Series Logo
Volume **00**, Number 00, Xxxx 19xx

COMPLETELY INVARIANT JULIA SETS OF POLYNOMIAL SEMIGROUPS

RICH STANKEWITZ

Abstract. Let G be a semigroup of rational functions of degree at least two, under composition of functions. Suppose that G contains two polynomials with non-equal Julia sets. We prove that the smallest closed subset of the Riemann sphere which contains at least three points and is completely invariant under each element of G, is the sphere itself.

1. Introduction

This paper addresses the issue of how one can extend the definition of a Julia set of a rational function of a complex variable of degree at least two to the more general setting of a Julia set of a rational semigroup.

It is possible to define the Julia set of a single rational function in two different, yet equivalent ways. The first definition of a Julia set is given as the complement of the set of normality of the iterates of the given function. The second definition of a Julia set is given as the smallest closed completely invariant set which contains three or more points. Each of these definitions can be naturally extended to the setting of an arbitrary rational semigroup G, but the extensions are not equivalent.

This paper will show that the extension of the first definition, denoted $J(G)$, is better for the purpose of achieving a situation where it is meaningful to study dynamics on the components of its complement, the Fatou set. (If one is studying dynamics from the point of view that complete invariance is required, then, of course, the extension of the second definition, denoted $E(G)$, is better.) We prove that for a semigroup generated by two polynomials, of degree greater than or equal to two, with non-equal Julia sets, we have $E(G) = \overline{\mathbb{C}}$, while $J(G)$ is known to be a compact subset of the complex plane $\mathbb{C}$. (If the Julia sets of the two generators are equal, then both $J(G)$ and $E(G)$ are equal to this common Julia set.)

2. Definitions and basic facts

In what follows all notions of convergence will be with respect to the spherical metric on the Riemann sphere $\overline{\mathbb{C}}$.

A rational semigroup G is a semigroup of rational functions of degree greater than or equal to two defined on the Riemann sphere $\overline{\mathbb{C}}$ with the semigroup operation

1991 Mthematics Subject Classification: Primary 30D05, 58F23. Key words and phrases. Polynomial semigroups, completely invariant sets, Julia sets. Research supported by a Department of Education GAANN fellowship and by the Research Board of the University of Illinois at Urbana-Champaign.

being functional composition. When a semigroup G is generated by the functions $\{f_1, f_2, \dots, f_n, \dots\}$, we write this as

$$G = \langle f_1, f_2, \dots, f_n, \dots \rangle.$$

In [2], p. 360 the definitions of the set of normality, often called the Fatou set, and the Julia set of a rational semigroup are as follows:

Definition 1. For a rational semigroup G we define the set of normality of G, $N(G)$, by

$$N(G) = \{z \in \overline{\mathbb{C}} : \exists \text{ a neighborhood of } z \text{ on which } G \text{ is a normal family}\}$$

and define the Julia set of G, $J(G)$, by

$$J(G) = \overline{\mathbb{C}} \setminus N(G).$$

Clearly from these definitions we see that $N(G)$ is an open set and therefore its complement $J(G)$ is a compact set. These definitions generalize the case of iteration of a single rational function and we write $N(\langle h \rangle) = N_h$ and $J(\langle h \rangle) = J_h$.

Note that $J(G)$ contains the Julia set of each element of G.

Definition 2. If h is a map of a set Y into itself, a subset X of Y is:

i) *forward invariant* under h if $h(X) \subset X$;

ii) *backward invariant* under h if $h^{-1}(X) \subset X$;

iii) *completely invariant* under h if $h(X) \subset X$ and $h^{-1}(X) \subset X$.

It is well known that the set of normality of h and the Julia set of h are completely invariant under h (see [1], p. 54), i.e.,

$$h(N_h) = N_h = h^{-1}(N_h) \text{ and } h(J_h) = J_h = h^{-1}(J_h). \tag{2.1}$$

In fact we have the following result.

Property 1. *The set J_h is the smallest closed completely invariant (under h) set which contains three or more points (see* [1]*, p. 67).*

In fact, this may be chosen as an alternate definition to the definition of J_h given in Definition 1.

From Definition 1, we get that $N(G)$ is forward invariant under each element of G and $J(G)$ is backward invariant under each element of G (see [2], p. 360). The sets $N(G)$ and $J(G)$ are, however, not necessarily completely invariant under the elements of G. This is in contrast to the case of single function dynamics as noted in (2.1). The question then arises, what if we required the Julia set of the semigroup G to be completely invariant under each element of G? We consider in this paper the consequences of such an extension, given in the following definition.

Definition 3. For a rational semigroup G we define

$$E = E(G) = \bigcap \{S : S \text{ is closed, completely invariant under each } g \in G, \#(S) \geq 3\}$$

where $\#(S)$ denotes the cardinality of S.

We note that $E(G)$ exists, is closed, is completely invariant under each element of G and contains the Julia set of each element of G by Property 1.

We now compare the sets $E(G)$ and $J(G)$.

Example 1 Suppose that $G = \langle f, g\rangle$ and $J_f = J_g$. Then $E = J_f = J_g$ since J_f is completely invariant under f and J_g is completely invariant under g. It is easily verified that if $J_f = J_g$, then $J(G) = J_f = J_g$, also.

The following example shows that we may have $J(G) \neq E(G)$.

Example 2 Let $a \in \overline{\mathbb{C}}, |a| > 1$ and $G = \langle z^2, z^2/a\rangle$. One can easily show that $J(G) = \{z : 1 \leq |z| \leq |a|\}$ (see [2], p.360) while $E = \overline{\mathbb{C}}$. Note that $J_{z^2} = \{z : |z| = 1\}$ and $J_{z^2/a} = \{z : |z| = |a|\}$.

The main results of this paper will show that, in some sense, Examples 1 and 2 illustrate the only two possibilities for polynomial semigroups. We prove the following theorems.

Theorem 1. *For polynomials f and g of degree greater than or equal to two, $J_f \neq J_g$ implies $E(G) = \overline{\mathbb{C}}$ where $G = \langle f, g\rangle$.*

The following theorem follows immediately.

Theorem 2. *For a rational semigroup G' which contains two polynomials f and g of degree greater than or equal to two, $J_f \neq J_g$ implies $E(G') = \overline{\mathbb{C}}$.*

3. Auxiliary results

Suppose that G is a semigroup generated by two polynomials, f and g, of degree k and l where $k, l \geq 2$. Note that $\infty \notin J(G)$, since any small neighborhood of ∞ which is forward invariant under both f and g is necessarily in $N(G)$. This G will remain fixed for the rest of the paper. Recall that $E(G) \supset J_f \cup J_g$.

We first begin with a lemma that will give a convenient description of E. Our next result shows how E is "built up" from J_f and J_g.

For a collection of sets $\mathcal{A}$, and a function h, we denote new collections of sets by $h(\mathcal{A}) = \{h(A) : A \in \mathcal{A}\}$ and $h^{-1}(\mathcal{A}) = \{h^{-1}(A) : A \in \mathcal{A}\}$.

Let us define the following countable collections of sets:

$$\mathcal{E}_0 = \{J_f, J_g\},$$
$$\mathcal{E}_{n+1} = f^{-1}(\mathcal{E}_n) \cup f(\mathcal{E}_n) \cup g^{-1}(\mathcal{E}_n) \cup g(\mathcal{E}_n),$$
$$\text{and } \mathcal{E} = \bigcup_{n=0}^{\infty} \mathcal{E}_n.$$

Since E is completely invariant under f and g and contains both J_f and J_g, we have $E \supset \bigcup_{A\in\mathcal{E}} A$. Since E is also closed, we have

$$E \supset \overline{\bigcup_{A\in\mathcal{E}} A}, \tag{3.1}$$

where $\overline{B}$ denotes the closure of the set B. The following lemma shows that these two sets are actually equal.

Lemma 1. *We have $E = \overline{\bigcup_{A\in\mathcal{E}} A}$.*

Proof. Because of (3.1) we need only show that $E \subset \overline{\bigcup_{A\in\mathcal{E}} A}$. Since the set on the right is closed and contains both J_f and J_g, it remains only to show that it is also completely invariant under both f and g. This can easily be shown by using the fact that f and g are continuous open maps. $\square$

Corollary 1. *The set E has no isolated points; i.e., E is perfect.*

Proof. Since J_f and J_g are perfect (see [1], p. 68) and backward and forward images of perfect sets under rational maps are perfect, we see that each set in $\mathcal{E}$ is perfect by a routine inductive argument. The corollary then follows since the closure of a union of perfect sets is perfect. $\square$

Lemma 2. *If E has nonempty interior, then $E = \overline{\mathbb{C}}$.*

Proof. Suppose that $E^o \neq \emptyset$, where B^o denotes the interior of the set B. It will be shown that this implies the existence of an open set $U \subset E$ such that U intersects J_f or J_g. Supposing that U intersects J_g, we may observe by the expanding property of Julia sets (see [1], p. 69) that we have $\overline{\bigcup_{n=1}^{\infty} g^n(U)} = \overline{\mathbb{C}}$. By the forward invariance of the closed set E under the map g we see that this implies that $E = \overline{\mathbb{C}}$.

We will use the following elementary fact:

For any sets B and C and any function h we have

$$B \cap h(C) \neq \emptyset \text{ if and only if } h^{-1}(B) \cap C \neq \emptyset. \tag{3.2}$$

Since $E^o \neq \emptyset$, there exists an open disc $\triangle \subset E$. By Lemma 1 we see then that there exists a set A in $\mathcal{E}_n$, say, such that $\triangle \cap A \neq \emptyset$. Since $A \in \mathcal{E}_n$, we see that it can be expressed as $A = h_n \cdots h_1(J_g)$, for example, where each $h_j \in \{f, f^{-1}, g, g^{-1}\}$. Considering each h_j as a map on subsets of $\overline{\mathbb{C}}$, as opposed to a map on points of $\overline{\mathbb{C}}$, we can define the "inverse" maps h_j^* accordingly, i.e., $h_1 = f$ implies $h_1^* = f^{-1}$ and $h_2 = g^{-1}$ implies $h_2^* = g$. The h_j^* are not true inverses since $f^{-1}(f(X))$ may properly contain X.

The fact (3.2) does imply, however, that

$$\begin{aligned} A \cap \triangle \neq \emptyset &\implies h_n \cdots h_1(J_g) \cap \triangle \neq \emptyset \\ &\implies h_{n-1} \cdots h_1(J_g) \cap h_n^*(\triangle) \neq \emptyset \\ &\quad \vdots \\ &\implies h_1(J_g) \cap h_2^* \cdots h_n^*(\triangle) \neq \emptyset \\ &\implies J_g \cap h_1^* \cdots h_n^*(\triangle) \neq \emptyset. \end{aligned}$$

Since each h_j^* maps open sets to open sets (as f, g, f^{-1}, g^{-1} do) we see that $U = h_1^* \cdots h_n^*(\triangle)$ is open. By the complete invariance of E under both f and g we see that each h_j^* takes subsets of E to subsets of E. Hence $U \subset E$ and the proof of Lemma 2 is complete. $\square$

Similar to the description of the set E given in Lemma 1 is the following description of the Julia set $J(G)$ of the semi-group $G = \langle f, g \rangle$. Consider the countable collection of sets

$$\begin{aligned} &\mathcal{F}_0 = \{J_f, J_g\}, \\ &\mathcal{F}_{n+1} = f^{-1}(\mathcal{F}_n) \cup g^{-1}(\mathcal{F}_n), \\ \text{and } &\mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{F}_n. \end{aligned}$$

Since $J(G)$ is backward invariant under f and g, closed, and contains both J_f and J_g (see [2], p. 360), we have $J(G) \supset \overline{\bigcup_{A \in \mathcal{F}} A}$.

Lemma 3. *We have* $J(G) = \overline{\bigcup_{A \in \mathcal{F}} A}$.

Proof. Since the set on the right is closed, backward invariant for both f and g (as follows as in the proof of Lemma 1) and clearly contains more that three points, it must contain $J(G)$ as the complement is then in the set of normality of G. □

From Lemma 3, noting that $\mathcal{F} \subset \mathcal{E}$, and Lemma 1 we get the following result.

Corollary 2. *The Julia set of the semigroup* G *is contained in* E*, i.e.,* $J(G) \subset E$.

Hence, combining Lemma 2 and Corollary 2, we get the following corollary.

Corollary 3. *If* $J(G)$ *has nonempty interior, then* $E = \overline{\mathbb{C}}$.

4. Proof of the main result

In this section we will prove the main result, Theorem 2, but first we establish the necessary lemmas.

Lemma 4. *If* f *and* g *are polynomials of degree greater than or equal to two and* $J_f \neq J_g$*, then* $\infty \in E$.

Proof. Denoting the unbounded components of the respective Fatou sets of f and g by F_∞ and G_∞, we recall (see [1], p. 54 and p. 82) that $J_f = \partial F_\infty$ and $J_g = \partial G_\infty$.

Since F_∞ and G_∞ are domains with nonempty intersection and $\partial F_\infty \neq \partial G_\infty$, we have $J_f \cap G_\infty \neq \emptyset$ or $J_g \cap F_\infty \neq \emptyset$.

Hence we may select $z \in J_g \cap F_\infty$, say. Denoting the nth iterate of f by f^n, we see that $f^n(z) \to \infty$, and by the forward invariance under the map f of the set E we get that each $f^n(z) \in E$. Since E is closed we see then that $\infty \in E$. □

Remark 1. Since it will be necessary later, we make special note of the fact used in the above proof that $J_f \neq J_g$ implies $J_f \cap G_\infty \neq \emptyset$ or $J_g \cap F_\infty \neq \emptyset$.

Remark 2. Note that the proof above shows also that ∞ is not an isolated point of E when $J_f \neq J_g$. This, of course, also follows from Corollary 1 and Lemma 4.

The disc centered at the point z with radius r will be denoted $\triangle(z, r)$.

Lemma 5. *Suppose that* $\triangle(0, r^*) = A \cup B$ *where* A *is open,* A *and* B *are disjoint, and both* A *and* B *are nonempty. If both* A *and* B *are completely invariant under the map* $L(z) = z^j$ *defined on* $\triangle(0, r^*)$ *where* $0 < r^* < 1$ *and* $j \geq 2$*, then the set* A *is a union of open annuli centered at the origin and hence* B *is a union of circles centered at the origin. Furthermore, each of* A *and* B *contains a sequence of circles tending to zero.*

Proof. Let $z_0 = re^{i\theta} \in A$. Since A is open we may choose $\delta > 0$ such that the arc $\alpha_{z_0} = \{re^{i\omega} : |\theta - \omega| \leq \frac{\delta}{2}\} \subset A$.

Fix a positive integer n such that $j^n \delta > 2\pi$. Since $L^n(z) = z^{j^n}$ we get

$$L^n(\alpha_{z_0}) = C(0, r^{j^n})$$

where $C(z, r) = \{\zeta : |\zeta - z| = r\}$.

By the forward invariance of A under L, we see that $C(0, r^{j^n}) \subset A$. But now by the backward invariance of A, we get

$$C(0, r) = L^{-n}(C(0, r^{j^n})) \subset A.$$

Thus for any $re^{i\theta} \in A$, we have $C(0,r) \subset A$. Hence A, being open, must be a union of open annuli centered at the origin and therefore B, being the complement of A in $\triangle(0,r^*)$, must be a union of circles centered at the origin.

We also note that if $C(0,r) \subset A$, then $C(0,r^{j^n}) \subset A$ is a sequence of circles tending to zero. Similarly we obtain a sequence of circles in B tending to zero. $\square$

Lemma 6. *Let $L : \triangle(0,r^*) \to \triangle(0,r^*)$, where $0 < r^* < 1$, be an analytic function such that $L(0) = 0$. Let B be a set with empty interior which is a union of circles centered at the origin and which contains a sequence of circles tending to zero. If B is forward invariant under the map L, then L is of the form*

$$L(z) = az^j$$

for some non-zero complex number a and some positive integer j.

Proof. Since $L(0) = 0$, we have, near $z = 0$,

$$\begin{aligned} L(z) &= az^j + a_1 z^{j+1} + \cdots \\ &= az^j[1 + \frac{a_1}{a} z + \cdots] \end{aligned}$$

for some non-zero complex number a and some positive integer j.

Let $h(z) = L(z)/az^j$ and note that $h(z)$ is analytic and tends to 1 as z tends to 0. We shall prove that $h(z) \equiv 1$ and the lemma then follows.

Let $C_n = C(0,r_n)$ be sequence of circles contained in B with $r_n \to 0$. We claim that each $L(C_n)$ is contained in another circle centered at the origin of, say, radius r_n'. If not, then the connected set $L(C_n)$ would contain points of all moduli between, say, r' and r''. This, however, would imply that B would contain the annulus between the circles $C(0,r')$ and $C(0,r'')$. Thus we have $L(C_n) \subset C(0,r_n')$.

So we see then that $h(C_n) \subset C(0, r_n'/|a|r_n^j)$.

But for large n we see that if h were non-constant, then $h(C_n)$ would be a path which stays near $h(0) = 1$ and winds around $h(0) = 1$. Since $h(C_n)$ is contained in a circle centered at the origin, this cannot happen. We thus conclude that h is constant. $\square$

Lemma 7. *If $B \subset \triangle(0,r^*)$ for $0 < r^* < 1$ is a nonempty relatively closed set which is completely invariant under the maps $H : z \mapsto z^j$ and $K : z \mapsto az^m$ defined on $\triangle(0,r^*)$ where a is a nonzero complex number and j,m are integers with $j,m \geq 2$, then $B = \triangle(0,r^*)$ or $|a| = 1$.*

Proof. We may assume that $|a| \leq 1$ by the following reasoning. Suppose that $|a| \geq 1$. Let b be a complex number such that $b^{m-1} = a$ and define $\psi(z) = bz$. Since $\psi \circ H \circ \psi^{-1}(z) = z^j/b^{j-1}$ and $\psi \circ K \circ \psi^{-1}(z) = z^m$, we see that the lemma would then imply that $\psi(B) = \triangle(0,|b|r^*)$ or $|b| = 1$. Since we know that $\psi(B) = \triangle(0,|b|r^*)$ exactly when $B = \triangle(0,r^*)$, and $|b| = 1$ exactly when $|a| = 1$, we may then assume that $|a| \leq 1$.

We will assume that $|a| < 1$ and show that this then implies that $B = \triangle(0,r^*)$.

We first note that by Lemma 5, B is a union of circles centered at the origin and B contains a sequence of circles tending to zero. If $C(0,\rho) \subset B$, then by the forward invariance of B under H, we see that $C(0,\rho^j) \subset B$. Also we get that if $C(0,\rho) \subset B$, then by the forward invariance of B under K, we have $C(0,|a|\rho^m) \subset B$. Using a change of coordinates $r = \log \rho$ this invariance can be stated in terms of the new functions

$$t(r) = jr \text{ and } s(r) = mr + c$$

where $c = \log |a| < 0$.

So the action of H and K on $\triangle(0, r^*)$ is replaced by the action of t and s on $I = [-\infty, \log r^*)$, respectively. In particular, we define

$$B' = \{\log \rho : C(0, \rho) \subset B\} \cup \{-\infty\}$$

keeping in mind that B is a union of circles centered at the origin. Then

$$s(B') \subset B', \tag{4.1}$$

$$s^{-1}(B') \cap I \subset B', \tag{4.2}$$

$$t(B') \subset B', \tag{4.3}$$

$$t^{-1}(B') \cap I \subset B', \tag{4.4}$$

$$B' \text{ is closed in the relative topology on } I. \tag{4.5}$$

In order to make calculations a bit easier we rewrite $s(r) = r_0 + m(r - r_0)$ where $r_0 = -c/(m - 1) > 0$.

Hence

$$\begin{aligned}
s^n(r) &= r_0 + m^n(r - r_0),\\
s^{-n}(r) &= r_0 + m^{-n}(r - r_0),\\
t^n(r) &= j^n r,\\
t^{-n}(r) &= j^{-n} r.
\end{aligned}$$

Consider

$$(t^{-n} \circ s^{-n} \circ t^n \circ s^n)(r) = r - r_0 + \frac{r_0}{j^n} + \frac{r_0}{m^n} - \frac{r_0}{m^n j^n}.$$

Let

$$d_n = \frac{r_0}{j^n} + \frac{r_0}{m^n} - \frac{r_0}{m^n j^n} = r_0 \frac{m^n + j^n - 1}{m^n j^n}$$

and note that $0 < d_n \le r_0$ with $d_n \to 0$ as $n \to \infty$.

We also note that $(t^{-n} \circ s^{-n} \circ t^n \circ s^n)(r) = r - r_0 + d_n$ implies that $(s^{-n} \circ t^{-n} \circ s^n \circ t^n)(r) = r + r_0 - d_n$ since these two functions are inverses of each other.

We claim that $(-\infty, \log r^* - r_0] \subset B'$.

Let us suppose that this is not the case, and suppose that $(r', \tilde{r})$ is an interval disjoint from B' with $-\infty < r' < \tilde{r} \le \log r^* - r_0$. Since B' is a closed subset of $[-\infty, \log r^* - r_0]$, we may assume that this interval is expanded so that $r' \in B'$. Note that here we used the fact that B contains a sequence of circles going to 0, hence B' contains a sequence of points going to $-\infty$.

Let $r'_n = (t^{-n} \circ s^{-n} \circ t^n \circ s^n)(r') = r' - r_0 + d_n$. We claim that each r'_n is in B'. This is almost obvious from the invariance of B' under s and t in (4.1) through (4.4), but some care needs to be taken to insure that each application of s, t, s^{-1}, and t^{-1} takes points to the right domain. By (4.1) we see that $s(r'), s^2(r'), \ldots, s^n(r') \in B'$. Hence by (4.3) we get $(t \circ s^n)(r'), (t^2 \circ s^n)(r'), \ldots, (t^n \circ s^n)(r') \in B'$.

Since $s^{-1}(r) > r$ for $r \in (-\infty, r_0)$ we see that because $(s^{-n} \circ t^n \circ s^n)(r')$ is clearly less than r' (as $t(r) < r$ for $r \in (-\infty, 0)$), also each of $(s^{-1} \circ t^n \circ s^n)(r'), \ldots, (s^{-n} \circ$

$t^n \circ s^n)(r')$ must be less than $r' < \log r^*$. Hence by (4.2) we see that each of these points lies in B'.

Similarly, since $t^{-1}(r) > r$ for $r \in (-\infty, 0)$ and $(t^{-n} \circ s^{-n} \circ t^n \circ s^n)(r') = r' - r_0 + d_n \leq r' < \log r^* < 0$, also each of $(t^{-1} \circ s^{-n} \circ t^n \circ s^n)(r'), \ldots, (t^{-n} \circ s^{-n} \circ t^n \circ s^n)(r')$ lies in $I = [-\infty, \log r^*)$. Hence by (4.4) each of these points is in B' and so each $r'_n \in B'$.

Hence we conclude that $r' - r_0 \in B'$ since B' is relatively closed in I and $r'_n \to r' - r_0 \in I$. Note also that $r'_n \searrow r' - r_0$.

Now we claim that for any $r'' \in B' \cap (-\infty, \log r^* - r_0)$, we have $r'' + r_0 \in B'$. Let $r''_n = (s^{-n} \circ t^{-n} \circ s^n \circ t^n)(r'') = r'' + r_0 - d_n$. Noting that each $r''_n < r'' + r_0 < \log r^*$ we may again use the invariance of B' under s and t in (4.1) through (4.4) in a similar fashion as above to obtain that each $r''_n \in B'$. Thus also the limit $r'' + r_0 \in B'$.

Consider again $r'_n \searrow r' - r_0$. By applying the above claim to each $r'_n \leq r' < \log r^* - r_0$, we get that each $r'_n + r_0 \in B'$. Since $r'_n + r_0 \searrow r'$ we then see that we have contradicted the statement that $(r', \tilde{r})$ is disjoint from B'.

So we conclude that $(-\infty, \log r^* - r_0] \subset B'$. Clearly then by the partial backward invariance of B' under the map t we get $[-\infty, \log r^*) \subset B'$. Hence we conclude that $\triangle(0, r^*) = B$. $\square$

In order to avoid some technical difficulties we will make use of the following well known result.

Theorem A. *A polynomial f of degree k is conjugate near ∞ to the map $z \mapsto z^k$ near the origin. More specifically, there exists a neighborhood U of ∞ such that we have a univalent*

$$\phi : U \to \triangle(0, r^*) \text{ for } 0 < r^* < 1 \text{ with } \phi(\infty) = 0 \text{ and } \phi \circ f \circ \phi^{-1}(z) = z^k.$$

Proof. After conjugating f by $z \mapsto 1/z$ we may apply Theorem 6.10.1 in [1], p. 150 to obtain the desired result. $\square$

We will denote the conjugate function of f by F, i.e.,

$$F(z) = \phi \circ f \circ \phi^{-1}(z) = z^k.$$

In order to further simplify some of the following proofs we will assume that $\phi(U) = D = \triangle(0, r^*)$. Note that U is forward invariant under f since $D = \triangle(0, r^*)$ is forward invariant under F. We may and will also assume that U is forward invariant under g as well.

We now define a corresponding function for g using the same conjugating map as we did for f. Let G be the function defined on $D = \triangle(0, r^*)$ given by

$$G = \phi \circ g \circ \phi^{-1}.$$

Note that $G(D) \subset D$.

Via this change of coordinates, we will use the mappings F and G to obtain information about the mappings f and g. In transferring to this simpler coordinate system we make the following definitions.

Let W be the complement of E, i.e.,

$$W = \overline{\mathbb{C}} \setminus E.$$

Note that W is open and it is also completely invariant under both f and g since it is the complement of a set which is completely invariant under both f and g.

Let W' denote the image of W under ϕ, i.e., $W' = \phi(U \cap W)$. Let E' denote the image of E under ϕ, i.e., $E' = \phi(U \cap E)$. Thus W' is open and E' is closed in the relative topology of D. Note that W' and E' are disjoint since W and E are disjoint and ϕ is univalent. Also since $W \cup E = \overline{\mathbb{C}}$ it easily follows that $W' \cup E' = \phi(U) = D$.

By the forward invariance of $W \cap U$ under f we see that

$$F(W') = F \circ \phi(W \cap U) = \phi \circ f(W \cap U) \subset \phi(W \cap U) = W'. \tag{4.6}$$

Similarly we get

$$F(E') \subset E'. \tag{4.7}$$

Since E' and W' are disjoint and forward invariant under F, and since $E' \cup W' = D$, we see that

$$F^{-1}(E') \cap D \subset E', \tag{4.8}$$

$$F^{-1}(W') \cap D \subset W'. \tag{4.9}$$

Note that in the same way as we obtained the results for F we get

$$G(W') \subset W', \tag{4.10}$$

$$G(E') \subset E', \tag{4.11}$$

$$G^{-1}(E') \cap D \subset E', \tag{4.12}$$

$$G^{-1}(W') \cap D \subset W'. \tag{4.13}$$

Lemma 8. *If $G(z) = az^l$ with $|a| = 1$, then $J_f = J_g$.*

Proof. The proof relies on the use of Green's functions. It is well known that the unbounded components F_∞ and G_∞ support Green's functions with pole at ∞ which we will denote by G_f and G_g respectively. It is also well known that on U we have

$$G_f(z) = -\log|\phi(z)|$$

since ϕ is a map which conjugates f to $z \mapsto z^k$ (see [1], p. 206).

Since for $\psi(z) = bz$ where $b^{l-1} = a$, the function $\psi \circ \phi$ conjugates g in U to $z \mapsto z^l$, we get in U,

$$G_g(z) = -\log|\psi \circ \phi(z)| = -\log|b\phi(z)| = -\log|\phi(z)|$$

where the last equality uses the fact that $|a| = 1$, and so $|b| = 1$.

Hence $G_f = G_g$ in U. Since G_f and G_g are each harmonic away from ∞ we get that $G_f = G_g$ on the unbounded component C of $F_\infty \cap G_\infty$.

We claim that this implies that $J_f = J_g$. Assuming that $J_f \neq J_g$, we see by Remark 1 that there exists a point which lies in the Julia set of one function, yet in the unbounded component of the Fatou set of the other function. Let us therefore suppose that $z_0' \in J_g \cap F_\infty$. Let γ be a path in F_∞ connecting z_0' to ∞. We see that γ must intersect ∂C somewhere, say at z_0. Since $z_0 \in F_\infty \cap \partial C$ we get $z_0 \in \partial G_\infty = J_g$.

We may select a sequence $z_n \in C$ such that $z_n \to z_0$. Since z_0 lies on the boundary of the domain of the Green's function G_g, i.e., $z_0 \in J_g$, we have $G_g(z_n) \to 0$ (see [1], p.207). Since z_0 lies in the domain of the Green's function G_f we see that

$G_f(z_n) \to G_f(z_0) > 0$. We cannot have both of these happen since $G_f(z_n) = G_g(z_n)$ and so we conclude that $J_g \cap F_\infty = \emptyset$. Hence we conclude that $J_f = J_g$. $\square$

We now are able to prove Theorem 1.

Proof of Theorem 1. Consider whether or not E has nonempty interior. If $E^o \neq \emptyset$, then by Lemma 2 we get $E = \overline{\mathbb{C}}$.

If $E^o = \emptyset$, then Lemma 5 implies that the set W' is a union of open annuli centered at the origin and hence E' is a union of circles centered at the origin. Since $E^o = \emptyset$, the set E' has empty interior.

Since we know by Remark 2 that there exists a sequence of points in E tending to infinity when $J_f \neq J_g$, also E' must contain a corresponding sequence of circles tending to zero. By Lemma 6 we see that the function G is of the form

$$G(z) = az^l$$

for some non-zero complex number a.

By considering the set E', we see that Lemma 7 implies that $|a| = 1$. We see that Lemma 8 then implies $J_f = J_g$. $\square$

References

[1] Alan F. Beardon. *Iterations of Rational Functions*. Springer-Verlag, New York, 1991.
[2] A. Hinkkanen and G.J. Martin. The dynamics of semigroups of rational functions I. *Proc. London Math. Soc.*, 3:358–384, 1996.

Department of Mathematics, University of Illinois, Urbana, Illinois 61801