UNIFORMLY PERFECT ANALYTIC AND CONFORMAL
NON-AUTONOMOUS ATTRACTOR SETS

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Abstract. Conditions are given which imply that certain non-autonomous
analytic iterated function systems (NIFS’s) in the complex plane \( \mathbb{C} \) have uni-
formly perfect attractor sets. Examples are given to illustrate the main the-
orem, as well as to indicate how it generalizes other results. Examples are
also given to illustrate how possible generalizations of corresponding results
for autonomous IFS’s do not hold in general in this more flexible setting.

1. Introduction

The aim of this paper is to obtain uniform perfectness results for attractors of certain
non-autonomous iterated function systems. When the maps are all analytic and the
IFS is autonomous, results of the type we seek are found in [14]. We also note that [7]
includes related results for similar systems (which require an open set condition).
Certain constructions in [15] are \textit{non-autonomous} iterated function systems shown
to have uniformly perfect attractors (though those examples were not presented as
attractors, but rather as Cantor-like constructions - see Example 4.1 in this paper),
while other examples there are not uniformly perfect. We look to generalize those
results here, and we begin by following [12] to introduce the main framework and
definitions (with some key differences) of \textit{non-autonomous iterated function systems}
(NIFS’s). We also note that attractors of NIFS’s are often Moran-set constructions
(see [17] for good exposition of such).

A \textit{non-autonomous iterated function system} (NIFS) \( \Phi \) on the pair \((U, X)\) is given by
a sequence \( \Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \ldots \), such that each \( \Phi^{(j)} \) is a collection of non-constant
functions \((\varphi_{i}^{(j)} : U \to X)_{i \in I^{(j)}}\), where each function maps the non-empty open
connected set \( U \subset \mathbb{C} \) into a compact set \( X \subset U \) such that there exists \( 0 < s < 1 \)

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and a metric $d$ on $U$ where $d(\varphi(z), \varphi(w)) \leq sd(z, w)$ for all $z, w \in X$ and all $\varphi \in \bigcup_{j=1}^{\infty} \Phi^{(j)}$. We also stipulate that $d$ induces the Euclidean topology on $X$. Thus this system is uniformly contracting on the metric space $(X, d)$.

We define a NIFS and its corresponding attractor set (see Definition 1.2) to be analytic (respectively, conformal) if all the maps are analytic (respectively, conformal) on $U$. Note that here and throughout conformal means analytic and one-to-one (globally on $U$, not just locally).

Important differences from [12] in the above setup are: 1) We do not impose that $X$ have other geometric properties such as convexity or a smooth boundary. 2) The maps do not need to be conformal. In fact, they do not even need to be locally conformal. 3) In [12], the focus is on certain measures and dimension of the attractor maps do not need to be conformal. In fact, they do not even need to be locally analytic. 4) We do not impose an open set condition, and, in fact, there can be substantial overlap in sets of the form $\varphi_a^{(j)}(X)$ and $\varphi_b^{(j)}(X)$. 5) The main object of interest to this paper is the analytic NIFS, and the condition imposed that each $\varphi$ map $U$ into $X$ allows us, under this condition of analyticity, to take the metric $d$ to be the hyperbolic metric on $U$ (see Section 3).

Given an NIFS, we wish to study the limit set (or attractor) which we can define after the next definition.

**Definition 1.1** (Words). For each $k \in \mathbb{N}$, we define the symbolic spaces

$$I^k := \prod_{j=1}^{k} I^{(j)} \quad \text{and} \quad I^\infty := \prod_{j=1}^{\infty} I^{(j)}.$$  

Note that a $k$-tuple $(\omega_1, \ldots, \omega_k) \in I^k$ may be identified with the corresponding word $\omega_1 \ldots \omega_k$. When $\omega^* \in I^\infty$ has $\omega_j^* = \omega_j$ for $j = 1, \ldots, k$, we call $\omega^*$ an extension of $\omega = \omega_1 \ldots \omega_k \in I^k$.

**Definition 1.2.** For all $k \in \mathbb{N}$ and $\omega = \omega_1 \ldots \omega_k \in I^k$, we define $\varphi_{\omega} := \varphi_{\omega_1} \circ \ldots \circ \varphi_{\omega_k}$ with

$$X_\omega := \varphi_{\omega}(X) \quad \text{and} \quad X_k := \bigcup_{\omega \in I^k} X_\omega.$$  

The limit set (or attractor) of $\Phi$ is defined as

$$J = J(\Phi) := \bigcap_{k=1}^{\infty} X_k.$$  

**Remark 1.1.** The attractor $J$ does not have to be compact. For example, $J$ is not compact for the autonomous system (see Section 2) given in Example 4.3 of [14]. However, if each index set $I^{(j)}$ is finite, then each $X_k$ is compact and hence so is $J$.

**Notation to be used throughout:** Let $q$ be a metric. For a set $F \subseteq \mathbb{C}$, we define its diameter to be $\text{diam}_q F = \sup\{q(z, w) : z, w \in F\}$ and $\epsilon$-ball about $F$ to be $B_q(F, \epsilon) = \{z : \text{dist}_q(z, F) < \epsilon\}$ where $\text{dist}_q(z, F) = \inf\{q(z, w) : w \in F\}$. Also, for $w \in \mathbb{C}$ and $r > 0$ we define the disk and circle, respectively, by $\Delta_q(w, r) = \{z : q(z, w) < r\}$ and $C_q(w, r) = \{z : q(z, w) = r\}$. If no metric is noted, then it is assumed that the metric is the Euclidean metric.
Remark 1.2 (Projection Map). Consider $\omega^* \in I^{\infty}$ and note that the compact sets $\varphi_{\omega^*_1 \cdots \omega^*_n}(X)$ decrease with $\text{diam}_d(\varphi_{\omega^*_1 \cdots \omega^*_n}(X)) \leq s^n \text{diam}_d(X) \to 0$ as $n \to \infty$. Hence $\bigcap_{n=1}^{\infty} \varphi_{\omega^*_1 \cdots \omega^*_n}(X)$ contains just a single point that we call $\pi(\omega^*)$. Note that $\pi(\omega^*) \in J$ since it clearly belongs to each $\varphi_{\omega^*_1 \cdots \omega^*_n}(X) \subseteq X_n$. We call $\pi_{\Phi} : I^{\infty} \to J$ the projection map.

Further note that for any non-empty compact $\bar{X} \subseteq X$ that is forward invariant under $\Phi$, i.e., $\varphi(\bar{X}) \subseteq \bar{X}$ for all $\varphi \in \bigcup_{j=1}^{\infty} \Phi^j$, we have that $\bigcap_{n=1}^{\infty} \varphi_{\omega^*_1 \cdots \omega^*_n}(\bar{X}) = \bigcap_{n=1}^{\infty} \varphi_{\omega^*_1 \cdots \omega^*_n}(X)$ since each is a singleton set with the left set being a subset of the right set. We summarize this by saying that the projection map $\pi_{\Phi}$ is independent of the choice of non-empty compact forward invariant set $X$.

Remark 1.3 (Pieces of $X_k$). The limit set $J = \bigcap_{k=1}^{\infty} X_k$ is a decreasing intersection of the $X_k$, but an important facet of the $X_k$ is that it is the union of what we call the pieces of $X_k$, each which must contain both a limit point and a fixed point. More precisely, note that for any $k \in \mathbb{N}$ and $\omega = \omega_1 \cdots \omega_k \in I^k$, we have that the piece $\varphi_{\omega}(X)$ of $X_k$, for which $\text{diam}_d(\varphi_{\omega}(X)) \leq s^k \text{diam}_d(X)$, contains both the fixed point of the contraction $\varphi_{\omega}$ and the point $\pi_{\Phi}(\omega^*) \in J$ for any extension $\omega^* \in I^{\infty}$ of $\omega$. Note also that the pieces of $X_k$ are not necessarily components of $X_k$ since the pieces may overlap in general.

In the NIFS systems studied in [12] (see Definition and Lemma 2.4 of [12], which makes key use of the open set condition - something we do not impose here), it must be the case that $\pi_{\Phi}(I^{\infty}) = J$. We do not necessarily have this in all cases (see Example 1.1), but we do have the following result.

**Lemma 1.1.** Let $J'(\Phi) = \{z : \phi_{\omega}(z) = z \text{ for some } \omega \text{ in some } I^k\}$ where $\Phi$ is a NIFS on $(U, X)$. Then $J(\Phi) \subseteq J'(\Phi)$, and hence $\overline{J(\Phi)} \subseteq \overline{J'(\Phi)}$. Also,

$$J(\Phi) = \pi_{\Phi}(I^{\infty}),$$

and so, if $\pi_{\Phi}(I^{\infty})$ is compact, then $J(\Phi) = \pi_{\Phi}(I^{\infty})$.

We note that in the non-autonomous case, unlike in the autonomous case (see Claim 2.1), $J'$ does not necessarily have to be a subset of $J$, or even of $\overline{J}$. See Example 4.2.

**Proof.** Let $z \in J$ and $\delta > 0$. Choose $k$ such that $s^k \text{diam}(X) < \delta$. Since $z \in J \subseteq X_k$, there exists $\omega \in I^k$ such that $z \in \varphi_{\omega}(X)$. Extend $\omega$ to any $\omega^* \in I^{\infty}$ and note that, as stated in Remark 1.3, $\varphi_{\omega}(X)$ contains both the fixed point of the contraction $\varphi_{\omega}$ and the point $\pi_{\Phi}(\omega^*) \in J$. Since $\varphi_{\omega}(X) \subseteq \Delta_d(z, s^k \text{diam}(X)) \subseteq \Delta_d(z, \delta)$, we conclude $J \subseteq J'(\Phi) \cap \overline{\pi_{\Phi}(I^{\infty})}$. This and the definition of $\pi_{\Phi}$ yield that $\overline{J} \subseteq \overline{\pi_{\Phi}(I^{\infty})} \subseteq J'$.

The final statement follows since if $\pi_{\Phi}(I^{\infty})$ is compact, we have $J(\Phi) \subseteq \overline{J(\Phi)} = \pi_{\Phi}(I^{\infty}) = \pi_{\Phi}(I^{\infty}) \subseteq J(\Phi)$. 

In certain examples, it is convenient to change the set $X$ to a more convenient forward invariant compact set. The following result shows that such a change to $X$, though it may affect $J$ (see Example 1.2), will not affect $\overline{J}$, the central object of study for this paper.
Lemma 1.2. Let $\tilde{X} \neq \emptyset$ be a compact subset of $X$ that is forward invariant under NIFS $\Phi$ on $(U, X)$, i.e., $\varphi(\tilde{X}) \subseteq X$ for all $\varphi \in \bigcup_{j=1}^{\infty} \Phi^{(j)}$. Then, calling $\tilde{X}_k := \bigcup_{\omega \in \Gamma} \varphi_\omega(\tilde{X})$, we have

$$J(\Phi) = \bigcap_{k=1}^{\infty} X_k = \bigcap_{k=1}^{\infty} \tilde{X}_k.$$  

Hence, if each $\tilde{X}_k$ is compact, then $J(\Phi) = \bigcap_{k=1}^{\infty} X_k = \bigcap_{k=1}^{\infty} \tilde{X}_k$.

Proof. Since, as was noted in Remark 1.2, the projection map $\pi_\Phi$ is independent of the choice of non-empty compact forward invariant set $X$, the first result follows immediately from Lemma 1.1.

When each $\tilde{X}_k$ is compact, the second result follows since $J(\Phi) \subseteq J(\Phi) = \bigcap_{k=1}^{\infty} X_k = \bigcap_{k=1}^{\infty} \tilde{X}_k = \bigcap_{k=1}^{\infty} \tilde{X}_k \subseteq \bigcap_{k=1}^{\infty} X_k = J(\Phi)$.

Example 1.1 (Projection map $\pi_\Phi : I^\infty \to J$ not onto). Let $X = [0, 1]$ be the unit interval. Let $\Phi^{(1)} = \{f_1, f_2, f_3, \ldots \}$ where $f_n(z) = \frac{3}{2} + \epsilon_n$ with $\epsilon_n = \frac{1}{3} - \frac{1}{2n^2}$. Note that $\epsilon_1 = 0$ and $0 < \epsilon_n < \frac{1}{3}$ for all $n \geq 2$. Let $\Phi^{(k)} = \{f_1\}$ for all $k \geq 2$.

Technically speaking, one should first establish an open set $U \subseteq \mathbb{C}$ (e.g., $\Delta(0, 10)$) and corresponding compact subset $X$ (e.g., $\Delta(0, 9)$) to satisfy the NIFS condition that each function map $U$ into $X$. And then afterwards use Lemma 1.2 to replace $X$ by the forward invariant interval $[0, 1]$ without altering the limit set $J$. However, in later examples we forgo such details leaving it for the reader to quickly check that such a procedure can be validly executed.

We now show $\frac{1}{3} \notin J \setminus \pi_\Phi(I^\infty)$. Since, for each $n \in \mathbb{N}$, we have $\frac{1}{3} \in [\epsilon_n, \frac{1}{3}] = [\epsilon_n, \frac{1}{3} + \epsilon_n] = f_n \circ f_1^{-1}(X) \subseteq X_n$, we see $\frac{1}{3} \in J$. However, for each $\omega \in I^\infty$ there must be some $f_n \in \Phi^{(1)}$ such that $\{\pi_\Phi(\omega)\} = \bigcap_{n=1}^{\infty} f_n \circ f_1^{-1}(X) = \bigcap_{n=1}^{\infty} [\epsilon_n, \frac{1}{3} + \epsilon_n] = \{\epsilon_n\} \neq \{\frac{1}{3}\}$. Hence $\pi_\Phi(I^\infty) = \{\epsilon_n : n \in \mathbb{N}\}$, and so $\pi_\Phi(I^\infty) \neq J = \{\epsilon_n : n \in \mathbb{N}\} \cup \{1/3\}$, where the equality follows from Lemma 1.1.

Example 1.2 (J depends on X). Let $X = [-1, 1]$ and $\tilde{X} = [0, 1]$. For each $n \in \mathbb{N}$, set $z_n = \frac{1}{n^2} > 0$ and $f_n(z) = \frac{1}{2}(z - z_n) + z_n$. Clearly, each of $X$ and $\tilde{X}$ is forward invariant under each contraction $f_n$. We consider the (autonomous) system generated where each $\Phi^{(k)} = \{f_n : n \in \mathbb{N}\}$. Considering $\tilde{X}_k$ given as in Lemma 1.2, it is clear that $0 \notin \tilde{X}_1$ since, for all $n$, we see $0 \notin \left[\frac{z_n^2}{2}, \frac{1+z_n}{2}\right] = f_n(\tilde{X})$. However, for all $n \in \mathbb{N}$, since the $n$-th iterate $f_n^n(z) = \frac{1}{n^2}(z - z_n) + z_n$, we see $0 \in [0, f_n^n(1)] = [f_n^n(-1), f_n^n(1)] = f_n^n(X) \subseteq X_n$. Hence $0 \in \bigcap_{n=1}^{\infty} X_n \setminus \bigcap_{n=1}^{\infty} \tilde{X}_n$, showing that $J$ does depend on the choice of forward invariant non-empty compact set $X$ (something which cannot happen in the NIFS systems studied in [12] where, as noted, $J = \pi_\Phi(I^\infty)$ must hold).

Given an NIFS $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \ldots$ on some $(U, X)$, we note that by excluding $\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(j-1)}$, the sequence $\Phi^{(j)}, \Phi^{(j+1)}, \Phi^{(j+2)}, \ldots$ also forms an NIFS (which formally would be $\tilde{\Phi}^{(1)}, \tilde{\Phi}^{(2)}, \tilde{\Phi}^{(3)}, \ldots$ where each $\tilde{\Phi}^{(k)} = \Phi^{(k+j-1)}$). The new NIFS would then induce sets as in Definition 1.2, which we denote as $X_{\omega}^{(j)}, X_k^{(j)}$, and $J^{(j)}$ with the superscript used to indicate the relationship to the original NIFS. In
particular, for the original NIFS the sets $X_k$ may also be denoted $X_k^{(1)}$. See Example 1.3, illustrated in Figure 1, noting that the superscript indicates the column and the subscript indicates the row where a given set resides.

Remark 1.4 (Invariance Condition). Note that for any $j \geq 1$ and $k \geq 0$, we unpack the relevant definitions (defining each $X_0^{(j)} = X$) to see the following invariance condition

\[(1.1) \quad \bigcup_{i \in I^{(j)}} \varphi_i^{(j)}(X_k^{(j+1)}) = X_{k+1}^{(j)},\]

which is illustrated in Figure 1 as a way of relating the diagonally adjacent sets $X_{k+1}^{(j)}$ and $X_k^{(j+1)}$.

Remark 1.5. Letting $k \to \infty$ in the invariance condition (1.1) leads one to wonder if we must always have $\bigcup_{i \in I^{(j)}} \varphi_i^{(j)}(J^{(j+1)}) = J^{(j)}$. While this is not true in general, we do always get the inclusion

\[
\bigcup_{i \in I^{(j)}} \varphi_i^{(j)}(J^{(j+1)}) = \bigcup_{i \in I^{(j)}} \varphi_i^{(j)}(X_k^{(j+1)}) \subseteq \bigcap_{k=1}^{\infty} \bigcup_{i \in I^{(j)}} \varphi_i^{(j)}(X_k^{(j+1)}) = \bigcap_{k=1}^{\infty} X_{k+1}^{(j)} = J^{(j)}.
\]

Now consider Example 1.1 to see that equality above does not follow. Since $J^{(2)} = \{0\}$, $\bigcup_{i \in I^{(1)}} \varphi_i^{(1)}(J^{(2)}) = \{e_n : n \in \mathbb{N}\} \neq \{e_n : n \in \mathbb{N}\} \cup \{1\} = J^{(1)}$. Additionally, $\bigcup_{i \in I^{(1)}} \varphi_i^{(1)}(J^{(2)}) \neq J^{(1)}$.

Additional hypotheses, however, lead to the following result.

**Lemma 1.3.** Let $\Phi$ be a NIFS on $(U, X)$. When $\Phi^{(j)}$ is finite, we have

\[\bigcup_{i \in I^{(j)}} \varphi_i^{(j)}(J^{(j+1)}) = J^{(j)}.\]

Hence, when $\Phi^{(j)}$ is finite and $J^{(j+1)}$ is compact (e.g., when all $\Phi^{(k)}$, for $k \geq j$, are finite), we see that $\bigcup_{i \in I^{(j)}} \varphi_i^{(j)}(J^{(j+1)}) = J^{(j)}$.

**Proof.** To prove the first statement it suffices consider $j = 1$. Letting $I_1^{\infty} = \prod_{k=1}^{\infty} I^{(k)}$ and $I_2^{\infty} = \prod_{k=2}^{\infty} I^{(k)}$, we define the respective projection maps $\pi_1 : I_1^{\infty} \to J^{(1)}$ and $\pi_2 : I_2^{\infty} \to J^{(2)}$. We first note that

\[\bigcup_{i \in I^{(1)}} \varphi_i^{(1)}(I_2^{\infty}) = \pi_1(I_1^{\infty}).\]
since

\[ \bigcup_{i \in I^{(1)}} \varphi_i^{(1)}(\pi_2(I_2^{(1)})) = \bigcup_{i \in I^{(1)}} \varphi_i^{(1)}(\bigcup_{\omega \in I_2^{(1)}} \{\pi_2(\omega)\}) = \bigcup_{i \in I^{(1)}} \varphi_i^{(1)}(\bigcap_{n=2}^{\infty} \varphi_{\omega_2 \cdots \omega_n}(X)) \]

\[ = \bigcup_{i \in I^{(1)}} \bigcup_{\omega \in I_2^{(1)}} \varphi_i^{(1)}(\bigcap_{n=2}^{\infty} \varphi_{\omega_2 \cdots \omega_n}(X)) \]

\[ = \bigcup_{i \in I^{(1)}} \bigcap_{n=1}^{\infty} \varphi_{i \omega_2 \cdots \omega_n}(X) = \bigcup_{\omega^* \in I_1^{(1)}} \{\pi_1(\omega^*)\} = \pi_1(I_1^{(1)}), \]

where Lemma 3.5 was used with regard to \( \varphi_i^{(1)} \) and the decreasing compact sets \( \varphi_{\omega_2 \cdots \omega_n}(X) \).

Then, using Lemma 1.1, we see

\[ \overline{J^{(1)}} = \overline{\pi_1(I^{(1)})} = \bigcup_{i \in I^{(1)}} \varphi_i^{(1)}(\pi_2(I_2^{(1)})) = \bigcup_{i \in I^{(1)}} \varphi_i^{(1)}(\pi_2(I_2^{(1)})) \]

\[ = \bigcup_{i \in I^{(1)}} \varphi_i^{(1)}(\pi_2(I_2^{(1)})) = \bigcup_{i \in I^{(1)}} \varphi_i^{(1)}(\overline{J^{(1)}}), \]

where we used the facts that the union is finite, each \( \varphi_i^{(1)} \) is continuous, and the set \( \pi_2(I_2^{(1)}) \) is compact.

The final statement of the lemma follows since, if \( \Phi^{(j)} \) is finite and \( J^{(j+1)} \) is compact, then \( J^{(j)} \subseteq \overline{J^{(j)}} = \bigcup_{i \in I^{(j)}} \varphi_i^{(j)}(\overline{J^{(j+1)}}) = \bigcup_{i \in I^{(j)}} \varphi_i^{(j)}(J^{(j+1)}) \subseteq J^{(j)} \), where the last inclusion is justified by Remark 1.5.

\[ \Box \]

**Example 1.3.** Let \( X = [0, 1] \) denote the closed unit interval. Consider a sequence \( (a_j) \) such that each \( 0 < a_j \leq 1/3 \), and define maps \( \varphi_1^{(j)}(z) = a_j z \) and \( \varphi_2^{(j)}(z) = a_j (z - 1) + 1 \). Then the families of maps \( \Phi^{(j)} = \{\varphi_1^{(j)}, \varphi_2^{(j)}\} \) define an NIFS. See Figure 1.

**Remark 1.6 (Combining Stages).** It will be useful later to analyze a limit set of some NIFS \( \Phi \) by first combining stages. Here we present what this means, in particular, showing that this does not alter the limit set. First, for families of maps \( \Gamma_1, \Gamma_2, \ldots, \Gamma_n \), we define \( \Gamma_1 \circ \Gamma_2 \circ \cdots \circ \Gamma_n \) to be \( \{f_1 \circ f_2 \circ \cdots \circ f_n : f_i \in \Gamma_i\} \).

Given an NIFS \( \Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, \ldots \) on some \( (U, X) \), we can create a new NIFS by combining finite strings of stages as follows. Consider any strictly increasing sequence \( (k_n)_{n=1}^{\infty} \) of positive integers and define a new NIFS \( \Phi \) by \( \Phi^{(1)} = \Phi^{(1)} \circ \cdots \circ \Phi^{(k_1)}, \Phi^{(2)} = \Phi^{(k_1+1)} \circ \cdots \circ \Phi^{(k_2)}, \) and, in general for \( n > 1 \), \( \Phi^{(n)} = \Phi^{(k_{n-1}+1)} \circ \cdots \circ \Phi^{(k_n)} \).

Notice that \( \Phi \) inherits all the defining properties of an NIFS from \( \Phi \). Furthermore, \( J(\Phi) = \bigcap_{n=1}^{\infty} X_{k_n} = \bigcap_{k=1}^{\infty} X_k = J(\Phi) \), since the sets \( X_k \) are decreasing.
2. Review of Autonomous Attractors and Statements of the Main Theorems

In this section we review known results for autonomous attractors and then state the main results for non-autonomous attractors in Theorems 2.1 and 2.2.

The system $\Phi$ is called autonomous (and thus just called an IFS) if $I^{(j)}$ and $\Phi^{(j)}$ are independent of $j$, i.e., each $\Phi^{(j)} = \{g_i : i \in I\}$ for some index set $I$. In such an instance we use the notation $A$ for the attractor instead of $J$ in order to give a notational reminder that we are in a very special (and previously well-studied) case. For such an autonomous system, we let $G = \langle g_i : i \in I \rangle$ denote the set of all finite compositions of generating maps $\{g_i : i \in I\}$, and, following [14], simply say $G = \langle g_i : i \in I \rangle$ is an IFS on $(U, X)$.

**Claim 2.1.** When $\Phi$ is autonomous, the attractor set $A = J$ given in Definition 1.2 satisfies $A \supseteq A'$ and $\overline{A} = \overline{A'}$, the closure of $A'$ in the Euclidean topology
(equivalently given by the metric $d$), where $A' = A'(G) := \{ z : \text{there exists } g \in G \text{ such that } g(z) = z \}$ is the set of (attracting) fixed points of $G$.

Note that in [14] the attractor set was defined to be $\overline{A'}$ and not defined in terms of $X_k$ as in Definition 1.2. This claim, however, shows that the closures of the sets given by the two definitions yield the same set.

Proof. Let $z \in A'$. Since the system is autonomous, there exist some $k \in \mathbb{N}$ and $\omega \in I^k$ such that $\phi_\omega(z) = z$. Clearly then for each $n$ we see that $z \in \phi^n_\omega(X) \subseteq X_{kn}$, where $\phi^n_\omega \in I^{kn}$ denotes the $n$th iterate of $\phi_\omega$ (note that the autonomous condition is used here). Hence $z \in \bigcap_{n=1}^\infty X_{kn} = \bigcap_{k=1}^\infty X_k = J = A$. Thus $A' \subseteq A$, and so $A' \subseteq A$.

The reverse inclusion follows from Lemma 1.1. □

If each $\Phi^{(j)} = \{g_1, \ldots, g_N\}$, a situation we call the finite autonomous case, then the attractor $A$ is the unique non-empty compact subset of $U$ that has the self-similarity property given by

$$A = \bigcup_{i=1}^N g_i(A)$$

(see [6], p. 724). We note that in this finite autonomous case, the sets $X^{(j)}_k$, and $J^{(j)}$ are all independent of $j$ (in Example 1.3 illustrated in Figure 1 this would amount to sets across rows being identical because $a_1 = a_2 = a_3 = \ldots$). Furthermore, the invariance shown in Remark 1.4 then becomes $\bigcup_{i=1}^N g_i(X_k) = X_{k+1}$, which by taking the limit as $k \to \infty$ in a suitable space produces (2.1)(see [6] or apply Lemma 1.3).

Remark 2.1. We also point out that in [8, 9] the limit set $J$ of a conformal IFS is defined a bit differently, but with a clear connection to our definition. See [8, 9] for a discussion on the Hausdorff dimension, packing dimension, and other properties of limit sets of their conformal IFS’s.

Uniformly perfect sets, which are defined in Section 3, were introduced by A. F. Beardon and Ch. Pommerenke in 1978 in [2]. Such sets cannot be separated by annuli that are too large in modulus (equivalently, large ratio of outer to inner radius). Thus, uniform perfectness, in a sense, measures how “thick” a set is near each of its points and is related in spirit to many other notions of thickness such as Hausdorff content and dimension, logarithmic capacity and density, Hölder regularity, and positive injectivity radius for Riemann surfaces. For an excellent survey of uniform perfectness and how it relates to these and other such notions see Pommerenke [11] and Sugawa [16].

In [13] certain autonomous conformal attractor sets are shown to be uniformly perfect, when the generating maps are Möbius. Then in [14] a collection of results regarding uniform perfectness are given for autonomous analytic attractor sets. The motivation for the current paper is to explore to what degree, if any, these results generalize to the non-autonomous case. Hence we first state the major results from [14].
Theorem A (Corollary 1.1 in [14]). Let $G = \langle g_i : i \in I \rangle$ be an analytic IFS on $(U, X)$ such that there exists $\eta > 0$ where $|g_i'| \geq \eta$ on $A$ for all $i \in I$. If $A$ has infinitely many points, then $A$ is uniformly perfect.

Theorem B (Corollary 1.2 in [14]). Let $G = \langle g_i : i \in I \rangle$ be a conformal IFS on $(U, X)$ such that there exist $\eta > 0$ where $|g_i'| \geq \eta$ on $A$ for all $i \in I$. If $A$ contains more than one point, then $A$ is uniformly perfect.

Theorem C (Corollary 1.3 in [14]). Let $G = \langle g_1, \ldots, g_N \rangle$ be a conformal IFS on $(U, X)$. If $A$ contains more than one point, then $A$ is uniformly perfect.

The proofs of Theorems A-C in [14], which consider only autonomous systems, heavily rely on the facts (i) $A' \subseteq A$, and (ii) $A$ is forward invariant under $G$, i.e., for every $a \in A$ and $g \in G$ we have $g(a) \in A$ (Lemma 2.2 in [14]). The main complicating features of the non-autonomous systems we wish to consider in this paper are that these properties do not hold or generalize in a way that allows for the techniques in [14] to be easily adapted to such more general systems (see Example 4.2 and Remark 4.3). In this paper, however, we do prove Theorem 2.1 regarding conformal NIFS’s and Theorem 2.2 regarding analytic NIFS’s.

**Theorem 2.1.** Let $\Phi$ be a conformal NIFS on $(U, X)$. Suppose

(i) (Möbius Condition) each map in $\varphi \in \cup_{j \in \mathbb{N}} \Phi^{(j)}$ is Möbius, and

(ii) (Two Point Separation Condition) there exists $\delta > 0$ such that each $\Phi^{(j)}$, for $j \in \mathbb{N}$, contains (not necessarily distinct) maps $\varphi_a^{(j)}$ and $\varphi_b^{(j)}$ such that for some (not necessarily distinct) $z_a, z_b \in J^{(j+1)}$ we have $|\varphi_a^{(j)}(z_a) - \varphi_b^{(j)}(z_b)| \geq \delta$, and

(iii) (Derivative Condition) there exists $\eta > 0$ such that for all $\varphi \in \cup_{j \in \mathbb{N}} \Phi^{(j)}$ we have $|\varphi'\varphi| \geq \eta$ on $X$.

Then each $\overline{J^{(j)}}$ is uniformly perfect. Furthermore, for a given $(U, X)$, the modulus of any annulus separating any $J^{(j)}$ is bounded above by a constant depending only on $\delta$ and $\eta$.

**Remark 2.2.** Instead of verifying the Two Point Separation Condition as stated, it is often easier to check any of the increasingly stronger conditions:

1. there exists $\delta > 0$ such that each $\Phi^{(j)}$, for $j \in \mathbb{N}$, contains at least two maps $\varphi_a^{(j)}$ and $\varphi_b^{(j)}$ such that for some $z \in J^{(j+1)}$ we have $|\varphi_a^{(j)}(z) - \varphi_b^{(j)}(z)| \geq \delta$,

2. there exists $\delta > 0$ such that each $\Phi^{(j)}$, for $j \in \mathbb{N}$, contains at least two maps $\varphi_a^{(j)}$ and $\varphi_b^{(j)}$ such that for all $z \in X$ we have $|\varphi_a^{(j)}(z) - \varphi_b^{(j)}(z)| \geq \delta$,

3. there exists $\delta > 0$ such that each $\Phi^{(j)}$, for $j \in \mathbb{N}$, contains at least two maps $\varphi_a^{(j)}$ and $\varphi_b^{(j)}$ such that the images $\varphi_a^{(j)}(X)$ and $\varphi_b^{(j)}(X)$ are at least a distance $\delta$ apart.

Note that (3) is much weaker than what in the literature is often called the Strong Separation Condition for finite autonomous systems, which can be equivalently stated as such: there exists $\delta > 0$ such that for all distinct maps $\varphi_a^{(j)}, \varphi_b^{(j)} \in \Phi^{(j)}$, for $j \in \mathbb{N}$, the images $\varphi_a^{(j)}(X)$ and $\varphi_b^{(j)}(X)$ are at least a distance $\delta$ apart.
We also note that this Two Point Separation Condition shows that, for each \( j \in \mathbb{N} \), \( \text{diam}(J^{(j)}) \geq \delta \) since for any \( z_a, z_b \in J^{(j+1)} \) and \( \varphi_a^{(j)}, \varphi_b^{(j)} \in \Phi^{(j)} \), we have, by the inclusion proved in Remark 1.5, \( \varphi_a^{(j)}(z_a), \varphi_b^{(j)}(z_b) \in J^{(j)} \). In the proof of Theorem 2.1, the Two Point Separation Condition is only used to obtain a uniform lower bound on \( \text{diam}(J^{(j)}) \).

Remark 2.3. Theorem 2.1 applies much more generally when we recall that one can combine stages in the manner described in Remark 1.6. Specifically, we may show \( \overline{J(\Phi)} \) is uniformly perfect by applying Theorem 2.1 to any \( \tilde{\Phi} \) created by combining stages in \( \Phi \). This is exactly the technique used to show uniform perfectness in Example 4.2.

Theorem 2.2. Suppose \( \Phi \) is an analytic NIFS such that \( \overline{J^{(n)}} \), for some integer \( n > 1 \), is uniformly perfect (e.g., when the NIFS given by \( \Phi^{(n)}, \Phi^{(n+1)}, \Phi^{(n+2)}, \ldots \), satisfies the hypotheses of Theorem 2.1). Suppose also that \( \overline{\Phi^{(j)}} = \Phi^{(1)} \circ \cdots \circ \Phi^{(n-1)} \) is finite. Then \( \overline{J(\Phi)} \) is uniformly perfect.

This paper is organized as follows. Section 3 contains basic lemmas and definitions. Section 4 presents some examples to demonstrate why the possible generalizations of Theorems A-C do not hold for general NIFS’s, in particular, showing that both (i) and (ii) can fail. Also, in Section 4 we show that our main result generalizes Theorem 4.1(2) of [15]. Section 5 is then used to prove the Theorems 2.1 and 2.2.

3. Definitions and basic facts

The main object of interest to this paper is the analytic NIFS. This allows us, via the next result used similarly in [14], to employ the hyperbolic metric in the definition of NIFS. In particular, any sequence \( \Phi^{(i)} \), \( \Phi^{(j)}, \Phi^{(j+2)}, \ldots \), such that each \( \Phi^{(j)} \) is a collection of non-constant analytic functions \( (\varphi_i^{(j)} : U \to X)_{i \in J^{(j)}} \), where each function maps the non-empty open connected set \( U \subset X \) into a compact set \( X \subset U \), will automatically be uniformly contracting the hyperbolic metric on \( U \).

Lemma 3.1 (Lemma 2.1 of [14]). If the analytic function \( \varphi \) maps an open connected set \( U \subset X \) into a compact set \( X \subset U \), then there exists 0 < \( s < 1 \), which depends on \( U \) and \( X \) only, such that \( d(\varphi(z), \varphi(w)) \leq sd(z, w) \) for all \( z, w \in X \) where \( d \) is the hyperbolic metric defined on \( U \).

Remark 3.1. Let \( \Phi \) be an analytic NIFS on \((U, X)\). Note that, for each \( x \in X \), the hyperbolic disk \( \Delta_d(x, 2 \cdot \text{diam}(X)) \subset U \) contains \( X \) and is connected (being the continuous image of a connected hyperbolic disk in \( \Delta(0, 1) \)). Hence, \( \overline{X} = \bigcup_{x \in X} \Delta_d(x, 2 \cdot \text{diam}(X)) \) is connected. Further, since \( X \) is forward invariant under \( \Phi \), then so is \( \overline{X} \) since analytic maps cannot increase hyperbolic distances. We note then that Lemma 1.2 (with the roles of \( X \) and \( \overline{X} \) inverted) allows us to replace \( X \) by the connected \( \overline{X} \) without altering \( \overline{J} \).

We call a doubly connected domain \( A \) in \( \mathbb{C} \) that can be conformally mapped onto a true (round) annulus \( \text{Ann}(w; R) = \{ z : r < |z - w| < R \} \), for some \( 0 < r < R \), a conformal annulus with the modulus of \( A \) given by \( \text{mod} A = \log(R/r) \), noting that \( R/r \) is uniquely determined by \( A \) (see, e.g., the version of the Riemann mapping theorem for multiply connected domains in [1]).
Definition 3.1. A conformal annulus $A$ is said to separate a set $F \subset \mathbb{C}$ if $F \cap A = \emptyset$ and $F$ intersects both components of $\mathbb{C} \setminus A$.

Definition 3.2. A compact subset $F \subset \mathbb{C}$ with two or more points is uniformly perfect if there exists a uniform upper bound on the modulus of each conformal annulus which separates $F$.

Remark 3.2. Because of the following well-known lemma (see, e.g., Theorem 2.1 of [10]), we can equivalently characterize uniformly perfect sets in terms of only true annuli: A compact subset $F \subset \mathbb{C}$ with two or more points is uniformly perfect if there exists a uniform upper bound on the modulus of each true annulus (centered at a point in $F$, if we choose) which separates $F$.

Lemma 3.2. Any conformal annulus $A \subset \mathbb{C}$ of sufficiently large modulus contains an essential true annulus $B$ (i.e., $B$ separates the boundary of $A$) with $\text{mod} \ A = \text{mod} \ B + O(1)$. Since, for any $R > 3r$ and any $w' \in \Delta(w,r)$, the true annulus $\text{Ann}(w';2r,R-r)$ is an essential annulus of $\text{Ann}(w;r,R)$, we may choose $B$ to be centered at any given point in the bounded component of $\mathbb{C} \setminus A$.

The concept of hereditarily non uniformly perfect was introduced in [15] and can be thought of as a thinness criterion for sets which is a strong version of failing to be uniformly perfect.

Definition 3.3. A compact set $E \subset \mathbb{C}$ is called hereditarily non uniformly perfect (HNUP) if no subset of $E$ is uniformly perfect.

Often a set is shown to be HNUP by showing it satisfies the following stronger property of pointwise thinness. This is done in several examples in [15], and will be done in Example 4.2. Also, certain non-autonomous Julia sets in [5] are shown to be HNUP this way (where it is worth noting that the limit set of a conformal NIFS is the Julia set).

Definition 3.4. A set $E \subset \mathbb{C}$ is called pointwise thin when for each $z \in E$ there exist $0 < r_n < R_n$ with $R_n/r_n \to +\infty$ and $R_n \to 0$ such that each true annulus $\text{Ann}(z;r_n,R_n)$ separates $E$.

Note that any pointwise thin compact set is HNUP since none of its points can lie in a uniformly perfect subset.

Lemma 3.3. Suppose $A = \text{Ann}(z;r,R)$, for some $z \in \mathbb{C}$ and $0 < r < R$, is a true annulus separating $J$, where $J = \cap_{k=1}^{\infty} X_k$ is the attractor of some NIFS $\Phi$. Fix $0 < \delta < \frac{R-r}{2}$. Then the annulus $B = \text{Ann}(z;r+\delta,R-\delta) \subset A$ separates some $X_k$. Hence, given any $0 < \epsilon < \text{mod} \ A$, we can choose $\delta > 0$ such that $\text{mod} \ B = \log(\frac{R-\delta}{r+\delta}) = \log(\frac{R}{r}) - \epsilon = \text{mod} \ A - \epsilon$, where $B$ separates some $X_k$.

Proof. Since $A$ separates $J$ and $B \subset A$, both components of $\mathbb{C} \setminus B$ must meet $J$, and therefore must meet each $X_k \supseteq J$. We complete the proof by showing that $B \cap X_k = \emptyset$ for some $k$. Suppose not. Now fix $k$ and choose $z_k \in X_k \cap B$. Hence there exists $\omega \in I^k$ such that $z_k \in \varphi_{\omega}(X)$. Since $\text{diam}_d(\varphi_{\omega}(X)) \leq s^k \text{diam}_d(X)$ (see Remark 1.3), we have that $\varphi_{\omega}(X) \subseteq \Delta_d(z_k,s^k \text{diam}_d(X)) \subset A$ for $k$ sufficiently large (since $z_k \in B \subset A$ and $d$ generates the Euclidean topology on $X$). Since
Let reference could not be found we provide a proof here. The following is a result that seems to be well understood by many, but since a

Lemma 3.4. Suppose \( A = \text{Ann}(z; r, R) \), for some \( z \in \mathbb{C} \) and \( 0 < r < R \), separates \( E \subseteq X \subseteq \mathbb{C} \) where \( \text{diam}(X) < \infty \) and \( R \geq 2 \cdot \text{diam}(X) \). Then \( \frac{B}{r} \leq 2 \).

Proof. Since \( A \) separates \( E \), there exist \( x_1, x_2 \in E \) with \( |x_1 - z| \geq R \) and \( |x_2 - z| \leq r \). Hence \( 2 \cdot \text{diam}(X) - r \leq R - r \leq |x_1 - x_2| \leq \text{diam}(E) \leq \text{diam}(X) \), which gives that \( \text{diam}(X) \leq r \). Again using that \( R - r \leq \text{diam}(X) \), we see that \( \frac{R-r}{r} \leq \frac{\text{diam}(X)}{r} \leq 1 \), which gives \( \frac{B}{r} \leq 2 \) as desired. \( \square \)

The following is a result that seems to be well understood by many, but since a reference could not be found we provide a proof here.

Proposition 3.1. Let \( f : U \to \mathbb{C} \) be non-constant and analytic on open connected \( U \subseteq \mathbb{C} \). Suppose that \( E \subseteq U \) is uniformly perfect. Then \( f(E) \) is uniformly perfect.

This result follows from the fact that locally non-constant analytic maps are either conformal or behave like \( z \mapsto z^k \) for some \( k \in \mathbb{N} \), which can distort the modulus of an annulus by at most a factor of \( k \).

Proof. The local behavior of non-constant analytic maps clearly implies that since \( E \) is perfect, so is \( f(E) \). We now suppose towards a contradiction that \( f(E) \) is not uniformly perfect. Hence there exists true annuli \( A_n = \text{Ann}(w_n; r_n, R_n) \) which separate \( f(E) \) with \( R_n/r_n \to \infty \).

By Lemma 3.2, we may assume each \( w_n \in f(E) \). Since \( f(E) \) is perfect, it follows that \( R_n \to 0 \) (see, e.g., Lemma 2.7 of [14]).

By compactness of both \( f(E) \) and \( E \), and passing to a subsequence if necessary, we may assume there exists \( w_0 \in f(E) \) such that \( w_n \to w_0 \) and \( z_0, z_n \in E \) such that \( z_n \to z_0 \) with each \( f(z_n) = w_n \).

Suppose \( f'(z_0) \neq 0 \). Thus there exists a local branch \( h \) of \( f^{-1} \) defined on some neighborhood of \( w_0 \). Hence, the conformal annuli \( h(A_n) \), for large \( n \), must then separate \( E \), which is a contradiction since \( E \) is uniformly perfect and \( \text{mod} \ h(A_n) = \text{mod} \ A_n \to \infty \).

Now suppose \( f'(z_0) = 0 \), and choose \( k \) such that \( f \) maps \( z_0 \) to \( w_0 \) with multiplicity \( k > 1 \). By pre- and post-composing with translations, we may assume \( z_0 = w_0 = 0 \), and so there exists a conformal map \( g \) defined on a neighborhood of 0 such that \( gf^{-1}(z) = z^k \) (see, e.g., Theorem 6.10.1 of [3]). It suffices to consider two cases: Case (i) Each \( A_n \) surrounds \( w_0 = 0 \), and Case (ii) No \( A_n \) surrounds \( w_0 = 0 \).

Case (i): From each conformal annulus \( g(A_n) \) of large modulus (and so for all large \( n \)), we apply Lemma 3.2 to extract an essential true annulus \( B_n = \text{Ann}(0; s_n, S_n) \subseteq g(A_n) \) of modulus \( \text{mod} B_n = \text{mod} A_n - K \), for some fixed \( K > 0 \). Since \( A_n' = \text{Ann}(0; s_n^{1/k}, S_n^{1/k}) \) maps by \( z \mapsto z^k \) onto \( B_n \subseteq g(A_n) \), we must have that each conformal annulus \( g^{-1}(A_n') \) surrounds \( z_0 = 0 \) and \( \text{mod} g^{-1}(A_n') = \text{mod} (A_n') = 1/k \text{mod} B_n \to \infty \), which is a contradiction since each \( g^{-1}(A_n') \) separates the uniformly perfect set \( E \).
Case (ii): Again for each conformal annulus $g(A_n)$ of large modulus (and so for all large $n$), we apply Lemma 3.2 to extract an essential true annulus $B_n = \text{Ann}(g(w_n); s_n, S_n) \subseteq g(A_n)$ of modulus $\text{mod } B_n = \text{mod } A_n - K$, for some fixed $K > 0$. Note that no $\Delta(g(w_n), S_n)$ contains 0. Hence, the map $z \mapsto z^k$ has $k$ well-defined inverse branches on $B_n$, one of which must map $B_n$ to a conformal annulus $B'_n$ surrounding $g(z_n)$. And so, $g^{-1}(B'_n)$ is a conformal annulus surrounding $z_n$ and separating $E$, with modulus mod $g^{-1}(B'_n) = \text{mod } B'_n = \text{mod } B_n = \text{mod } A_n - K$. This is a contradiction since $E$ is uniformly perfect and $\text{mod } A_n \to \infty$. \hfill $\Box$

The following result can easily be shown.

Lemma 3.5. Suppose $f : X \to Y$ is continuous and compact sets $A_n \subseteq X$ form a decreasing sequence. Then $f(\cap_{n=1}^{\infty} A_n) = \cap_{n=1}^{\infty} f(A_n)$.

4. Examples

In this section we provide examples to show that possible generalizations of Theorems A-C to the non-autonomous case do not hold. Specifically, we show that none of the following Statements 1-3 hold. Examples to illustrate Theorem 2.1 are also given, along with an analysis of how this theorem generalizes Theorem 4.1(2) of [15].

Statement 1: (Generalization of Theorem A) Let $\Phi^{(1)}, \Phi^{(2)}, \ldots$ be an analytic NIFS on $(U, X)$ such that there exists $\eta > 0$ with $|\varphi'| \geq \eta$ on $X$ for all $\varphi \in \cup_{j=1}^{\infty} \Phi^{(j)}$. If $J$ has infinitely many points, then $J$ is uniformly perfect.

Statement 2: (Generalization of Theorem B) Let $\Phi^{(1)}, \Phi^{(2)}, \ldots$ be a conformal NIFS on $(U, X)$ such that there exists $\eta > 0$ with $|\varphi'| \geq \eta$ on $X$ for all $\varphi \in \cup_{j=1}^{\infty} \Phi^{(j)}$. If $J$ contains more than one point, then $J$ is uniformly perfect.

Statement 3: (Generalization of Theorem C) Let $\Phi^{(1)}, \Phi^{(2)}, \ldots$ be a conformal NIFS on $(U, X)$ such that there is a uniform bound on the cardinality of $\Phi^{(j)}$. If $J$ contains more than one point, then $J$ is uniformly perfect.

Example 4.1. Each set $I_a$ in Theorem 4.1 of [15] is a limit set of a NIFS suitably chosen as follows. Set $X = [0, 1]$, fix $m \in \{2, 3, \ldots\}$, and choose $0 < a \leq \frac{1}{m+1}$. Fix a sequence $\vec{a} = (a_1, a_2, \ldots)$ such that $0 < a_k \leq a$ for $k = 1, 2, \ldots$. For each $k \in \mathbb{N}$, set $\Phi^{(k)}$ to be the collection $\{\varphi_{1}^{(k)}, \ldots, \varphi_{m}^{(k)}\}$ of linear maps, each with derivative $a_k$, such that the images $\varphi_{1}^{(k)}(X), \ldots, \varphi_{m}^{(k)}(X)$ are $m$ equally spaced subintervals of $X$ with $\varphi_{1}^{(k)}(X) = [0, a_k]$ and $\varphi_{m}^{(k)}(X) = [1 - a_k, 1]$. Example 1.3, illustrated in Figure 1, is such an NIFS (with $m = 2$). Each set $X_k$ then coincides with what [15] calls $I_k$, and consists of $m^k$ basic intervals. And the limit set $J$ then coincides with what [15] calls $I_{\vec{a}}$.

Theorem 4.1(2) of [15] shows that $J$ is uniformly perfect when $\lim \inf a_k > 0$. This also follows from Theorem 2.1, noting that we may choose $\eta = \inf \varphi_{\infty} > 0$ to satisfy the Derivative Condition and choose $\delta = 1 - 2a$ to satisfy the Two Point Separation
Condition (even when \( \liminf a_k = 0 \)) since the images \( \varphi_1^{(k)}(X) \) and \( \varphi_m^{(k)}(X) \) are always a distance \( 1 - 2a_k \) apart.

We also note that when \( \liminf a_k > 0 \), Theorem 2.1 shows \( J \) is uniformly perfect even when the strict setup above is considerably relaxed. For example, the sets \( \varphi_1^{(k)}(X), \ldots, \varphi_m^{(k)}(X) \) do not need to be equally spaced subintervals of \( X \). In fact, these sets could even overlap, as long as the Two Point Separation Condition is met (and \( \liminf a_k > 0 \)), and \( J \) would still be uniformly perfect.

Lastly we note that Theorem 4.1(1) of [15] shows that \( J \) is perfect but pointwise thin (and thus HNUP - see Definition 3.3) when \( \liminf a_k = 0 \).

**Remark 4.1.** Note that Example 1.3, with each \( a_j = \frac{1}{j+1} \), shows that Statement 3 does not hold since \( J \) would then be perfect but also be HNUP. It also illustrates that the Derivative Condition in Theorem 2.1 is critical, even when all the other conditions are met.

**Example 4.2.** Again, let \( X = [0,1] \). Set \( f_1(z) = \frac{z}{3}, f_2(z) = \frac{z+2}{3} \) and \( f_3(z) = \frac{1}{3}(z - \frac{1}{2}) + \frac{1}{2} \). We fix a sequence of positive integers \( (l_j) \), and then create \( \Phi \) by choosing \( \Phi^{(1)} = \{ f_1, f_2 \}, \Phi^{(2)} = \Phi^{(3)} = \ldots = \Phi^{(1+l_1+1)} = \{ f_3 \}, \Phi^{(1+l_1+2)} = \{ f_1, f_2 \}, \Phi^{(1+l_1+3)} = \ldots = \Phi^{(1+l_1+1+l_2)} = \{ f_3 \} \), etc. Hence, defining \( L_0 = 0 \) and \( L_n = \sum_{j=1}^n (1 + l_j) \), we have, for each \( n = 0, 1, 2, \ldots \), \( \Phi^{(L_n+1)} = \{ f_1, f_2 \} \) and \( \Phi^{(L_n+i)} = \{ f_3 \} \) for \( 2 \leq i \leq 1 + L_{n+1} \).

We prove the following dichotomy.

**Claim:** We have that \( \sup l_j = +\infty \) implies \( J \) is perfect but pointwise thin (and thus HNUP), whereas \( \sup l_j < +\infty \) implies \( J \) is uniformly perfect.

We now replace \( \Phi \) with a related NIFS \( \Phi \) such that \( J(\Phi) = J(\Phi) \) by combining stages of consecutive \( \Phi^{(j)} \) which equal \( \{ f_3 \} \) (see Remark 1.6). Specifically, we have \( \Phi^{(1)} = \Phi^{(1)} = \{ f_1, f_2 \}, \Phi^{(2)} = \Phi^{(3)} = \ldots = \Phi^{(1+l_1+1)} = \{ f_3 \}, \Phi^{(3)} = \Phi^{(1+l_1+1)} = \{ f_3 \}, \Phi^{(4)} = \{ f_3 \}, \ldots \), noting each iterate \( f_3^{(n)}(z) = \frac{1}{3}(z - \frac{1}{2}) + \frac{1}{2} \). More succinctly we have for each \( n \in \mathbb{N} \), \( \Phi^{(2n-1)} = \{ f_1, f_2 \} \) and \( \Phi^{(2n)} = \{ f_3^{(2n)} \} \). We now replace \( \Phi \) by \( \Phi \), hence the \( X_n^{(j)} \) and \( I \) below formally are constructed in reference to \( \Phi \).
Figure 2. Table illustrating $\tilde{\Phi}$ in Example 4.2, where $l_1 = 1$ and $l_2 = 2$.

Note that $X_2^{(3)}$ (see Figure 2) consists of two components $f_1(f_2^2(X))$ and $f_2(f_1^2(X))$ which are separated by a true annulus $A$ centered at $f_2(\frac{1}{2}) = \frac{5}{6}$ whose inner radius is $\frac{1}{2} \text{diam}_2(f_2^2(X)) = \frac{1}{2 \cdot 3^{2\tau}}$ and outer radius is $\frac{5}{6} - \frac{1}{2} = \frac{1}{3}$. Further note that annulus $f_1^2(A)$ separates the two components of $X_3^{(2)}$, and annulus $f_2(f_1^2(A))$, with modulus $\text{mod}_2 A = \log(2 \cdot 3^{l_2})$, separates one component of $X_4^{(1)}$ from its other three components. In general, one can see that $X_2^{(2n-1)}$ is separated by an annulus of modulus $\log(2 \cdot 3^{l_n})$, which is then mapped by $\varphi_\omega$, for each $\omega \in I^{2n-2}$, to an annulus of the same modulus which separates $X_4^{(1)}$. In fact, it is clear that each component of $X_2^{(1)}$ is separated from each other component by an annulus of modulus $\log(2 \cdot 3^{l_n})$.

We now show that $J = J(\Phi)$ is pointwise thin according to Definition 3.4 when $\sup l_j = +\infty$. Let $z \in J$. From above $z$ lies in the bounded component of the complement of an annulus of modulus $\log(2 \cdot 3^{l_n})$ which separates $X_2^{(1)}$ (and therefore separates $J$ since every component of $X_2^{(1)}$ clearly contains a point of $J$). Since the outer radii of such annuli clearly shrink to zero as $n \to \infty$ and the modulus is unbounded (when $\sup l_j = +\infty$), we have pointwise thinness of $J$.

Perfectness follows from the fact that the diameter of each component of $X_2^{(1)}$ shrinks to zero as $n \to \infty$ and each component of $X_2^{(1)}$ contains two components of $X_{2n+2}^{(1)}$. 
We now suppose \( \sup l_j < +\infty \) and prove \( J \) is uniformly perfect. Again we combine stages, this time doing so in order to utilize Theorem 2.1. Create NIFS \( \Psi \) with \( J(\Psi) = J(\Phi) = J(\tilde{\Phi}) \) by stipulating that, for each \( k \in \mathbb{N} \), \( \Psi^{(k)} = \tilde{\Phi}^{(2k-1)} \circ \tilde{\Phi}^{(2k)} = \{ f_1 \circ f_1^{(k)}, f_2 \circ f_2^{(k)} \} \). Since the images \( f_1 \circ f_1^{(k)}(X) \subseteq f_1(X) = [0,1/3] \) and \( f_2 \circ f_2^{(k)}(X) \subseteq f_2(X) = [2/3, 1] \) are always separated by \( \delta = 1/3 \), we see that the Two Point Separation Condition (with respect to \( \Psi \)) is met. Further the Derivative Condition (with respect to \( \Psi \)) is also met (when \( \sup l_j < +\infty \), but not when \( \sup l_j = +\infty \)) since each map in \( \Psi^{(k)} \) is linear with derivative \( \frac{1}{2} \). From Theorem 2.1 it then follows that \( J(\Psi) \) is uniformly perfect.

Remark 4.2. Note that it is not the case for all NIFS that an annulus which separates some \( X_{j}^{(i)} \) will map by every (or even any) function in \( \Phi^{(i-1)} \) to an annulus that separates \( X_{j+1}^{(i-1)} \). In Example 4.2, however, this happens for the annuli involved because of the large separation between the sets \( f_1(X) \) and \( f_2(X) \).

Remark 4.3. Example 4.2 shows that (when \( \sup l_j = +\infty \)) \( J(\Phi) \) can be perfect yet fail to be uniformly perfect even when \( \Phi \) (but not the modified NIFS \( \tilde{\Phi} \)) satisfies both the Derivative Condition and Möbius Condition of Theorem 2.1. This example shows that the Two Point Separation Condition in Theorem 2.1 is critical, and also shows that none of the above Statements 1-3 hold. We also note that \( J' = \{ z : \phi_\omega(z) = z \) for some \( \omega \) in some \( I^k \} \) is not a subset of \( J \) (e.g., 0 is a fixed point of \( f_1 \) but is not in \( J \)). Hence, also \( J \) is not forward invariant under the maps \( \phi_\omega \) for \( \omega \in I^k \). Compared with statements (i) and (ii) as given for autonomous IFSs just before the statement of Theorem 2.1, we note that the non-autonomous situation is far more delicate.

5. Proof of the Main Theorems

In this section we prove the Theorems 2.1 and 2.2. We begin by first proving a crucial lemma that will be key in providing a uniform Lipschitz constant for certain locally defined inverse maps.

Lemma 5.1. Let \( F \) be a collection of analytic functions mapping non-empty open set \( U \subset \mathbb{C} \) into compact set \( X \subset U \) such that there exists \( \eta > 0 \) where for all \( f \in F \) we have \( |f'| \geq \eta \) on \( X \). Then there exists \( r_0 > 0 \) such that for every \( f \in F \) and \( x \in X \), we have \( |g'| \leq \frac{\eta}{\delta} \) on \( \Delta(f(x), r_0) \) where \( g \) is the local branch of the inverse of \( f \) such that \( g(f(x)) = x \).

Note that this lemma does not require the maps \( f \in F \) to be Möbius, or even globally conformal on \( U \).

Proof. First note that by compactness, there exists \( r > 0 \) such that for all \( x \in X \) we have \( \Delta(x, r) \subseteq U \). Applying Lemma 2.3 of [14], where \( M > 0 \) is taken large enough so that \( X \subset \Delta(0, M) \), we see that for some \( \rho > 0 \) each \( f \in F \) is one-to-one on \( \Delta(x, \rho) \) for every \( x \in X \). (Note that \( \rho \) is independent of \( f \in F \) and \( x \in X \).) By the Koebe distortion theorem (see, e.g., Theorem 1.6 of [14]), there exists \( 0 < r_1 < \rho \) such that for every \( f \in F \) and \( x \in X \), we have \( |f'| \geq \frac{\eta}{\delta} \) on \( \Delta(x, r_1) \). By the Koebe 1/4 Theorem, for each \( x \in X \) we then see that \( f(\Delta(x, r_1)) \supseteq \Delta(f(x), \frac{\eta}{4\delta}) \). Hence,
calling \( r_0 = \frac{1}{4\eta} \) we have that a branch \( g \) of \( f^{-1} \) is defined on \( \Delta(f(x), r_0) \) such that 
\( g(f(x)) = x \) and has \( |g'| \leq \frac{2}{\eta} \) there.

Remark 5.1. Under the hypotheses of Theorem 2.1, the Derivative Condition along with the distortion theorems used in the proof of the above lemma yield that 
\( \inf \{ \text{diam}(\varphi(X)) : \varphi \in \cup_{j \in \mathbb{N}} \Phi^{(j)} \} > 0 \). To see this, choose \( x_0 \in X \) and \( r > 0 \) such that \( \Delta(x_0, r) \subset X \) (note that \( X \) must have interior since it contains the open sets \( \varphi(U) \) for all \( \varphi \in \cup_{j \in \mathbb{N}} \Phi^{(j)} \)). Fixing \( r < \rho \) from the above proof, we see that by the Koebe 1/4 Theorem, \( \varphi(X) \supset \varphi(\Delta(x_0, r)) \supset \Delta(\varphi(x_0), \frac{r}{4}) \) for all \( \varphi \in \cup_{j \in \mathbb{N}} \Phi^{(j)} \), which justifies the claim.

Proof of Theorem 2.1. We begin by replacing \( X \), if it is not connected, by the connected \( \tilde{X} \subset U \) as in Remark 3.1, noting that the hypotheses are still met. Indeed, the Möbius and Two Point Separation Conditions are clearly still satisfied with respect to \( \tilde{X} \supset X \). The Derivative Condition also still holds with respect to \( \tilde{X} \supset X \) though not as trivially. We show this by contradiction. Assume \( \varphi_n(z_n) \to 0 \) as \( n \to \infty \) where each \( z_n \in \tilde{X} \) and each \( \varphi_n \in \cup_{j \in \mathbb{N}} \Phi^{(j)} \). By compactness we may suppose \( z_n \to z_0 \in \tilde{X} \). Since by Montel’s Theorem, the family \( \cup_{j \in \mathbb{N}} \Phi^{(j)} \) is normal on \( U \), we may suppose \( \varphi_n \) converges normally on \( U \) to some map \( \varphi \). Hence, we must have \( \varphi(z_0) = 0 \). Since each map in \( \cup_{j \in \mathbb{N}} \Phi^{(j)} \) is Möbius, and thus one-to-one on \( U \), we see by Hurwitz’s Theorem that \( \varphi \) must be constant. This implies that for any \( x \in X \), we must have \( \varphi_n(x) \to \varphi(x) = 0 \), but this contradicts the Derivative Condition on \( X \) which gives that each \( |\varphi_n'(x)| \geq \eta \).

It suffices to prove \( \overline{J^{(1)}} \) is uniformly perfect since clearly each sub-NIFS of \( \Phi \) which generates \( J^{(j)} \) must also satisfy conditions (i)-(iii). First note that by Remark 2.2 we see that \( \text{diam}(J^{(1)}) \geq \delta \) and so \( J = J^{(1)} \) has more than one point. Recalling Remark 3.2 and Lemma 3.2, we consider a true annulus \( A_1 \) which separates \( J \) and which has modulus large enough so that any conformal annulus \( B \) with mod \( B \geq \text{mod} A_1 - 1 \) contains an essential true annulus \( B' \subset B \) such that mod \( B' = \frac{1}{2} \text{mod} B \). Since true annulus \( A_1 \) must also separate \( J = \cap_{k=1}^\infty X_k \), we apply Lemma 3.3 to obtain a true annulus \( A \subset A_1 \) which separates some \( X_{k_0} \) and has mod \( A = \text{mod} A_1 - 1 \). We complete the proof by showing that there exists an upper bound on mod \( A \).

Recall the superscript notation of Section 1, in particular, that \( X_{k_0}^{(1)} = X_{k_0} \). By the invariance condition (1.1) in Remark 1.4, we have \( \bigcup_{i \in I'} \varphi_i^{(1)}(X_{k_0-1}^{(2)}) = X_{k_0}^{(1)} \), and so there must be some \( \varphi_i^{(1)} \in \Phi^{(1)} \) such that \( A \) surrounds some point of \( \varphi_i^{(1)}(X_{k_0-1}^{(2)}) \) (i.e., the bounded component of \( \mathbb{C} \setminus A \) contains a point of \( \varphi_i^{(1)}(X_{k_0-1}^{(2)}) \)). Since \( A \) separates \( X_{k_0}^{(1)} \), we must have one of two cases: Case (I) \( A \) surrounds all of \( \varphi_i^{(1)}(X_{k_0-1}^{(2)}) \), or Case (II) \( A \) separates \( \varphi_i^{(1)}(X_{k_0-1}^{(2)}) \). See Figure 3.

Case (I): Write \( A = \text{Ann}(z; r, R) \) and suppose it surrounds all of \( \varphi_i^{(1)}(X_{k_0-1}^{(2)}) \). From Lemma 3.4 it follows that we only need to consider cases where \( R < 2 \cdot \text{diam}(X) \). We now establish an upper bound for mod \( A = \log(R/r) \) by finding a positive lower bound for \( r \).
Figure 3. Illustration of the proof of Theorem 2.1 using the system of Example 1.3. Note that $A$ and $A_2 = (\phi_2^{(1)})^{-1}(A)$ are both of Case (II) type, whereas $A'$ is of Case (I) type, but $(\phi_1^{(1)})^{-1}(A')$ is of Case (I) type.

Since $(\phi_i^{(1)})^{-1}$ is a Möbius map, the conformal annulus $A_2 = (\phi_i^{(1)})^{-1}(A)$ must surround all of $X_{k_0-1}^{(2)}$. This, however, implies that $(\phi_i^{(1)})^{-1}(\Delta(z,r)) \supseteq X_{k_0-1}^{(2)} \supseteq J^{(2)}$, which by the Two Point Separation Condition (see Remark 2.2) gives that $\text{diam} \left( (\phi_i^{(1)})^{-1}(\Delta(z,r)) \right) \geq \delta$. Notice that due to the Derivative Condition and Lemma 5.1, there exists $r_0 > 0$ such that for any $x \in X$ and $\varphi \in \cup_{j \in \mathbb{N}} \Phi^{(j)}$, we have $|(\varphi^{-1})'| \leq \frac{2}{\eta}$ on $\Delta(\varphi(x), r_0)$.

We now suppose $r < \min\{\frac{\delta \eta}{4}, \frac{\eta}{2}\}$, from which we derive a contradiction, thus producing a lower bound for $r$ and completing the proof for Case (I). Since $\Delta(z,r)$ meets $\varphi_i^{(1)}(X_{k_0-1}^{(2)})$, we may choose $x_0 \in X_{k_0-1}^{(2)} \subseteq X$ such that $\varphi_i^{(1)}(x_0) \in \Delta(z,r) \subset \Delta(\varphi_i^{(1)}(x_0), 2r) \subset \Delta(\varphi_i^{(1)}(x_0), r_0)$. Since $\left| \left( (\varphi_i^{(1)})^{-1} \right)' \right| \leq \frac{2}{\eta}$ on $\Delta(\varphi_i^{(1)}(x_0), r_0)$ which contains the convex set $\Delta(z,r)$, we see that $\text{diam} \left( (\varphi_i^{(1)})^{-1}(\Delta(z,r)) \right) \leq \frac{4\eta}{\eta} < \delta$, which is a contradiction.

**Case (II):** Suppose $A$ separates $\varphi_i^{(1)}(X_{k_0-1}^{(2)})$. Hence, the conformal annulus $A_2 = (\phi_i^{(1)})^{-1}(A)$ must separate $X_{k_0-1}^{(2)}$ and must have mod $A_2 = \text{mod} A$. In terms of
Figure 3, we have constructed an annulus $A_2$ which separates $X^{(2)}_{k_0-1}$ in the picture diagonally up and right of the picture of $X^{(1)}_{k_0}$.

Hence we may repeat our process as follows. Since $A_2$ separates the set $X^{(2)}_{k_0-1} = \bigcup_{i \in I^{(2)}} \varphi^{(2)}_i (X^{(3)}_{k_0-2})$, we must have one of two cases: Case (I') $A_2$ surrounds all of some $\varphi^{(2)}_i (X^{(3)}_{k_0-2})$, or Case (II') $A_2$ separates some $\varphi^{(2)}_i (X^{(3)}_{k_0-2})$. If Case (I') holds, extract an essential true annulus $A'_2 \subset A_2$ with mod $A'_2 = 1$ mod $A_2 = 1$ mod $A$, which must surround all of $\varphi^{(2)}_i (X^{(3)}_{k_0-2})$, and then bound mod $A'_2$ as in Case (I) above. If Case (II') holds, we repeat the process of Case (II) above, noting that we do not need to first extract a true annulus from $A_2$.

This process must then end by eventually applying the method of Case (I), or by eventually producing (after $k_0$ steps) an annulus $A_{k_0}$, with the same modulus as of $A$, which separates $X^{(k_0)}_1$. The proof is thus concluded by showing that such a modulus is uniformly bounded independent of the choice of $k_0$. First, extract an essential true annulus $A'_{k_0} = \text{Ann}(z', r', R') \subset A_{k_0}$ with mod $A'_{k_0} = 1$ mod $A_{k_0} = 1$ mod $A$, which necessarily separates $X^{(k_0)}_1$. Again by Lemma 3.4, it is then clear that we only need to produce a lower bound for $r'$. This follows easily from Remark 5.1 by noting that $\Delta(z', r')$ would need to contain the connected set $\varphi(X)$ for some $\varphi \in \Phi^{(k_0)}$.

Examination of the above proof shows that mod $A_1$ is bounded above by a constant which depends only on $\delta$ and $\eta$. $\square$

Note that the step of extracting a true annulus of half the modulus is done only at most once in the above proof.

Proof of Theorem 2.2. By Proposition 3.1, for each $\varphi \in \tilde{\Phi}^{(1)}$ the set $\phi \left( \overline{J^{(n)}} \right)$ is uniformly perfect. Lemma 1.3 gives that $\overline{J^{(1)}} = \bigcup_{\varphi \in \tilde{\Phi}^{(1)}} \varphi \left( \overline{J^{(n)}} \right)$, and the result follows since the finite union of uniformly perfect sets is uniformly perfect. $\square$

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References


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