \setminus E(G)$. For any integer $n \geq 1$ we define

$$\mu_n^a = \frac{1}{d^n} \sum_{\substack{g(z)=a \\ l(g)=n}} \delta_z \tag{5.1}$$

where here $d = d_1 + \cdots + d_k$ and the sum is taken over all solutions, counted according to multiplicity, to the equations $g(z) = a$ as g ranges over all words in G of length n. Since there are exactly d^n such solutions, the measure μ_n^a is a probability measure. (Note that in this sum we may have multiple words representing the same group element. This is fine, as this mirrors how the computer algorithm would work. In most cases this will not be an issue as an arbitrary rational semigroup is likely to be free on its generating set.)

The results we wish to discuss follow.

Theorem 5.2 ([7], Theorem 1)**.** *Let $G = \langle f_1, \dots, f_k \rangle$ be a finitely generated rational semigroup with $\deg f_j = d_j \geq 2$ and $d = d_1 + \cdots + d_k$. Then the measures μ_n^a defined by (5.1) converge weakly to a regular Borel probability measure $\mu = \mu_G$ independently of and locally uniformly in $a \in \overline{\mathbb{C}} \setminus E(G)$. The closed support of μ is $J(G)$. Further, μ satisfies the following inequalities. For any Borel set $E \subset \overline{\mathbb{C}}$,*

$$\begin{aligned} \mu(E) + \sum_{i=1}^{k} \frac{1}{d}\mu\left(f_i\left(\bigcup_{j=1}^{k} f_j^{-1}(E)\right) \setminus E\right) &\leq \mu\left(\bigcup_{j=1}^{k} f_j^{-1}(E)\right) \\ &\leq \mu(E) + \sum_{i=1}^{k} \frac{d_j}{d}\mu\left(f_i\left(\bigcup_{j=1}^{k} f_j^{-1}(E)\right) \setminus E\right) \end{aligned} \tag{5.2}$$

and also

$$\mu(E) \geq \frac{1}{d}\sum_{j=1}^{k} \mu\left(f_j(E)\right). \tag{5.3}$$

We also have the following corollary, indicating conditions that guarantee that the measure μ_G is invariant under the generating set of G.

Corollary 5.1 ([7], Corollary 1)**.** *The equalities*

$$\mu(E) = \mu\left(\bigcup_{j=1}^{k} f_j^{-1}(E)\right) = \sum_{j=1}^{k} \mu(f_j^{-1}(E)) \tag{5.4}$$

hold for every Borel set $E \subset J(G)$ if for all integers $1 \leq i, j \leq k$, $i \neq j$,

$$\mu\left(f_i^{-1}(J(G)) \cap f_j^{-1}(J(G))\right) = 0.$$

5.1. **Discussion of the inequalities** (5.2)**.** Given a Borel set $E \subset \overline{\mathbb{C}}$, a point $a \in \overline{\mathbb{C}} \setminus E(G)$, and a positive integer n, the measure $\mu_n^a(E)$ is the proportion of the total number of preimages of a under length n words of G that lie in E. Consider the measure $\mu_{n+1}^a\left(\bigcup_{j=1}^{k} f_j^{-1}(E)\right)$.

Each preimage of a under a length n word lying in E has a total of d preimages under the generators f_j that lie in $\bigcup_{j=1}^k f_j^{-1}(E)$. Thus from (5.1) we see that

$$\mu_n^a(E) \leq \mu_{n+1}^a\left(\bigcup_{j=1}^k f_j^{-1}(E)\right) \tag{5.5}$$

However, it is possible that some preimage of a under a length n word that lies outside of E will itself have a preimage in $\bigcup_{j=1}^k f_j^{-1}(E)$ under some generator, assuming that k, the number of generators, is at least 2. Hence the inequality in (5.5) may be strict. The sums found in (5.2) represent a lower and upper bound on this error for the limiting measure.

5.2. **Discussion and Proof of Corollary 5.1.** We first note that as it is possible that $f_j^{-1}(N(G)) \cap J(G) \neq \emptyset$ for some generator f_j, we cannot expect (5.4) to hold for every Borel set E, since $\mu(N(G)) = 0$ and the μ-measure of any open set meeting $J(G)$ is positive. Thus some restriction is necessary.

We give a proof of Corollary 5.1, assuming the truth of Theorem 5.2.

Examining the inequalities (5.2), we see that $\mu(E) = \mu\left(\bigcup_{j=1}^k f_j^{-1}(E)\right)$ if and only if $\mu\left(f_i\left(\bigcup_{j=1}^k f_j^{-1}(E)\right) \setminus E\right) = 0$ for $i = 1, \ldots, k$. Assume that

$$\mu\left(f_i^{-1}(J(G)) \cap f_j^{-1}(J(G))\right) = 0$$

for all $i \neq j$. Hence given any Borel subset $E \subset J(G)$, and $l \in \{1, \ldots, k\}$, it follows that we need to show

$$\mu\left(f_l\left(\bigcup_{j=1}^k f_j^{-1}(E)\right) \setminus E\right) = 0. \tag{5.6}$$

It suffices to show that

$$\left(f_l\left(\bigcup_{j=1}^k f_j^{-1}(E)\right) \setminus E\right) \cap J(G)$$

has μ-measure 0. However,

$$\left(f_l\left(\bigcup_{j=1}^k f_j^{-1}(E)\right) \setminus E\right) \cap J(G) \subset \bigcup_{j\neq l}\left(f_l(f_j^{-1}(E)) \cap J(G)\right)$$

since any point in $f_l\left(\bigcup_{j=1}^k f_j^{-1}(E)\right) \setminus E$ must lie in some $f_l(f_j^{-1}(E))$ for $j \neq l$. Any point in $f_l(f_j^{-1}(E)) \cap J(G)$ is the image under f_l of a point in $f_j^{-1}(E) \cap f_l^{-1}(J(G))$. Since $E \subset J(G)$, we have shown that

$$\bigcup_{j \neq l} \left(f_l(f_j^{-1}(E)) \cap J(G) \right) \subset \bigcup_{j \neq l} f_l \left(f_j^{-1}(J(G)) \cap f_l^{-1}(J(G)) \right).$$

We are assuming that $f_j^{-1}(J(G)) \cap f_l^{-1}(J(G))$ has μ-measure 0 for all $j \neq l$. By examining the inequality (5.3) of Theorem 5.2 we may conclude that if $\mu(F) = 0$ for any Borel set F, then $\mu(g(F)) = 0$ for all $g \in G$. Hence the set

$$\bigcup_{j \neq l} f_l \left(f_j^{-1}(J(G)) \cap f_l^{-1}(J(G)) \right)$$

also has μ-measure 0. The above inclusions now imply (5.6) and so we also have $\mu(E) = \mu\left(\bigcup_{j=1}^k f_j^{-1}(E)\right)$ as claimed. It is an easy exercise to show that

$$\mu\left(\bigcup_{j=1}^{k} f_j^{-1}(E)\right) = \sum_{j=1}^{k} \mu(f_j^{-1}(E))$$

under the assumptions of the theorem. □

Remark 5.1. We believe that the sufficient conditions of the corollary are also necessary, but as of this writing a complete proof has not been established.

Example 5.1. Let $f(z) = z^2$ and let $g(z) = z^2/a$ for some $a > 1$. Let $G = \langle f, g \rangle$. It is shown in [15], Example 1, that $J(G) = \{z : 1 \leq |z| \leq a\}$. We explicitly construct μ for this semigroup and show that μ satisfies the conditions of Corollary 5.1.

The preimages of $z_0 = -\sqrt{a} = e^{\log(\sqrt{a}) + i\pi}$ under f and g are

$$\{e^{\frac{1}{2}\log(\sqrt{a}) + i\frac{\pi}{2}}, e^{\frac{1}{2}\log(\sqrt{a}) + i\frac{3\pi}{2}}, e^{\frac{3}{2}\log(\sqrt{a}) + i\frac{\pi}{2}}, e^{\frac{3}{2}\log(\sqrt{a}) + i\frac{3\pi}{2}}\}.$$

We inductively calculate the preimages of z_0 under length n words. Assume that the preimages of z_0 under the length n words of G are

$$z_{j,k}^n = \exp\left(\frac{2j-1}{2^n}\log\left(\sqrt{a}\right) + i\frac{(2k-1)\pi}{2^n}\right)$$

for $j, k = 1, \ldots, 2^n$. The preimages under f and g of a given point $z_{j,k}^n$ are

$$\left\{\exp\left(\frac{2j-1}{2^{n+1}}\log\left(\sqrt{a}\right) + i\frac{(2k-1)\pi}{2^{n+1}}\right),\right.$$

$$\exp\left(\frac{2j-1}{2^{n+1}}\log\left(\sqrt{a}\right) + i\frac{(2k-1)\pi}{2^{n+1}} + i\pi\right),$$

$$\exp\left(\frac{2j-1}{2^{n+1}}\log\left(\sqrt{a}\right) + \log\left(\sqrt{a}\right) + i\frac{(2k-1)\pi}{2^{n+1}}\right),$$

$$\exp\left(\frac{2j-1}{2^{n+1}}\log\left(\sqrt{a}\right)+\log\left(\sqrt{a}\right)i\frac{(2k-1)\pi}{2^{n+1}}+i\pi\right)\bigg\}.$$

After reordering, induction yields that the preimages of z_0 under the length n words of G for all $n \geq 1$ are

$$z_{j,k}^n = \exp\left(\frac{2j-1}{2^n}\log\left(\sqrt{a}\right)+i\frac{(2k-1)\pi}{2^n}\right)$$

for $j,k = 1,\dots,2^n$. Thus

$$\mu_n^{-\sqrt{a}} = \frac{1}{4^n}\sum_{j,k=1}^{2^n}\delta_{z_{j,k}^n}.$$

Let

$$R = \{w = u + iv : 0 \leq u \leq \log a, 0 \leq v < 2\pi\}.$$

We think of R as the set $\log(J(G))$. Let

$$w_{j,k}^n = \log(z_{j,k}^n) = \frac{2j-1}{2^{n+1}}\log(a) + i\frac{(2k-1)\pi}{2^n}$$

and let

$$\bar{\mu}_n = \frac{1}{4^n}\sum_{j,k=1}^{2^n}\delta_{w_{j,k}^n}.$$

Note that for any set $E \subset J(G)$, we have that $\mu_n^{-\sqrt{a}}(E) = \bar{\mu}_n(\log(E))$. The measures $\bar{\mu}_n$ converge weakly to $m/(2\pi\log(a))$ where m is Lebesgue measure restricted to R. To see this one need just consider the definition of the Riemann integral. This implies that

$$\mu(E) = \frac{m(\log(E))}{2\pi\log(a)}$$

where μ is the measure from the conclusion of Theorem 5.2. Since

$$f^{-1}(J(G)) \cap g^{-1}(J(G)) = \{z : |z| = \sqrt{a}\},$$

we have $\mu\left(f^{-1}(J(G)) \cap g^{-1}(J(G))\right) = 0$ and so G satisfies the conditions of Corollary 5.1.

We further remark that it is easy to construct examples of rational semigroups $G = \langle f_1,\dots,f_k\rangle$ where $f_i^{-1}(J(G)) \cap f_j^{-1}(J(G)) = \emptyset$ for all $i \neq j$. Such G clearly satisfy the conditions of Corollary 5.1.

5.3. **Proof of Theorem 5.2.** We break the proof up into several parts, dealing with each of the statements of Theorem 5.2 separately.

5.4. **Proof of the weak convergence of μ_n^a.** Consider the Banach space $C(K)$ of continuous real valued functions on a compact set $K \subset \overline{\mathbb{C}} \setminus E(G)$ with norm $\|\phi\| = \max\{|\phi(z)| : z \in K\}$. For our purposes it will suffice to consider such K where K contains at least three points and where $f_j^{-1}(K) \subset K$ for each generator f_j. These assumptions guarantee that $g^{-1}(K) \subset K$ for all $g \in G$ and hence $J(G) \subset K$ by Remark 1.1.

For $z \in K$, define the function

$$(T\phi)(z) = \int_K \phi(\zeta) d\mu_1^z(\zeta) = \frac{1}{d}\sum_{j=1}^{d} \phi(z_j) \tag{5.7}$$

where the points z_j are the solutions of $f_i(w) = z$, listed according to multiplicity, for all i with $i = 1, \dots, k$.

The function $T\phi$ is continuous on the compact set K. For if $\epsilon > 0$ is given, we may choose $\delta_1 > 0$ so that if $q(a, b) < \delta_1$, where $q(\cdot, \cdot)$ is the chordal metric, and $a, b \in K$ then $|\phi(a) - \phi(b)| < \epsilon$ and we may further choose $\delta_2 > 0$ so that if $q(z, z') < \delta_2$ the solutions of the equations $f(w) = z$, $f(w) = z'$ may be ordered so that $q(z_j, z'_j) < \delta_1$. Then $|(T\phi)(z) - (T\phi)(z')| \le d^{-1}\sum_{j=1}^{d} |\phi(z_j) - \phi(z'_j)| < \epsilon$. The action of T on $C(K)$ is clearly linear, hence $T : C(K) \to C(K)$ is a linear operator. By considering $\phi \equiv 1$, it is immediate that the operator norm $\|T\|$ of T satisfies

$$\|T\| = \sup\{\|T\phi\| : \|\phi\| = 1\} = 1.$$

Hence T is a continuous linear operator from $C(K)$ to itself.

Recursively define $T^m\phi$ via $T^m\phi = T(T^{m-1}\phi)$. Then

$$(T^m\phi)(z) = \frac{1}{d^m}\sum_{j=1}^{d^m} \phi(z_j^m) = \int_K \phi(\zeta)\, d\mu_m^z(\zeta) \tag{5.8}$$

where here the points z_j^m are the solutions to the equations $g(w) = z$, listed according to multiplicity where g ranges over the length m words of G. We see this as follows. Assume 5.8 holds for $m - 1$. Then

$$(T^m\phi)(z) = (T(T^{m-1}\phi))(z) = \frac{1}{d}\sum_{i=1}^{d}(T^{m-1}\phi)(z_i^1) = \frac{1}{d}\sum_{i=1}^{d}\frac{1}{d^{m-1}}\sum_{k=1}^{d^{m-1}} \phi(z_{i,k}^{m-1})$$

where the points z_i^1 are the solutions to $f(w) = z$ under length one words of G and the points $z_{i,k}^{m-1}$ represent the solutions of the equations $g(w) = z_i^1$ where g ranges over the length $m - 1$ words of G, which thus in total also represent the solutions of $h(w) = z$ as h ranges over the length m words of G. Hence $(T^m\phi)(z) = d^{-m}\sum_{j=1}^{d^m}\phi(z_j^m) = \int_K \phi(\zeta)\,d\mu_m^z(\zeta)$ by induction.

5.5. **Relationships between ϕ and $T^m\phi$.** We establish some relationships between ϕ and $T^m\phi$ when $\|\phi\| \neq 0$. Recall that K is compact and backwards invariant under the generators of the semigroup. For a given integer $m \geq 1$, choose $z_0 = z_0(m) \in K$ so that $\|T^m\phi\| = |(T^m\phi)(z_0)|$. Then

$$\|T^m\phi\| = |(T^m\phi)(z_0)| \leq \frac{1}{d^m}\sum_{j=1}^{d^m}|\phi((z_0)_j)| \leq \|\phi\|$$

with equality if and only if $\phi((z_0)_j) = \pm\|\phi\|$ where the sign depends only on m and not on j. We remark that if ϕ is not identically $\|\phi\|$ or $-\|\phi\|$ on $J(G)$, then there is an integer m such that $\|T^l\phi\| < \|\phi\|$ for all $l \geq m$. We see this as follows. If there is a point $w \in J(G)$ and a neighborhood U of w such that $|\phi(u)| < \|\phi\|$ for all $u \in U \cap K$, then, since $w \in J(G)$, Proposition 1.6 implies that there exists an integer N such that for all $n \geq N$, we have $K \subset \bigcup_{l(g)=n} g(U)$. In particular, if $m \geq N$ there is a solution $(z_0)_j$ to $g(w) = z_0$ in U for some word g of length m. Similarly if there are points $w_1, w_2 \in J(G)$ with $\phi(w_1) = \|\phi\|$ and $\phi(w_2) = -\|\phi\|$, then there are disjoint neighborhoods U_i of w_i on which $|\phi|$ is close to $\pm\|\phi\|$ respectively, such that for each i the equation $g(z) = z_0$ has a solution in U_i for some word g of length m. Hence in either case we see that if ϕ is not identically $\|\phi\|$ or $-\|\phi\|$ on $J(G)$, then $\|T^m\phi\| < \|\phi\|$ for this and all larger m. Clearly for constant ϕ, we have $\|T^m\phi\| = \|\phi\|$ for all m.

Similarly we see that the minimal value of $T^m\phi$ is nondecreasing in m as follows. Choose $z_0 \in K$ so that $\min\{(T\phi)(z) : z \in K\} = (T\phi)(z_0)$. Then

$$\min\{(T\phi)(z) : z \in K\} = (T\phi)(z_0) = \frac{1}{d}\sum_{j=1}^{d}\phi((z_0)_j) \geq \min\{\phi(z) : z \in K\}.$$

Further, equality holds if and only if $\phi((z_0)_j) = \min\{\phi(z) : z \in K\}$ for all j. Then by induction, for all $m \geq 1$, we have that

$$\min\{(T^{m+1}\phi)(z) : z \in K\} \geq \min\{(T^m\phi)(z) : z \in K\} \geq \min\{\phi(z) : z \in K\}.$$

Our goal is to show that the sequence $T^m\phi$ converges uniformly on K to a constant function l_ϕ. To do this we need the following lemma.

Lemma 5.2. *For every $\phi \in C(K)$, the family $\{T^m\phi : m = 1, 2, \ldots\}$ is equicontinuous.*

The proof of Lemma 5.2, which is somewhat technical, comes later. For now assume that the lemma has been established.

5.6. **Convergence of $T^m\phi$.** Given the lemma, we show the uniform convergence of the functions $T^m\phi$ as follows. The Arzelà–Ascoli Theorem gives that every subsequence of $T^m\phi$ has a uniformly convergent subsequence which must converge to a continuous function. Thus assuming that $T^{n_k}\phi$ converges uniformly on K to the continuous function ψ and by passing to a further subsequence, if necessary, we may assume that $m_k = n_{k+1} - n_k \to \infty$ and that $T^{m_k}\psi$ converges uniformly to a continuous function χ. Then

$$\|T^{n_{k+1}}\phi - T^{m_k}\psi\| = \|T^{m_k}(T^{n_k}\phi - \psi)\| \le \|T^{n_k}\phi - \psi\| \to 0.$$

Thus $\chi = \psi$, i.e., $T^{m_k}\psi \to \psi$. It follows from previous considerations that for all m, $\|T^m\psi\| \le \|\psi\|$. Also for $m_k \ge m$,

$$\|\psi\| = \lim_{m_k\to\infty} \|T^{m_k}\psi\| = \lim_{m_k\to\infty} \|T^{m_k-m}(T^m\psi)\| \le \|T^m\psi\|$$

hence $\|T^m\psi\| = \|\psi\|$ for all m. We may then conclude, as noted above, that either $\psi \equiv \|\psi\|$ or $\psi \equiv -\|\psi\|$ on $J(G)$. For convenience, assume that $\psi \equiv \|\psi\|$ on $J(G)$. We will show that in fact, $\psi \equiv \|\psi\|$ on all of K. Assume that this is not the case. (We are necessarily now assuming that $K \setminus J(G) \neq \emptyset$.) Then

$$c = \min\{\psi(z) : z \in K\} < \|\psi\|.$$

Recall that T does not decrease the minimum value. Hence we have that

$$\min\{(T^{m_k}\psi)(z) : z \in K\}$$

is nondecreasing in k. This implies that since $T^{m_k}\psi \to \psi$, we must have that

$$\min\{(T^{m_k}\psi)(z) : z \in K\} = c$$

for all k. Choose $z_k \in K$ so that $T^{m_k}(z_k) = c$. First assume that infinitely many z_k are the same point. By passing to a subsequence if necessary, we may then assume that $z_k = z_0$ for all k. As we have seen before, $(T^{m_k}\psi)(z_0) = \min\{\psi(z) : z \ \in K\} = c$ if and only if $\psi(z) = c$ on every preimage of z_0 under all words of G of length m_k. Recall that we are assuming $\psi \equiv \|\psi\|$ on $J(G)$. Let U be a neighborhood of $J(G)$ so that $\psi(z) > c$ for all $z \in U \cap K$. By Proposition 1.6 there exists an integer N such that $z_0 \in \bigcup_{l(g)=n} g(U)$ for all integers $n \ge N$. In particular, z_0 has a preimage in U under some length m_k word when $m_k \ge N$. Hence $(T^{m_k}\psi)(z_0) > c$ which is a contradiction. For the next case, since K is compact, by passing to a subsequence if necessary, we may assume that $z_k \to z_0 \in K$. Take a small closed neighborhood D

of z_0 in K. Again by Proposition 1.6 there exists an integer N such that

$$D \subset \bigcup_{l(g)=n} g(U)$$

for $n \geq N$. Assume that $z_k \in D$ for all $m_k \geq N_1 \geq N$. Then z_k has a preimage in U under some length m_k word whenever $m_k \geq N_1$. Again this contradicts the assumption that $T^{m_k}(z_k) = c$. Thus $\psi \equiv l_\phi$ is constant. Further, for $m > n_k$,

$$\|T^m\phi - l_\phi\| = \|T^{m-n_k}(T^{n_k}(\phi - l_\phi))\| \leq \|T^{n_k}(\phi - l_\phi)\| \to 0$$

so we can see that $T^m\phi \to l_\phi$ uniformly on K as $m \to \infty$, as claimed.

5.7. **The existence and regularity of $\boldsymbol{\mu}$.** For functions ϕ and ϕ' continuous on K, we have

$$|l_\phi - l_{\phi'}| = \lim_{m\to\infty} \|T^m(\phi - \phi')\| \leq \|\phi - \phi'\|,$$

l may be considered as a continuous linear functional on $C(K)$ and by the Riesz Representation Theorem may be represented uniquely in the form

$$l_\phi = \int_K \phi(\zeta) d\mu_K(\zeta)$$

for a regular Borel measure μ_K.

Hence for all $\phi \in C(K)$ and for all $a \in K$,

$$\lim_{m\to\infty} \int_K \phi(\zeta)\, d\mu_m^a(\zeta) = \int_K \phi(\zeta)\, d\mu_K(\zeta)$$

where the convergence is uniform in $a \in K$. However, as $J(G) \subset K$ for all sets K under consideration, we see that $\mu_K = \mu$ is independent of K and so can be considered as a measure on $\overline{\mathbb{C}}$. Setting $\phi \equiv 1$, we see that μ is a probability measure. This will complete our proof that the measures μ_n^a converge weakly to the regular Borel probability measure μ, independently of and localy uniformly in a, once we have proven Lemma 5.2.

5.8. **Proof of Lemma 5.2.** We now proceed with a proof of Lemma 5.2. It suffices to prove local equicontinuity.

Let C_j denote the set of critical points of f_j. Then $CV_1 = \bigcup_{j=1}^k f_j(C_j)$ is the set of the critical values of the length one words of G,

$$CV_2 = \bigcup_{j_1,j_2=1}^{k} f_{j_2}(f_{j_1}(C_{j_1})) \cup \bigcup_{j=1}^{k} f_j(C_j) = \bigcup_{j=1}^{k} f_j(CV_1) \cup CV_1$$

is the set of the critical values of the length two words of G and in general,

$$CV_n = \bigcup_{j=1}^{k} f_j(CV_{n-1}) \cup CV_{n-1} = \bigcup_{m=1}^{n} \bigcup_{j_1,j_2,\dots,j_m=1}^{k} (f_{j_m} \circ \cdots \circ f_{j_1})(C_{j_1})$$

is the set of the critical values of the length n words of G.

Let U be a simply connected domain. Assume that CV_l is disjoint from U. Then for $n \le l$ there are exactly d^n single valued analytic inverse functions on U from length n elements of G. In general, let $\sigma_n(U)$ denote the total number of single valued analytic inverse functions on U from the length n words of G. Let τ denote the total number of distinct critical values of the functions f_j. We claim that for all n,

$$d^n - \sigma_n \le \tau k^l \sum_{j=1}^{n-l} d^j k^{n-l-j}. \tag{5.9}$$

The claim is clearly true for $n \le l$. For a given function $f_{i_1} \circ \cdots \circ f_{i_n}$, let $\sigma_{(i_1,\dots,i_n)}(U)$ denote the number of its single valued analytic inverse functions on U. Then $\sigma_n(U) = \sum_{i_1,\dots,i_n=1}^{k} \sigma_{(i_1,\dots,i_n)}(U)$. The images of U under the inverses of $f_{i_1} \circ \cdots \circ f_{i_n}$ are mutually disjoint and simply connected. Then at least $\sigma_{(i_1,\dots,i_n)} - \tau$ of them contain no critical value of any f_i. Thus $f_{i_1} \circ \cdots \circ f_{i_n} \circ f_i$ has $\sigma_{(i_1,\dots,i_n,i)}(U) \ge d_i(\sigma_{(i_1,\dots,i_n)}(U) - \tau)$ inverses on U. In particular,

$$\sigma_{n+1} = \sum_{i=1}^{k} \sum_{i_1,\dots,i_n}^{k} \sigma_{(i_1,\dots,i_n,i)} \ge \sum_{i=1}^{k} \sum_{i_1,\dots,i_n}^{k} d_i(\sigma_{(i_1,\dots,i_n)} - \tau) = d(\sigma_n - k^n\tau).$$

Then by induction,

$$\sigma_{n+1} \ge d(\sigma_n - k^n\tau) \ge d(d^n - \tau k^l \sum_{j=1}^{n-l} d^j k^{n-l-j} - k^n\tau)$$

$$= d^{n+1} - \tau k^l \sum_{j=1}^{n+1-l} d^j k^{n+1-l-j}$$

which gives the claim. Note that this estimate depends on U only in the fact that U misses CV_l.

Let $\epsilon > 0$ be given. There is a positive integer L depending on ϵ so that if U is a simply connected domain that misses CV_L, then for all $n \ge 1$,

$$\frac{d^n - \sigma_n(U)}{d^n} \le \tau \left(\frac{k}{d}\right)^L \sum_{j=1}^{n-L} \left(\frac{k}{d}\right)^{n-L-j} \le \tau \left(\frac{k}{d}\right)^L \sum_{i=0}^{\infty} \left(\frac{k}{d}\right)^i$$

$$= \tau \left(\frac{k}{d}\right)^L \frac{d}{d-k} < \frac{\epsilon}{4\|\phi\|},$$

recalling that $d \geq 2k$.

Case 1. Assume that $z_0 \in K$ and $z_0 \notin CV_L$. Let $U = \Delta(z_0, 2\rho)$ be a chordal neighborhood of z_0 that misses CV_L. Let $h_1^n, \dots, h_{\sigma_n}^n$ denote the single valued analytic inverses on U from the length n elements of G. It is shown in [17], Corollary 2.2, that the collection $\{h_j^n : n \geq 1, 1 \leq j \leq \sigma_n\}$ forms a normal family on U. Hence given $\eta > 0$, we may choose $\delta > 0$ so that if $z, \zeta \in K$, $q(z,\zeta) < \delta$, $q(z, z_0) < \rho$, and $q(\zeta, z_0) < \rho$, then $q(h_j^n(z), h_j^n(\zeta)) < \eta$ for all j and n. Further assume that η was chosen so that $|\phi(u) - \phi(v)| < \epsilon/2$ whenever $q(u,v) < \eta$ and $u, v \in K$.

Thus when $z, \zeta \in K$, $q(z,\zeta) < \delta$, $q(z, z_0) < \rho$, and $q(\zeta, z_0) < \rho$, we have

$$|(T^n\phi)(z) - (T^n\phi)(\zeta)| = |d^{-n} \sum_{j=1}^{d^n} (\phi(z_j) - \phi(\zeta_j))|$$

$$\leq d^{-n} \sum_{j=1}^{\sigma_n} |\phi(h_j^n(z)) - \phi(h_j^n(\zeta))| + 2\|\phi\| \frac{d^n - \sigma_n}{d^n} < \epsilon.$$

We make the remark that this technique will work for $\epsilon' < \epsilon$ if one looks at points in $K \setminus CV_M$ for a sufficiently large choice of $M > L$.

Case 2. Now assume that $z_0 \in K \cap CV_L$. Choose $M > L$ so that for all $n \geq 1$, as in Case 1,

$$|(T^n\phi)(u) - (T^n\phi)(v)| < \epsilon/2$$

for points u, v in suitable neighborhoods of points in $K \setminus CV_M$. Recall that $CV_L \subset CV_M$ whenever $L < M$. As $K \cap CV_M$ is finite, one can show that there exists an integer κ such that for any $a \in K$, there exists a word g of length κ such that $g(z) = a$ has at least one solution outside of $K \cap CV_M$. Recall that $g^{-1}(K) \subset K$ for all $g \in G$. We wish to find, as in Case 1, a $\delta > 0$ such that if $u, v \in K$, $q(u, z_0) < \delta/2$, and $q(v, z_0) < \delta/2$, then

$$|(T^n\phi)(u) - (T^n\phi)(v)| < \epsilon$$

for all $n \geq 1$.

Given a positive integer n, let S_n denote the set of solutions, listed according to multiplicity, of $g(z) = z_0$ where g ranges over the length $n\kappa$ words of G. The cardinality of S_n is $d^{n\kappa}$. We inductively divide the sets S_n as follows.

The integer κ was chosen such that there is at least one solution in S_1 outside of $K \cap CV_M$. Let $S_1(1)$ consist of this single point and let $S_1(2)$ consist of the remaining $d^\kappa - 1$ points. For $n = 2$, let $S_2(1)$ be the d^κ preimages of $S_1(1)$, let $S_2(2)$ be the $d^\kappa - 1$ preimages under length κ words of the points in $S_1(2)$ guaranteed to be outside of $K \cap CV_M$, and let $S_2(3)$ be the remaining $(d^\kappa - 1)^2$ points in S_2. Note that $d^\kappa + (d^\kappa - 1) + (d^\kappa - 1)^2 = d^{2\kappa}$. Suppose that the sets S_l have been broken into sets $S_l(j)$ for $1 \le l \le n-1$ and $1 \le j \le l+1$ where $S_l(j)$ contains $d^{(l-j)\kappa}(d^\kappa - 1)^{j-1}$ points for $j = 1, \dots, l$ and $S_l(l+1)$ contains $(d^\kappa - 1)^l$ points. In particular, we assume that for $j = 1, \dots, l-1$, $S_l(j)$ consists of the preimages of the set $S_j(j)$ under words of length $(l-j)\kappa$. We also assume that $S_l(l)$ consists of the preimages of $S_{l-1}(l)$ under length κ words guaranteed to be outside of $K \cap CV_M$. We then define $S_n(j)$ for $1 \le j \le n+1$ as follows. For $j = 1, \dots, n-1$, the set $S_n(j)$ consists of the $d^{(n-j)\kappa}(d^\kappa - 1)^{j-1}$ points which are preimages of the set $S_{n-1}(j)$ under length κ words. Note that $S_n(j)$ consists of the preimage of $S_j(j)$ under all length $(n-j)\kappa$ words. We define the set $S_n(n)$ to consist of $(d^\kappa - 1)^{n-1}$ points which are preimages of $S_{n-1}(n)$ guaranteed to be outside of $K \cap CV_M$. The set $S_n(n+1)$ then consists of the remaining $(d^\kappa - 1)^n$ preimages of $S_{n-1}(n)$. Note that the sum of the number of elements in the sets $S_n(j)$ is given by

$$\sum_{j=1}^{n} d^{(n-j)\kappa}(d^\kappa - 1)^{j-1} + (d^\kappa - 1)^n = d^{n\kappa}.$$

We choose N large enough so that

$$\left(\frac{d^\kappa - 1}{d^\kappa}\right)^N < \frac{\epsilon}{4\|\phi\|}.$$

With N and κ now fixed we may choose $\eta > 0$ so that for all $n \ge 1$,

$$|(T^n\phi)(a) - (T^n\phi)(b)| < \frac{\epsilon}{2}$$

for all points $a, b \in K$ such that $q(a, w) < \eta$ and $q(b, w) < \eta$ whenever $w \in S_j(j)$ for $j = 1, \dots, N$, recalling that $S_j(j) \cap CV_M = \emptyset$ and hence we have the estimate from our choice of M.

Pick $u, v \in K$ such that $q(u, z_0) < \delta/2$ and $q(v, z_0) < \delta/2$ where δ is yet to be specified. Consider

(5.10)

$$|(T^{n+N\kappa}\phi)(u) - (T^{n+N\kappa}\phi)(v)| = d^{-N\kappa}\left|\sum_{j=1}^{d^{N\kappa}}(T^n\phi)(z_j^N(u)) - (T^n\phi)(z_j^N(v))\right|$$

where $z_j^N(u)$ are the solutions to $g(z) = u$ for length $N\kappa$ words and $z_j^N(v)$ is defined similarly. We now choose δ small enough so that for each $j = 1, \dots, N$, and each point w in $S_j(j)$ there is a solution to

$g(z) = u$ and $g(z) = v$ for some word g of length $j\kappa$ within q-distance η of w. We denote the collection of these pairs of preimages of u and v collectively by $\bar{S}_j(j)$. As the elements in $S_j(j)$ do not arise as preimages of points in $S_i(i)$ under length $(j-i)\kappa$ words for $i < j$, the number δ may be chosen so that the same may be said of the sets $\bar{S}_j(j)$. We now divide the sum in (5.10) as follows. Recall that $S_N(j)$ consists of the preimages of $S_j(j)$ under length $(N-j)\kappa$ words. Let $\bar{S}_N(j)$ denote the preimages under all length $(N-j)\kappa$ words of the pairs of points in $\bar{S}_j(j)$ for $j = 1, \ldots, N$ and let $\bar{S}_N(N+1)$ denote the remaining preimages of u and v under length $N\kappa$ words. The remark after the construction of the sets $\bar{S}_j(j)$ shows that the sets $\bar{S}_N(j)$ are well defined and account for all preimages of u and v under length $N\kappa$ words. We remark that the cardinality of $\bar{S}_l(j)$ was constructed to be the same as that of $S_l(j)$ for $1 \le l \le N$ and $1 \le j \le l+1$. We let $\sum_{\bar{S}_N(j)}$ denote the sum over the pairs in $\bar{S}_N(j)$. Note that for $j = 1, \ldots, N$,

$$\sum_{\bar{S}_N(j)} (T^n\phi)(z^N(u)) - (T^n\phi)(z^N(v))$$

$$= d^{(N-j)\kappa} \sum_{\bar{S}_j(j)} (T^{n+(N-j)\kappa}\phi)(z^j(u)) - (T^{n+(N-j)\kappa}\phi)(z^j(v))$$

where $z^j(u)$ stand for solutions of $g(z) = u$ for length $j\kappa$ words and $z^j(v)$ is defined similarly.

Thus if $u, v \in K$, $q(u, z_0) < \delta/2$ and $q(v, z_0) < \delta/2$, then

$$|(T^{n+N\kappa}\phi)(u) - (T^{n+N\kappa}\phi)(v)| \le d^{-N\kappa} \sum_{i=1}^{d^{N\kappa}} |(T^n\phi)(z_i^N(u)) - (T^n\phi)(z_i^N(v))|$$

$$= d^{-N\kappa} \sum_{j=1}^{N+1} \sum_{\bar{S}_N(j)} |(T^n\phi)(z^N(u)) - (T^n\phi)(z^N(v))|$$

$$= \sum_{j=1}^{N} d^{-j\kappa} \sum_{\bar{S}_j(j)} |(T^{n+(N-j)\kappa}\phi)(z^j(u)) - (T^{n+(N-j)\kappa}\phi)(z^j(v))|$$

$$+ d^{-N\kappa} \sum_{\bar{S}_N(N+1)} |(T^n\phi)(z^N(u)) - (T^n\phi)(z^N(v))|$$

$$\le \sum_{j=1}^{N} \frac{(d^\kappa - 1)^{j-1}}{d^{j\kappa}} \frac{\epsilon}{2} + \left(\frac{d^\kappa - 1}{d^\kappa}\right)^N 2\|\phi\|$$

$$< \frac{\epsilon}{2d^\kappa} \sum_{j=0}^{\infty} \left(\frac{d^\kappa - 1}{d^\kappa}\right)^j + \frac{\epsilon}{2} = \epsilon$$

for all $n \geq 1$. We then use the continuity of the functions $T^n\phi$ to get the estimate $|(T^m\phi)(u) - (T^m\phi)(v)| < \epsilon$ for $m = 1, \ldots, N\kappa$, shrinking δ if necessary.

Thus Cases 1 and 2 show that $\{T^n\phi\}$ is equicontinuous at each point of K. This completes the proof of Lemma 5.2 and hence establishes the weak convergence of the measures μ_n^a independently and locally uniformly in $a \in \overline{\mathbb{C}} \setminus E(G)$. The regularity of the limiting measure μ has also been established.

5.9. **Proof of the inequalities** (5.2)**.** The inequalities (5.2) of Theorem 5.2 are established as follows. For any continuous real valued function ϕ on $\overline{\mathbb{C}}$ and for $a \in J(G)$,

$$\int \phi(\zeta)\, d\mu(\zeta) = \lim_{m\to\infty} \int \phi(\zeta)\, d\mu_m^a(\zeta) = \lim_{m\to\infty} d^{-m} \sum_{\substack{g(z)=a \\ l(g)=m}} \phi(z)$$

(5.11)

$$= \lim_{m\to\infty} d^{-m} \sum_{j=1}^{k} \sum_{\substack{h(f_j(z))=a \\ l(h)=m-1}} \phi(z) = \sum_{j=1}^{k} \frac{d_j}{d} \lim_{m\to\infty} d^{-m+1} \sum_{\substack{h(z)=a \\ l(h)=m-1}} \phi_j(z)$$

$$= \sum_{j=1}^{k} \frac{d_j}{d} \lim_{m\to\infty} \int \phi_j(\zeta)\, d\mu_{m-1}^a(\zeta) = \sum_{j=1}^{k} \frac{d_j}{d} \int \phi_j(\zeta)\, d\mu(\zeta)$$

where $\phi_j(z) = d_j^{-1} \sum_{i=1}^{d_j} \phi(z_{i,j}(z))$ with the $z_{i,j}(z)$ ranging over the solutions of $f_j(w) = z$. As the solutions depend continuously on z, the function $\phi_j(z)$ is continuous for each $j = 1, \ldots, k$.

Remark 5.2. This shows that

$$\int \phi(\zeta)\, d\mu(\zeta) = \int (T\phi)(\zeta)\, d\mu(\zeta)$$

where T is the continuous linear operator defined on the space of continuous functions on $\overline{\mathbb{C}}$ defined by (5.7). Hence the measure μ is T^* invariant.

For any compact set K, let the functions ϕ^n be continuous and uniformly bounded, and let ϕ^n decrease to $\chi_{\bigcup_{j=1}^k f_j^{-1}(K)}$ as $n \to \infty$. Thus

$$\int \phi^n(\zeta)\, d\mu(\zeta) \to \int \chi_{\bigcup_{j=1}^k f_j^{-1}(K)}(\zeta)\, d\mu(\zeta) = \mu(\bigcup_{j=1}^{k} f_j^{-1}(K)). \tag{5.12}$$

Recall from (5.11) that

$$\int \phi^n(\zeta)\, d\mu(\zeta) = \sum_{j=1}^{k} \frac{d_j}{d} \int \phi_j^n(\zeta)\, d\mu(\zeta). \tag{5.13}$$

We remark that for any point $z \in \overline{\mathbb{C}}$, and for any $j = 1, \ldots, k$,

$$\lim_{n\to\infty} \phi_j^n(z) = \lim_{n\to\infty} d_j^{-1} \sum_{i=1}^{d_j} \phi^n(z_{i,j}(z)) = d_j^{-1} \sum_{i=1}^{d_j} \chi_{\bigcup_{l=1}^k f_l^{-1}(K)}(z_{i,j}(z)).$$

For any $z \in K$, and $j = 1, \ldots, k$, as each $z_{i,j}(z) \in \bigcup_{l=1}^k f_l^{-1}(K)$, it follows that $\phi_j^n(z)$ decreases to $\chi_K(z)$. Hence

$$\lim_{n\to\infty} \int_K \phi_j^n(\zeta)\, d\mu(\zeta) = \int_K \chi_K(\zeta)\, d\mu(\zeta) = \mu(K) \tag{5.14}$$

for each $j = 1, \ldots, k$.

For $z \in f_j\left(\bigcup_{i=1}^k f_i^{-1}(K)\right) \setminus K$ the number counted according to multiplicity of elements in $f_j^{-1}(z) \cap \bigcup_{i=1}^k f_i^{-1}(K)$ can be any integer between 1 and d_j. This integer could vary as z ranges over $f_j\left(\bigcup_{i=1}^k f_i^{-1}(K)\right) \setminus K$. Thus for such z,

$$\begin{aligned} &\frac{1}{d_j} \chi_{f_j\left(\bigcup_{i=1}^k f_i^{-1}(K)\right)\setminus K}(z) \\ &\qquad \le \lim_{n\to\infty} \phi_j^n(z) = d_j^{-1} \sum_{l=1}^{d_j} \chi_{\bigcup_{i=1}^k f_i^{-1}(K)}(z_{l,j}(z)) \\ &\qquad\qquad \le \chi_{f_j\left(\bigcup_{i=1}^k f_i^{-1}(K)\right)\setminus K}(z) \end{aligned}$$

and so

$$\begin{aligned} &\frac{1}{d_j} \mu\left(f_j\left(\bigcup_{i=1}^k f_i^{-1}(K)\right) \setminus K \right) \\ &\qquad \le \lim_{n\to\infty} \int_{f_j\left(\bigcup_{i=1}^k f_i^{-1}(K)\right)\setminus K} \phi_j^n(\zeta)\, d\mu(\zeta) \\ &\qquad\qquad \le \mu\left(f_j\left(\bigcup_{i=1}^k f_i^{-1}(K)\right) \setminus K \right). \end{aligned} \tag{5.15}$$

For any $z \notin f_j\left(\bigcup_{i=1}^k f_i^{-1}(K)\right)$ we obviously have $f_j^{-1}(z) \cap \bigcup_{i=1}^k f_i^{-1}(K) = \emptyset$ and so $\lim_{n\to\infty} \phi_j^n(z) = 0$ for $j = 1, \ldots, k$. Hence

$$\lim_{n\to\infty} \int_{\overline{\mathbb{C}} \setminus f_j\left(\bigcup_{i=1}^k f_i^{-1}(K)\right)} \phi_j^n(\zeta)\, d\mu(\zeta) = 0. \tag{5.16}$$

Together, (5.12), (5.13), (5.14), (5.15), and (5.16) yield the inequalities (5.2) for any compact subset K of $\overline{\mathbb{C}}$.

We also have the inequalities (5.2) for any open set U. We repeat the above argument replacing K by U and replacing the functions ϕ^n by continuous, uniformly bounded functions ψ^n which increase to $\chi_{\bigcup_{j=1}^k f_j^{-1}(U)}$. Routine, but tedious, arguments, making use of the regularity of μ, may now be used to extend the inequalities (5.2) to all Borel sets.

5.10. **Proof of the inequality** (5.3)**.** Now we establish the inequality (5.3) of Theorem 5.2. Let U be open and let the functions ϕ^n be nonnegative, continuous on $\overline{\mathbb{C}}$, and increase to χ_U. What we have shown in (5.11) is that

$$\int \phi^n(\zeta)\, d\mu(\zeta) = \frac{1}{d} \int \sum_{j=1}^{k} \sum_{i=1}^{d_j} \phi^n(z_{i,j}(\zeta))\, d\mu(\zeta)$$

where the points $z_{i,j}(\zeta)$ are the solutions to $f_j(z) = \zeta$. In particular, when $\zeta \in f_j(U)$, there is at least one solution to $f_j(z) = \zeta$ in U. Hence for $\zeta \in f_j(U)$,

$$\lim_{n\to\infty} \sum_{i=1}^{d_j} \phi^n(z_{i,j}(\zeta)) \geq \chi_{f_j(U)}(\zeta)$$

and so

$$\mu(U) = \int \chi_U(\zeta)\, d\mu(\zeta) = \lim_{n\to\infty} \int \phi^n(\zeta)\, d\mu(\zeta)$$

$$\geq \lim_{n\to\infty} \frac{1}{d} \sum_{j=1}^{k} \int_{f_j(U)} \sum_{i=1}^{d_j} \phi^n(z_{i,j}(\zeta))\, d\mu(\zeta) \geq \frac{1}{d} \sum_{j=1}^{k} \int_{f_j(U)} \chi_{f_j(U)}(\zeta)\, d\mu(\zeta)$$

$$= \frac{1}{d} \sum_{j=1}^{k} \mu(f_j(U)).$$

This establishes (5.3) for open sets. Now let E be a Borel set and let the set U be open and contain E. Then

$$\mu(U) \geq \frac{1}{d} \sum_{j=1}^{k} \mu(f_j(U)) \geq \frac{1}{d} \sum_{j=1}^{k} \mu(f_j(E)).$$

Taking the infimum over such sets U establishes (5.3) by the regularity of μ.

5.11. **The Support of μ.** We now prove the statement about the closed support of μ. Since the measure μ is independent from the initial point a, if we take $a \in J(G)$ it is immediate that $\mathrm{supp}(\mu) \subset J(G)$. Given $\epsilon > 0$ and $z \in J(G)$, let ϕ be continuous on $\overline{\mathbb{C}}$ with $0 \leq \phi \leq 1$ on $\overline{\mathbb{C}}$, $\phi \equiv 1$ on $\overline{\Delta(z, \epsilon/2)}$ and $\phi \equiv 0$ off of $\Delta(z, \epsilon)$. The expanding property established in Proposition 1.6 guarantees that there is an integer N such that $J(G) \subset \bigcup_{l(g)=N} g(\Delta(z, \epsilon/2))$. Recalling that $h^{-1}(J(G)) \subset J(G)$ for all $h \in G$, as g ranges over the words of length $m+N$, the equations $g(w) = a$ for fixed $a \in J(G)$ have at least d^m solutions in $\Delta(z, \epsilon/2)$ for any positive integer m. Thus

$$\mu(\Delta(z,\epsilon)) = \int \chi_{\Delta(z,\epsilon)}(\zeta)\, d\mu(\zeta) \geq \int \phi(\zeta)\, d\mu(\zeta) = \lim_{m\to\infty} \int \phi(\zeta)\, d\mu^a_{m+N}(\zeta)$$

$$\geq \lim_{m\to\infty} \int_{\overline{\Delta(z,\frac{\epsilon}{2})}} \phi(\zeta)\, d\mu^a_{m+N}(\zeta) \geq d^{-N}.$$

Hence $\mathrm{supp}(\mu) = J(G)$.

With this, we have completed the proof of Theorem 5.2.

6. The Filled-in Julia Set for Polynomial Semigroups of Finite Type

The material in this section is taken from [6].

For a polynomial f of degree at least two, the filled-in Julia set, denoted by $K(f)$, is defined to be the set of points $z \in \mathbb{C}$ such that the forward orbit $\{f^n(z)\}$ is bounded. The complement of $K(f)$, the set of points which tend to ∞ under iteration of f, is called the basin of attraction of ∞ and is denoted $A_\infty(f)$. It is well known that $K(f)$ is the union of $J(f)$ and the bounded components of $N(f)$. Also, $A_\infty(f)$ is the component of $N(f)$ containing ∞. Further, we have $J(f) = \partial K(f) = \partial A_\infty(f)$. (See [10], §III.4.) We have the following proposition.

Proposition 6.1. *Let f be a polynomial of degree at least two. Then the following are equivalent:*

1. *$A_\infty(f)$ is simply connected.*
2. *$J(f)$ is connected.*
3. *$K(f)$ is connected.*
4. *f has no finite critical point in $A_\infty(f)$.*

For a proof see [5], Theorem 9.5.1 together with the fact that a domain is simply connected if and only if its complement is connected.

Proposition 6.1 is one of the many instances where the critical points of a rational function play a strong role in its dynamics. In this chapter we point out one of the differences in the role played by the critical points of the functions in a rational semigroup.

6.1. **Polynomial Semigroups of Finite Type.** Hinkkanen and Martin considered the generalization of the filled-in Julia set to more general polynomial semigroups. However, there are some questions about the proper generalization.

If G is a polynomial semigroup, one need not have any point z where the set $G(z) = \{g(z) : g \in G\}$ is bounded. See Remark 6.5 below. However, it is also possible to construct polynomial semigroups where for every point $z \in \mathbb{C}$, the set $G(z)$ has a finite accumulation point. To see this consider $G = \langle z^2, z^2/2, z^2/3, \dots \rangle$. Hinkkanen and Martin invented the concept of a polynomial semigroup of finite type as a natural compromise between the two extremes. It turns out that various one-complex-dimansional moduni spaces for discrete groups can be described as the complements of the filled-in Julia sets (defined below) for certain polynomial semigroups of finite type. See [15] for a discussion.

We summarize the definition and main theorem on polynomial semigroups of finite type found in [15].

Definition 6.1. We say that a polynomial semigroup G is of **finite type** if it satisfies the following conditions:

(1) For any positive integer N, there are only finitely many polynomials in G whose degree is less than N.
(2) There is a domain D in $\mathbb{C}$, whose complement $\mathbb{C}\setminus D$ is a bounded continuum, such that each $g \in G$ maps D into itself, that is $g(D) \subset D$.

Remark 6.1. It is easy to see that every finitely generated polynomial semigroup where the generators have degree at least two is of finite type. More generally, if G is of finite type, G can only have finitely many generators of a given degree. Lastly, any degree 1 elements in G must be generated by a single elliptic Möbius transformation, i.e., must be finite order rotations around some point in $\mathbb{C}$.

Definition 6.2. Let G be a polynomial semigroup of finite type. The **filled-in Julia set** of G, denoted $K(G)$, is the closure of the set of points $z \in \mathbb{C}$ such that $G(z) = \{g(z) : g \in G\}$ has a finite limit point. The complement of $K(G)$, denoted $A(G) = \overline{\mathbb{C}} \setminus K(G)$, is the basin of attraction of ∞ for G.

Remark 6.2. It is clear that $K(g) \subset K(G)$ for all $g \in G$ and that $g^{-1}(K(G)) \subset K(G)$ for all $g \in G$. Either of these two statements imply that $J(G) \subset K(G)$; the first by Theorem 1.2 and the second by Property 1.1.

The following example, provided by Hinkkanen and Martin, shows that the set $A(G)$ need not be connected.

Example 6.1. Let $D_j = \{z : |z - a_j| \leq r_j\}$ for $1 \leq j \leq 3$ be disks that are tangent to each other outside (with disjoint interiors). Let B be a very large disk containing all the D_j well in its interior. Define $G = \langle g_1, g_2, g_3 \rangle$, where $g_j(z) = a_j + (c_j(z - a_j))^{n_j}$ and $c_j > 0$ is chosen so that $J(g_j) = \partial D_j$ while the positive integer $n_j \geq 2$ is so chosen that for a suitable $\epsilon > 0$, to be specified, we have $g_j^{-1}(B) \subset B_j = \{z : |z - a_j| \leq r_j + \epsilon\}$. Choose ϵ so small that there is still an open set W in between (in the interstice of) the disks B_j for $1 \leq j \leq 3$. Now it is easily seen that $K(G) \subset \bigcup_{j=1}^3 B_j$. Furthermore, if $z \in W$, then any g_j maps z outside B so that $G(z)$ clusters only to infinity. So in this case the complement of $K(G)$ has a bounded component and is not connected.

Hinkkanen and Martin's main result on polynomial semigroups of finite type in [15] is the following.

Proposition 6.2 ([15], Theorem 7.2)**.** *Let G be a polynomial semigroup of finite type. Then there is a domain $V \supset D$, where D is as in Definition 6.1, containing a neighborhood of ∞, such that V coincides with the unbounded component of the complement of the set*

$$\overline{\bigcup_{g \in G} g^{-1}\left(\bigcup_{h \in G} K(h)\right)},$$

and has the following property: for any $z \in V$ (and hence any $z \in D$) and for any compact subset K of $\mathbb{C}$, there are only finitely many $g \in G$ such that $g(z) \in K$, and, furthermore, V is the largest domain containing the point at infinity that has this property.

Our first result is that V, which may be thought of as the immediate basin of attraction for infinity, arises from simpler sets than the one in Proposition 6.2.

Theorem 6.1 ([6], Theorem 5.1.7)**.** *Let G be a polynomial semigroup of finite type and let V be as in Proposition 6.2. Then V is also the unbounded component of the complements of $K(G)$, $\overline{\bigcup_{g \in G} K(g)}$, and $J(G)$. In particular, V is a component of $N(G)$.*

Proof. That V is the unbounded component of $A(G) = \overline{\mathbb{C}} \setminus K(G)$ was shown in the proof of Proposition 6.2. For notational simplicity let

$$K_1(G) = \overline{\bigcup_{g\in G} g^{-1}\left(\bigcup_{h\in G} K(h)\right)}. \tag{6.1}$$

We remark that $K_1(G)$ is the smallest, closed set containing $K(g)$ for all $g \in G$ that is backwards invariant under each $g \in G$. Also let

$$K_2(G) = \overline{\bigcup_{g\in G} K(g)}. \tag{6.2}$$

We remark that

$$J(G) \subset K_2(G) \subset K_1(G) \subset K(G). \tag{6.3}$$

The first inclusion follows from the fact that

$$J(G) = \overline{\bigcup_{g\in G} J(g)} \subset \overline{\bigcup_{g\in G} K(g)} = K_2(G).$$

See Theorem 1.2. The second inclusion of (6.3) follows from the fact that

$$g^{-1}(K(g)) = K(g)$$

for all polynomials g. The final inclusion follows from Remark 6.2.

Thus if V' and V'' are the unbounded components of $\overline{\mathbb{C}} \setminus K_2(G)$ and $\overline{\mathbb{C}} \setminus J(G)$, respectively, we have the inclusions

$$V \subset V' \subset V''.$$

Recall that V is maximal with respect to the property that given any point $z \in V$ and any compact set $K \in \mathbb{C}$, there are only finitely many $g \in G$ such that $g(z) \in K$. We wish to show that V'' also has this property, and hence $V = V' = V''$. The proof closely follows that of Proposition 6.2. We reproduce the relevant facts here.

Recall that $J(G) = \overline{\bigcup_{g\in G} J(g)}$ (Theorem 1.2). For each $g \in G$ of degree at least two, let $S_g(z)$ denote the Green's function of $A_\infty(g)$ with pole at infinity. For any $z \in V''$, z lies in $A_\infty(g)$ so $S_g(z) > 0$ for all $g \in G$. Further,

$$S_g(g(z)) = (\deg g) S_g(z) \tag{6.4}$$

for all $z \in A_\infty(g)$ (see [10], p. 35.) See the same reference to establish the fact that the logarithmic capacity of $J(g)$ satisfies $cap(J(g)) = 1/M^{1/(n-1)}$ where $n = \deg g$ and M is the modulus of the leading coefficient of g. Hence the logarithmic capacity of $J(g)$ is positive. Let $T(z)$ denote the Green's function of V'' with pole at infinity. As V'' is contained in the complement of $J(g)$ and as both $T(z)$ and $S_g(z)$

have logarithmic singularities at ∞, we see that for each $g \in G$ the function $T(z) - S_g(z)$ is bounded and harmonic in V'', non-positive on the boundary of V'' and hence non-positive on V''. Thus $S_g(z) \geq T(z)$. Hence for each $z \in V''$

$$\inf\{S_g(z) : g \in G\} \geq T(z) > 0.$$

Let $N \geq 1$. By assumption, there are only finitely many $g \in G$ with $\deg g \leq N$. Thus for $z \in V''$, the numbers $S_g(g(z)) = (\deg g) S_g(z) \geq (\deg g) T(z)$ tend to ∞ as g runs over the elements of G so that the numbers $\deg g$ are non-decreasing. We wish to show that this implies that the numbers $g(z)$ also tend to ∞.

For each $g \in G$ of degree at least two, the set $J(g)$ is compact and lies in a fixed disk of radius R centered at the origin. Here R depends on V'' only. We know from the above that $cap(J(g))$ is positive, so $c = -\log(cap(J(g)))$ is well defined. There is a probability measure m on $J(g)$ such that

$$S_g(z) - c = \int_{J(g)} \log|z - w|\, dm(w). \tag{6.5}$$

See [33], §1.5. Suppose that $|g(z)| < r$. If $cap(J(g)) > L > 0$, then using (6.5), we see that

$$S_g(g(z)) = c + \int_{J(g)} \log|g(z) - w|\, dm(w) < -\log(L) + \log(R + r)$$

since $|g(z)| < r$, $J(g) \subset \Delta(0, R)$, and m is a probability measure on $J(g)$. If there were a positive lower bound on $cap(J(g))$ for $g \in G$ and $|g(z)| < r$ for infinitely many $g \in G$, we would have an upper bound on $S_g(g(z))$ for infinitely many $g \in G$ which is a contradiction.

The point $z \in V''$ is at a fixed positive distance d from every $J(g)$. For a given large r, consider two disjoint subsets of elements of G. First consider those $g \in G$ for which $cap(K(g)) > d/2$ and $|g(z)| < r$. From the comments above, we see that there can be only finitely many such g.

Next, consider those $g \in G$ for which $|g(z)| < r$ and $cap(J(g)) \leq d/2$. Let g be a member of this latter set and set $n = \deg g$ and $L = cap(J(g))$. Then

$$\int_{J(g)} \log|z - w|\, dm(w) \geq \int_{J(g)} \log d\, dm(w) = \log d$$

since the distance between z and $J(g)$ is at least d, while

$$\int_{J(g)} \log|g(z) - w|\, dm(w) < \log(R + r)$$

since $|g(z)| < r$ and $J(g) \subset \Delta(0, R)$. Since $S_g(g(z)) = nS_g(z)$ we see that

$$\log(R + r) - \log(L) > S_g(g(z)) = nS_g(z)$$
$$= n \left(\int_{J(g)} \log |z - w| \, dm(w) - \log(L) \right) \geq n(\log(d) - \log(L)),$$

hence

$$\log(R + r) > n \log d + (n - 1)(-\log L) \tag{6.6}$$
$$\geq n \log d + (n - 1)(-\log(d/2)) \tag{6.7}$$
$$= n \log 2 + \log(d/2). \tag{6.8}$$

For a given r this last equation implies an upper bound for $n = \deg g$, say $n < n(r, d, R)$. By assumption, there are only finitely many $g \in G$ whose degree is at most $n(r, d, R)$, and so in this second class there are at most finitely many $g \in G$. Hence for any large r and any $z \in V''$ there are only finitely many $g \in G$ for which $|g(z)| < r$. This shows that $V = V''$ and hence $V = V' = V''$. □

We have just shown that the sets $J(G)$, $K_2(G)$, $K_1(G)$ and $K(G)$ share the same unbounded component of their complements. We have also shown that

$$J(G) \subset K_2(G) \subset K_1(G) \subset K(G).$$

We now show through a series of examples that the inclusions above can be strict.

Example 6.2. Let $G = \langle z^2, z^2/a \rangle$ with $a > 1$. Then $J(G) = \{z : 1 \leq |z| \leq a\}$ (see [15], Example 1) but $K_2(G) = K(G) = \{z : |z| \leq a\}$.

Example 6.3. Let $f_1(z) = z^2$, $f_2(z) = (z-10)^2+10$, and let $G = \langle f_1, f_2 \rangle$. Note that $K(f_1) = \overline{\Delta(0, 1)}$ and $K(f_2) = \overline{\Delta(10, 1)}$ where $\Delta(a, r)$ is the disk of radius r centered at a. We will show that $K_2(G)$ is a proper subset of $K_1(G)$.

Let $U = \overline{\mathbb{C}} \setminus \overline{\Delta(5, 9)}$. We will show that

$$f_j(U) \subset U \text{ for } j = 1, 2$$

from which it follows that $g(U) \subset U$ for all $g \in G$. A straightforward calculation yields that on ∂U,

$$|f_1(9e^{i\theta} + 5) - 5| \geq 11.$$

Thus by the minimum principle, $f_1(U) \subset U$. A similar calculation also yields

$$|f_2(9e^{i\theta} + 5) - 5| \geq 11,$$

hence $f_2(U) \subset U$ as well.

For any $g \in G$, this shows that $g(U) \subset U$ and hence $U \subset N(g)$ by Montel's Theorem (Proposition 1.1). In particular, $U \subset A_\infty(g)$ for all $g \in G$. Hence $U \subset V$, where V is the domain from Proposition 6.2.

Let $N = \overline{\Delta\left(\sqrt{10}, 1/(3\sqrt{10})\right)}$. We claim that $N \subset K_1(G)$. To see this, we can show that on ∂N,

$$\left| f_1\left(\frac{e^{i\theta}}{3\sqrt{10}} + \sqrt{10}\right) - 10 \right| < 1$$

and so by the maximum principle we see that $f_1(N) \subset K(f_2) = \Delta(10,1)$ and thus $N \subset K_1(G)$ by definition.

We also have that $f_2(N) \subset U$, for on ∂N,

$$\left| f_2\left(\frac{e^{i\theta}}{3\sqrt{10}} + \sqrt{10}\right) - 5 \right| > 43.$$

We claim that this shows that $N \cap K(g) = \emptyset$ for all $g \in G$. Recall that $g(U) \subset U$ for all $g \in G$. Let $g = g_n \circ g_{n-1} \circ \cdots \circ g_1$ where each g_j equals f_1 or f_2. If $g_1 = f_2$, then $g(N) \subset U$ since $f_2(N) \subset U$. If $g_1 = f_1$, then $g^2(N) \subset U$. We see this as follows. We first show that $f_1(K(f_2)) = f_1(\overline{\Delta(10,1)}) \subset U$ by calculating

$$|f_1(e^{i\theta} + 10) - 5| \geq 76.$$

Thus by the minimum principle we see that $f_1(K(f_2)) \subset U$. Since $g_1 = f_1$, then $g^2 = h \circ f_1 \circ f_2^k \circ f_1$ for some $k \geq 0$ and some $h \in G$ or $h(z) = z$. What we have shown above gives that

$$g^2(N) = (h \circ f_1 \circ f_2^k \circ f_1)(N) \subset (h \circ f_1 \circ f_2^k)(K(f_2)) = h(f_1(K(f_2))) \subset h(U) \subset U.$$

Hence $N \cap K(g) = \emptyset$ for all $g \in G$, i.e. $N \cap \left(\bigcup_{g \in G} K(g)\right) = \emptyset$. In particular, this shows that $\sqrt{10} \in K_1(G) \setminus K_2(G)$, so $K_2(G)$ is a proper subset of $K_1(G)$ as claimed.

Example 6.4. Let G be the semigroup from Example 6.3. We construct a subsemigroup H of G such that $K_1(H)$ is a proper subset of $K(H)$.

Let $h_0 = f_1$, $h_1 = f_2 \circ f_1$, and in general let $h_n = f_2^n \circ f_1$. Let $H = \langle h_0, h_1, \dots \rangle$. Note that any subsemigroup of a polynomial semigroup of finite type is itself a polynomial semigroup of finite type. Let N and U be as in the previous example. We have shown that $h_n(N) \subset \Delta(10,1)$ for $n = 0, 1, 2, \dots$. Thus $N \subset K(H)$ by definition. However we have also shown in the previous example that $h_n(h_m(N)) \subset U$ for all m, $n \geq 0$ and that $h(U) \subset U$ for all $h \in H$. Recall that $U \cap K(g) = \emptyset$ for all $g \in G$. Hence if $h, g \in H$, we have $g(N) \cap K(h) = \emptyset$. Thus

$$N \cap \left(\bigcup_{g \in H} g^{-1} \left(\bigcup_{h \in H} K(h) \right) \right) = \emptyset$$

and in particular, $K_1(H)$ is a proper subset of $K(H)$.

Remark 6.3. This last example gives an infinitely generated polynomial semigroup H of finite type such that $K(H) \setminus K_1(H) \neq \emptyset$. No such example for a finitely generated polynomial semigroup is as of yet known.

6.2. **Relationship Between Critical Points and $K(G)$.** As stated in Proposition 6.1, for a polynomial f of degree at least 2, the set $A_\infty(f)$ is simply connected if and only if it contains no finite critical point of f. We show now through two examples that for a polynomial semigroup G of finite type, there is in general no relationship between the connectivity of the set V from Proposition 6.2 and the location of the critical points of the elements of G.

Example 6.5. In this example we construct a finitely generated polynomial semigroup such that V is not simply connected, yet the finite critical points of every element g in G lie in $K(G)$.

Let G be the semigroup constructed in Example 6.3, i.e, $G = \langle f_1, f_2 \rangle$ where $f_1(z) = z^2$ and $f_2(z) = (z - 10)^2 + 10$. Recall that $K(f_1) = \overline{\Delta(0, 1)}$ and that $K(f_2) = \overline{\Delta(10, 1)}$. By Proposition 6.2, there exists a number $R > 0$ so that the set $G(z) = \{g(z) : g \in G\}$ clusters only at infinity for all $z \in \{z : |z - 5| \geq R\}$.

Let $S = \{z = x + iy : |x - 5| \leq 1, |y| \leq R + 1\}$. Note that $S \subset A_\infty(f_j)$ for $j = 1, 2$. Thus we may choose integers $n_1, n_2 \geq 1$ so that $f_j^{n_j}(S) \subset \mathbb{C} \setminus \Delta(5, R)$ for $j = 1, 2$.

Let $G' = \langle f_1^{n_1}, f_2^{n_2} \rangle$. We have shown that $G'(z)$ clusters only to infinity for $z \in S \cup (\mathbb{C} \setminus \Delta(5, R))$. Hence $S \cup (\mathbb{C} \setminus \Delta(5, R)) \subset V$ where V is the unbounded component of the complement of $K(G')$. In particular, since $K(f_j^{n_j}) \subset K(G')$ for $j = 1, 2$, this shows that V is multiply connected.

However, all of the finite critical points for elements in G' lie in $K(G')$. The chain rule shows that any finite critical point for an element of G' is a preimage of the critical points of the generators, namely 0 and 10. These two points are in $K(G')$ which is backwards invariant under any element of G'. Hence all finite critical points of elements of G' are in $K(G')$ as claimed.

Example 6.6. For our next example, we construct a finitely generated polynomial semigroup G such that the set V from Proposition 6.2 is simply connected, yet contains a finite critical point of an element of G.

Let $g_0(z) = z^2+1$. As is easily seen, $g_0^n(0) \to \infty$ hence $J(g_0)$ is totally disconnected (see [10], Theorem 4.2). Further, we see that the set of purely real and purely imaginary numbers belong to $N(g_0) = A_\infty(g_0)$ as follows. For x real, $|x^2+1| = x^2+1 > |x|$ and hence $g_0^n(x) \to \infty$. Since $N(g_0)$ is completely invariant under g_0 and $g_0(iy) = -y+1 \in \mathbb{R}$ for $y \in \mathbb{R}$, we have $i\mathbb{R} \subset N(g_0)$ as well.

We may find positive numbers r_1, r_2 and δ so that $J(g_0)$ is contained in the compact set

$$C = \{re^{i\theta} : 0 < r_1 \leq r \leq r_2,\ 0 < \delta \leq |\theta| \leq \pi\}$$

The set C may be covered by finitely many disks $\Delta(\alpha_j, \epsilon_j)$, $j = 1, \dots, n$, so that the union $\bigcup_{j=1}^n \overline{\Delta(\alpha_j, \epsilon_j)}$ is connected and does not contain the set $\{z + iy : x \geq 0,\ y = 0\}$. Define

$$g_j(z) = \frac{(z-\alpha_j)^2}{\epsilon_j} + \alpha_j$$

for $j = 1, \dots, n$. Note that $K(g_j) = \overline{\Delta(\alpha_j, \epsilon_j)}$.

Let $G = \langle g_0, g_1, \dots, g_n \rangle$. By Proposition 6.2 there exists a number $R > 0$ so that the forward orbit $G(z)$ accumulates only at infinity for all z with $|z| \geq R$. We may construct a domain D that contains 0 such that $\{z : |z| \geq R\} \subset \overline{D}$, yet $\overline{D} \cap \left(\bigcup_{j=0}^n K(g_j)\right) = \emptyset$. For each $j = 0, \dots, n$, choose an integer $m_j \geq 1$ so that $g_j^{m_j}(D) \subset \{z : |z| > R\}$. Let $G' = \langle g_0^{m_0}, g_1^{m_1}, \dots g_n^{m_n} \rangle$. Let V be the set from Proposition 6.2 for the semigroup G'. Note that $0 \in D \subset V$. In particular, V contains a finite critical point for an element from G', namely $g_0^{m_0}$. We now show that V, which is the unbounded component of the complement of $K_1(G')$ (see (6.1)), is simply connected. To do this, we need the following lemmas.

Lemma 6.1 ([5], Lemma 5.7.2)). *Let g be a rational function of degree d and let K be a compact connected subset of $\mathbb{C}$. Then $g^{-1}(K)$ has at most d components and each is mapped onto K by g.*

Lemma 6.2 ([6], Lemma 5.2.4). *Let $G = \langle f_1, f_2, \dots, f_k \rangle$ be a finitely generated polynomial semigroup such that the set*

$$E_0 = \bigcup_{j=1}^k K(f_j)$$

is connected. Then the set

$$K_1(G) = \overline{\bigcup_{g \in G} g^{-1}\left(\bigcup_{h \in G} K(h)\right)}$$

is also connected.

Proof. Let

$$E_1 = \bigcup_{j=1}^{k} f_j^{-1}(E_0).$$

Note that $E_0 \subset E_1$ since $K(f_j) = f_j^{-1}(K(f_j)) \subset f_j^{-1}(E_0)$ for $1 \leq j \leq k$. Further note that E_1 is connected since each connected component of $f_j^{-1}(E_0)$, of which there are only finitely many by Lemma 6.1, must meet $K(f_j)$ and hence must meet the connected set E_0. A finite union of connected sets each meeting a given connected set such that the union contains this set must itself be connected. Hence E_1 is connected.

In general, define

$$E_m = \bigcup_{j=1}^{k} f_j^{-1}(E_{m-1}).$$

As before, we can show that $E_{m-1} \subset E_m$ and E_m is connected for all m. Eachg set E_m clearly is compact in $\mathbb{C}$.

Let $E_\infty \equiv \bigcup_{m=0}^{\infty} E_m$. We see that E_∞ is connected, for if there were open sets A and B such that $\overline{A} \cap B = \emptyset = A \cap \overline{B}$ and such that $E_\infty \subset A \cup B$, since each E_m is connected and $E_m \subset E_{m+1}$, the set E_∞ would lie completely in A or in B. Thus E_∞ is connected.

We remark that

$$E_\infty = \bigcup_{g \in G} g^{-1}\left(E_0\right) \tag{6.9}$$

since by construction, E_m consists of the preimages of E_0 under the length n words of G for $n \leq m$.

We now complete the proof that $K_1(G)$ is connected. Let

$$K_0(G) = \bigcup_{g \in G} g^{-1}\left(\bigcup_{h \in G} K(h)\right),$$

so $K_1(G) = \overline{K_0(G)}$. Note that $E_\infty \subset K_0(G)$. We will show that $\overline{E_\infty} \cup K_0(G)$ is connected. From there we see that $K_1(G)$ is connected, since

$$K_1(G) = \overline{\overline{E_\infty} \cup K_0(G)}$$

and the closure of a connected set is connected.

Now for the proof that $\overline{E_\infty} \cup K_0(G)$ is connected. First we see that $J(G) \subset \overline{E_\infty}$, for $J(G) = \overline{\bigcup_{g\in G} J(g)}$ (Theorem 1.2) and $J(g) \subset \overline{\bigcup_{n=0}^{\infty} g^{-n}(z)}$ for all but at most two $z \in \overline{\mathbb{C}}$ (Proposition 1.4), in particular for some $z \in E_0$. We then use (6.9) to conclude that $J(G) \subset \overline{E_\infty}$. Next we see that every component of $g^{-1}(K(h))$ meets $J(G)$ and hence meets $\overline{E_\infty}$ for any g, $h \in G$ since $J(G)$ is backwards invariant under all elements of G. Thus the union of $\overline{E_\infty}$ and the components of $g^{-1}(K(h))$ for all g, $h \in G$, i.e., $\overline{E_\infty} \cup K_0(G)$, is connected and hence $K_1(G)$ is connected. □

Remark 6.4. Some questions remain about Lemma 6.2, namely must any of $J(G)$, $K_2(G)$ or $K(G)$ be connected under the assumptions of the lemma?

The construction of Example 6.6 is concluded for our semigroup

$$G' = \langle g_0^{m_0}, g_1^{m_1}, \ldots, g_n^{m_n} \rangle$$

since by construction $\bigcup_{j=1}^{n} K(g_j^{m_j})$ is connected and so $K_1(G')$ is also connected by Lemma 6.2. As V is a component of the complement of the closed, connected set $K_1(G)$, it is simply connected.

6.3. **Alternative Definitions for $K(G)$.** The following theorem, which appears in [14], provides another characterization of $K(G)$ and relates it to the set of points whose orbit under G is bounded.

Recall that $G(z) = \{g(z) : g \in G\}$.

Proposition 6.3 (in [14]). *If G is a polynomial semigroup of finite type, and if V is as in Proposition 6.2, so that V is the unbounded component of the complement of $K(G)$, and if $R > 0$ is such that V contains $\{z : |z| > R\}$, then*

$$B(G) \equiv \{z : G(z) \text{ bounded}\} = \bigcap_{g\in G} \{z : |g(z)| \le R\} \tag{6.10}$$

is a compact set whose complement is connected. Furthermore,

$$B(G) \subset K(G) = \overline{\bigcup_{n>R} \bigcap_{N\ge 2} \bigcup_{\substack{g\in G \\ \deg g \ge N}} \{z : |g(z)| \le n\}}. \tag{6.11}$$

Remark 6.5. It is often the case that $B(G) = \emptyset$. If $f(z) = z^2$, $g(z) = (z-10)^2 + 10$ and $G = \langle f, g\rangle$, then since $K(f) = \overline{\Delta(0,1)}$ and $K(g) = \overline{\Delta(10,1)}$, it is easy to see that $B(G) = \emptyset$.

The following question was posed by Hinkannen and Martin. Assume there exists a number $R_1 > R$ so that if $G(z)$ clusters to a finite point, then it clusters to a point w with $|w| \leq R_1$. In this case, the characterization of $K(G)$ in Proposition 6.3 simplifies to

$$K(G) = \overline{\bigcap_{N \geq 2} \bigcup_{\substack{g \in G \\ \deg g \geq N}} \{z : |g(z)| \leq R_1\}}. \tag{6.12}$$

We see this as follows. Assume that

$$z \in \bigcap_{N \geq 2} \bigcup_{\substack{g \in G \\ \deg g \geq N}} \{z : |g(z)| \leq R_1\}.$$

In particular, this implies that there exists a sequence of distinct elements $g \in G$ such that $|g(z)| \leq R_1$. Hence $G(z)$ has a finite accumulation point, i.e., $z \in K(G)$. Since $K(G)$ is closed, we see that

$$\overline{\bigcap_{N \geq 2} \bigcup_{\substack{g \in G \\ \deg g \geq N}} \{z : |g(z)| \leq R_1\}} \subset K(G).$$

Now assume that z is such that $G(z)$ has a finite accumulation point. We are assuming that it must accumulate somewhere in $\overline{\Delta(0, R_1)}$. Hence there is a sequence of elements $g_n \in G$ so that $|g(z)| \leq R_1$. Since G is of finite type, we see that the degree of the functions g_n must tend to infinity as $n \to \infty$. Hence

$$z \in \bigcap_{N \geq 2} \bigcup_{\substack{g \in G \\ \deg g \geq N}} \{z : |g(z)| \leq R_1\}.$$

Since $K(G)$ was defined to be the closure of such points, we see that

$$K(G) \subset \overline{\bigcap_{N \geq 2} \bigcup_{\substack{g \in G \\ \deg g \geq N}} \{z : |g(z)| \leq R_1\}}$$

and so we have established (6.12) assuming the existence of the number R_1. Must such a number R_1 always exist? When G is finitely generated, the answer is affirmative.

Theorem 6.2 ([6], Theorem 5.3.3)**.** *Let G be a finitely generated polynomial semigroup where the degree of the generators is at least two. Let V be as in Proposition 6.2 and let $R > 0$ be such that V contains $\{z : |z| > R\}$. Then there exists a number $R_1 > R > 0$ so that if z is any point such that $G(z)$ has a finite cluster point, then $G(z)$ clusters to some point w such that $|w| \leq R_1$. Hence*

$$K(G) = \overline{\bigcap_{N \geq 2} \bigcup_{\substack{g \in G \\ \deg g \geq N}} \{z : |g(z)| \leq R_1\}}.$$

Proof. The set equality was established above, assuming the existence of the number R_1. Assume that no such number R_1 exists, i.e., given any $R_1 > R > 0$, there is a point $z_0 \in \mathbb{C}$ so that $G(z_0)$ accumulates in $\mathbb{C}$ but not in the closed disk of radius R_1. Viewing the semigroup G as words in the generators $\{g_1, \dots, g_k\}$, we see that there must be an integer M so that if the length of g is at least M, then $|g(z_0)| > R_1 > R$. Let $h_1, \dots, h_{k^M}$ be the words in G of length M in G. If $G(z_0)$ is to have a finite accumulation point, then $G(h_i(z_0))$ must also have a finite accumulation point for some i, $1 \le i \le k^M$. To see this we simply note that the words of length at least M are given by

$$\bigcup_{i=1}^{k^M} G \circ h_i$$

where $G \circ h_i = \{g \circ h_i : g \in G\}$, and any sequence from this collection must have an infinite subsequence from some $G \circ h_i$. But from our original assumption, $|h_i(z_0)| > R_1 > R$ for $1 \le i \le k^M$, so $G(h_i(z_0))$ accumulates only to infinity. This is a contradiction. □

Remark 6.6. Theorem 6.2 provides the basis for a computer algorithm for generating an approximate picture of $K(G)$ when G is finitely generated. Namely, for a suitable number R_1 and a suitable positive integer N, one colors the pixel p black if and only if for each integer $1 \le n \le N$, at least one word g of length n satisfies $|g(p)| \le R_1$.

Remark 6.7. We have shown that if G is a finitely generated polynomial semigroup, and z is such that $G(z)$ has a finite accumulation point, then it must have an accumulation point in a disk centered at 0 of radius R_1, where R_1 is independent of z. We make the simple remark that $G(z)$ need not have all its accumulation points in this disk.

Let $f(z) = z^2$ and $g(z) = (z-10)^2 + 10$. Let $G = \langle f, g \rangle$. If $w \in \Delta(10, 1)$, then

$$\lim_{n\to\infty} g^n(w) = 10.$$

Then for any fixed $k \ge 1$,

$$\lim_{n\to\infty} f^k(g^n(w)) = 10^{2^k},$$

so the accumulation points of $G(w)$ accumulate to infinity.

7. Ahlfors Theory of Covering Surfaces

Let $f(z)$ be meromorphic on a domain Ω. We define the spherical derivative by

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}.$$

If $f(z)$ is meromorphic in $|z| \leq r$, denote

$$L(r) = \int_{|z|=r} f^{\#}(z)|dz| = \int_0^{2\pi} \frac{|f'(re^{i\theta})|r}{1+|f(re^{i\theta})|^2} d\theta,$$

and

(7.1)

$$S(r) = \frac{1}{\pi} \iint_{|z|<r} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} dx\, dy \quad = \frac{1}{\pi} \int_0^{2\pi} \int_0^r \frac{|f'(\rho e^{i\theta})|^2 \rho}{(1+|f(\rho e^{i\theta})|^2)^2} d\rho\, d\theta$$

i.e., $L(r)$= the length of the image of the circle $|z| = r$ on the Riemann sphere; $S(r) = (1/\pi)*$area of the image of the disk $|z| < r$ on the Riemann sphere, determined with regard to multiplicity.

Let us suppose $f(z)$ is meromorphic in $|z| \leq r$. Let D be a domain on $\overline{\mathbb{C}}$, and let $I(r, D)$ denote the area of the image $f(\{|z| \leq r\})$ which lies over D (with regard to multiplicity). Let $I_0(D)$ denote the area of D. In this section all domains will be taken to be Jordan domains each of which is bounded by a sectionally analytic (s.a.) Jordan curve (see [12], p.126). Setting $S(r, D) = \frac{I(r,D)}{I_0(D)}$ (see [28], p. 29) we state

Theorem 7.1 (First Fundamental Theorem)**.** *There is a constant $h_1 = h_1(D)$ such that*

$$|S(r) - S(r, D)| \leq h_1 L(r).$$

Furthermore, suppose $\triangle$ is a subdomain of $|z| < r$, with $\overline{\triangle} \cap \{|z| = r\} = \emptyset$, which is mapped by $f(z)$ in a p-to-one fashion onto D. Then $\triangle$ is called an island over D of multiplicity p, and in this instance, such an island contributes the quantity p to $S(r, D)$. If $p = 1$, we say $\triangle$ is a simple island.

Theorem 7.2 (Second Fundamental Theorem)**.** *Let $D_1, \ldots, D_q$, $q \geq 3$, be Jordan domains on the w-sphere having disjoint closures. Then there exists a constant h_2 depending only on the domains D_j such that*

$$\sum_{j=1}^{q} (S(r) - \overline{n}(r, D_j)) \leq 2S(r) + h_2 L(r),$$

where $\overline{n}(r, D)$ is the total number of distinct islands over D in $|z| < r$ without regard to multiplicity.

Proof. This is almost the same statement as is Theorem 5.5 in [12]. We will translate the necessary notations. Suppose that the domain D_j is covered by the islands D_j^i for $i = 1, \ldots, k(j)$. Each island D_j^i is mapped by f onto D_j such that each point is covered equally often (counting multiplicity). Let $n(D_j^i)$ denote this multiplicity. Letting $\rho(D_j^i)$ denote the Euler characteristic of D_j^i we define the excess $n_1(D_j^i)$ of the island D_j^i by

$$n_1(D_j^i) = n(D_j^i) + \rho(D_j^i).$$

Writing $n_1 = (n-1)+(\rho+1) = (n-1)+(l-1)$ where $l(D)$ denotes the number of components of $\overline{\mathbb{C}} \setminus D$, we see that n_1 is equal to the excess of the multiplicity of the island over 1 plus the excess of the connectivity of the island over 1. If $n = 1$, the map is univalent and the island is necessarily simply connected so that $n_1 = 0$. Otherwise $n_1 > 0$.

Let $n(D_j) = \sum_{i=1}^{k(j)} n(D_j^i)$ and $n_1(D_j) = \sum_{i=1}^{k(j)} n_1(D_j^i)$.

So $n_1(D_j) - n(D_j) = \sum_{i=1}^{k(j)} \rho(D_j^i) \geq \sum_{i=1}^{k(j)}(-1) = -k(j) = -\overline{n}(r, D_j)$. Hence

$$\sum_{j=1}^{q}(S(r) - \overline{n}(r, D_j)) \leq \sum_{j=1}^{q}(S(r) - n(D_j) + n_1(D_j)) \leq 2S(r) + h_2 L(r)$$

where the last inequality is the statement in Theorem 5.5 in [12]. □

Theorem 7.3. *Let $D_1, \ldots, D_5$ be Jordan domains on the w-sphere having disjoint closures. Let f be meromorphic on the unit disc with no simple islands over any of the D_j. Then there exists an H depending only on the D_j's such that*

$$S(r) < HL(r)$$

for all $0 \leq r < 1$.

Proof. Since each island over D_j has multiplicity greater than or equal to two, $S(r, D_j) \geq 2\overline{n}(r, D_j)$ for each j.

By the First Fundamental Theorem (Theorem 7.1) $S(r, D_j) \leq S(r) + h_j L(r)$ where h_j is the constant depending only on D_j.

So $\overline{n}(r.D_j) \leq \frac{1}{2} S(r.D_j) \leq \frac{1}{2}(S(r) + h_j L(r))$ and so

$$\sum_{j=1}^{5} \overline{n}(r, D_j) \leq \frac{5}{2} S(r) + \frac{1}{2} \sum_{j=1}^{5} h_j L(r).$$

So by the Second Fundamental Theorem (Theorem 7.2), we have

$$5S(r) \leq \sum_{j=1}^{5} \overline{n}(r, D_j) + 2S(r) + \tilde{h}L(r) \leq \frac{5}{2} S(r) + \frac{1}{2} \sum_{j=1}^{5} h_j L(r) + 2S(r) + \tilde{h}L(r)$$

and so for $H = 2\tilde{h} + \sum_{j=1}^{5} h_j$ we have $S(r) < HL(r)$. □

Lemma 7.1. *Let f be meromorphic on the unit disc such that $S(r) < HL(r)$ for all $0 \le r < 1$, then there exists a constant h_2 depending only on H such that*

$$F^{\#}(0) < h_2.$$

Proof. See [28], p. 84. □

Theorem 7.4 (Ahlfors Five Island Theorem)**.** *Let $D_1, \dots, D_5$ be Jordan domains on the w-sphere having disjoint closures. Let f be meromorphic on the unit disc. Then there exists a constant C depending only on the domains D_j and not on $f(z)$ such that if*

$$f^{\#}(0) > C$$

then $f(z)$ maps an island in the unit disc univalently onto some D_j. If $|z| < R$ is used instead of the unit disk, then for the latter conclusion we require $f^{\#}(0) > \frac{C}{R}$.

Proof. The conclusion follows immediately from Theorem 7.3 and Lemma 7.1. □

Theorem 7.5 (Ahlfors Three Island Theorem)**.** *Let $D_1, \dots, D_3$ be bounded Jordan domains on the w-sphere having disjoint closures. Let f be analytic on the unit disc. Then there exists a constant C depending only on the domains D_j and not on $f(z)$ such that if*

$$f^{\#}(0) > C$$

then $f(z)$ maps an island in the unit disc univalently onto some D_j. If $|z| < R$ is used instead of the unit disk, then for the latter conclusion we require $f^{\#}(0) > \frac{C}{R}$.

Proof. Let D_4 be a Jordan domain containing ∞ that is mutually disjoint from each of D_1, D_2 and D_3. Since f is analytic, there are no islands over D_4, i.e., $\overline{n}(r, D_4) = 0$. As in the proof of Theorem 7.3 we suppose that there are no simple islands and so $\overline{n}(r, D_j) \le \frac{1}{2} S(r, D_j) \le \frac{1}{2}(S(r) + h_j L(r))$ for $j = 1, \dots, 3$. Hence by the Second Fundamental Theorem (Theorem 7.2) we see that

$$4S(r) \le \sum_{j=1}^{4} \overline{n}(r, D_j) + 2S(r) + \tilde{h}L(r) \le \frac{3}{2}S(r) + \frac{1}{2}\sum_{j=1}^{3} h_j L(r) + 2S(r) + \tilde{h}L(r)$$

and so for $H = 2\tilde{h} + \sum_{j=1}^{3} h_j$ we have $S(r) < HL(r)$.

Lemma 7.1 can now be used to finish the proof. □

For similar *existence of a simple island* results for $f(z)$ with regularly exhaustible Riemann surfaces see [12], p. 148 and [28], p. 30.

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