COMPLEX DYNAMICS OF MÖBIUS SEMIGROUPS

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Abstract. We study the dynamics of semigroups of Möbius transformations on the Riemann sphere, especially their Julia sets and attractors. This theory relates to the dynamics of rational functions, rational semigroups, and Möbius groups and we compare and contrast these theories. We particularly examine Caruso's family of Möbius semigroups, based on a random dynamics variant of the Fibonacci sequence.

Contents

1. Introduction

This paper examines a random dynamics variant of the Fibonacci ratios. Specifically, we let z_n be a recursively defined sequence such that $z_{n+1} = z_{n-1} \pm \beta z_n$, where the terms and the nonzero constant β are complex numbers and the sign $\pm$ is chosen at random. The ratio of two consecutive terms $r_n = z_n/z_{n-1}$ then satisfies the recursion relation $r_{n+1} = \pm\beta + 1/r_n$. When $\beta = 1$ and we always choose the sign $+$ (as in the Fibonacci sequence) the limit as $n \to \infty$ of r_n is the *golden ratio* $\phi = (1+\sqrt{5})/2$, as long as $r_0 \neq (1-\sqrt{5})/2$. By introducing the randomness mentioned above, and allowing for other values of β, we will see that the set of possible long term states may be a complicated and beautiful fractal set. In particular, the set of possible long term states is often the attractor set $\mathcal{A}_\beta$ of a Contracting Iterated

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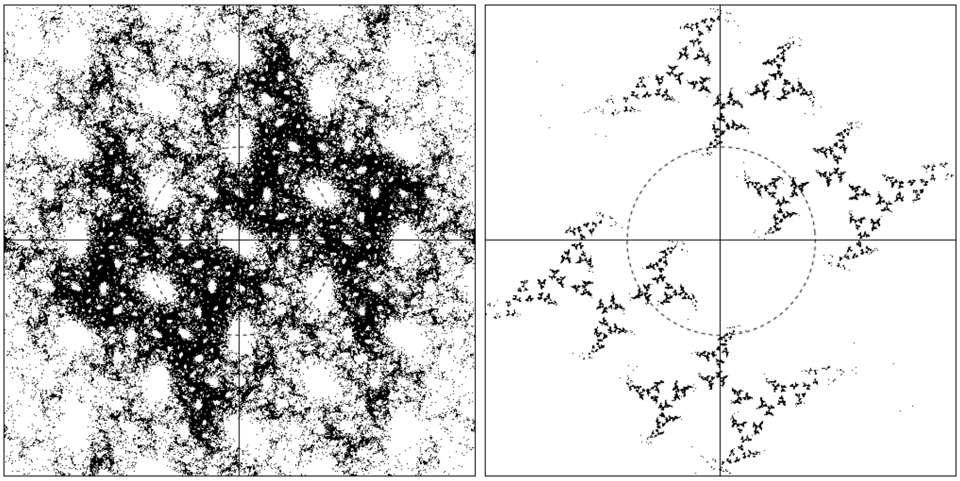


FIGURE 1. The sets obtained by randomly iterating an arbitrary point in the complex plane $\mathbb{C}$ using the semigroup generated by $\{\beta + 1/z, -\beta + 1/z\}$. We plot a number of points in the randomly generated orbit, dropping the first few iterates as transient. In some cases, the pictures that we get do not seem to depend on the chosen seed or the chosen composition sequence. The unit circle (dashed) is shown for reference. On the left, the parameter $\beta = e^{i\pi/6}$, and on the right $\beta = 1.25e^{i\pi/3}$. In both cases we computed one million points and dropped the first hundred points as transient. See Remark 5.13.

Function System (CIFS) generated by $\{\beta+1/z, -\beta+1/z\}$.[1] See Figure 1. Some background on these relationships can be found in [8, 14, 40].

This paper is motivated by Caruso's problem: given a value of β, in what sense does there exist an attractor set $\mathcal{A}_\beta$, how does such a set depend on β, and what are the properties of $\mathcal{A}_\beta$? In [8], the set of long term states was described by means of numerical experiments. The frequency distributions of the modulus and argument of these states were presented for particular values of the parameter β. In this paper we deal instead with the topological properties of the set of possible long term states.

It turns out that the attractor $\mathcal{A}_\beta$ (when it exists) of the CIFS generated by $\{\beta+1/z, -\beta+1/z\}$ is the Julia set of the semigroup S'_β generated under function composition by the inverse transformations $1/(z-\beta)$ and $1/(z+\beta)$ of these CIFS generators. Thus, one of our goals in this paper is to characterize the Julia set of S'_β for different values of the complex parameter β.

During the course of our investigation it became apparent that the study of S'_β is intimately related to the study of two other algebraic structures, namely, the semigroup S_β and the group G_β generated by $\beta + 1/z$ and $-\beta + 1/z$. Thus, we also focus on the topological characteristics of the Julia set of S_β and the limit set of G_β for different values of β. The study of these structures, as they arise in Caruso's problem, led us to find several interesting

[1]Notice that the sequences generated with this system are also generalized continued fractions of the form

$$\pm\beta + \cfrac{1}{\pm\beta + \frac{1}{\pm\beta + \dots}}.$$

general results about semigroups of Möbius transformations. We have expanded our focus to present these results.

The article is organized in the following manner. In Sections 2 and 3, we develop, in a general setting, the background material relevant to our problem. We introduce non-elementary semigroups and explain their basic properties and characteristics. For example, Theorem 3.3 shows that when the Julia set $J(S)$ of a non-elementary Möbius semigroup S is connected, the limit set of the group generated by S is also connected. The concepts of attractors and blocks are discussed in Sections 4 and 5. In particular, in Theorem 5.7 we give necessary and sufficient conditions for a finitely generated Möbius semigroup to have what we call a *thick* attractor, which attracts any random orbit whose initial value is near the attractor. Further properties and comparisons between limit sets and various sorts of Julia sets are discussed via examples presented in Section 6.

In Section 7 we study the Caruso semigroups S_β. In Section 7.1 we prove that each S_β is non-elementary and we study the symmetries that arise in dynamical and parameter planes. In Section 7.2 we study the connectedness of the Julia sets $J(S_\beta)$. In particular, Corollary 7.12 shows that $J(S_\beta)$ must be bounded and disconnected when a thick attractor exists. We present sufficient conditions for $J(S_\beta)$ to equal the limit set of G_β in Section 7.3. In Section 7.4 we calculate this limit set when G_β is not discrete. We use reflection groups in Section 8 to treat the cases when β^4 is real.

In a subsequent paper we will use Riley groups to study the Caruso problem when β^4 is not real.

We would like to thank Linda Keen for introducing us to reference [19] while visiting Boston University in 2005. We also thank Dan Goodman for sharing his computer programs and for giving detailed descriptions of how they work.

2. Rational semigroups

The dynamics of iteration of a complex analytic map has been studied deeply in various contexts, such as rational, entire, and meromorphic maps. This theory generalizes to the setting where the map may be changed at each point of the orbit, exactly as in a random walk. More precisely, instead of repeatedly applying the same map over and over again, one may start with a family of maps $\{h_i : i \in I\}$, and consider the dynamics of any iteratively defined *composition sequence* of maps, that is, any sequence $h_{i_n} \circ \cdots \circ h_{i_1}$ where $i_1, ..., i_n \in I$. Randomly choosing the map at each stage is the setting for *random dynamics* (see [11, 4, 6, 7, 5, 37, 34]). Restricting one's attention to the case where all h_i are rational, one is lead to study the dynamics of rational semigroups.

A *rational semigroup* is a nonempty semigroup of nonconstant rational functions defined on the Riemann sphere[2] $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with the semigroup operation being functional composition.[3] When a semigroup S is generated by a family of functions h_i, $i \in I$, for some nonempty index set I, we write $S = \prec h_i : i \in I \succ$. Thus S is exactly the set of all finite compositions of the maps h_i. Note that many papers use the notation $\langle \cdot \rangle$ for rational semigroups but, since we discuss both groups and semigroups, we will reserve the notation

[2]As usual, the letter $\mathbb{C}$ denotes the complex plane and $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$.

[3]Beware that some authors require that a rational semigroup contain at least one rational map of degree two or more, but we do not impose such a restriction here.

$\langle\cdot\rangle$ for groups, so $\langle h_i : i \in I\rangle$ denotes the group generated by $h_i,\ i \in I$. We say S is *cyclic* when S is generated by some one function, so $S = \prec h \succ$.

A rational mapping $m(z)$ of $\widehat{\mathbb{C}}$ is invertible if and only if it is *Möbius*, meaning that the degree of $m(z)$ is one. Clearly the Möbius transformations form a group $\mathcal{M}$. A rational semigroup S whose elements are invertible lies in $\mathcal{M}$ and is called a *Möbius semigroup*. A *Möbius group* is a subgroup of $\mathcal{M}$.

We recall the connection between $\widehat{\mathbb{C}}$ and hyperbolic 3-space H^3. The upper halfspace model of hyperbolic 3-space H^3 is the subset of R^3 consisting of triples (x_1, x_2, x_3) with $x_3 > 0$. In this model, $\widehat{\mathbb{C}}$ is the boundary of H^3. $\mathcal{M}$ acts by isometries on H^3 and this action extends by continuity to the the Möbius action on $\widehat{\mathbb{C}}$. This Möbius action on H^3 is called the *Poincare extension* of the Möbius action on $\widehat{\mathbb{C}}$ [2].

Each $m \in \mathcal{M}$ has the form $m(z) = (az + b)/(cz + d)$, where the vector of coefficients (a, b, c, d) lies in the open subset $V \subset \mathbb{C}^4$ defined by $ad - bc \neq 0$. The mapping $m(z)$ only determines (a, b, c, d) up to a nonzero scalar factor. It follows that $\mathcal{M}$ is a Lie group of real dimension 6 diffeomorphic to the quotient space $V/\mathbb{C}^*$. This quotient topology on $\mathcal{M}$ is also the topology of uniform convergence on $\widehat{\mathbb{C}}$ ([2], p. 78). See Theorem 3.1 for a classification of Möbius maps and their dynamical behavior under iteration.

After its inception in the paper [16] of Hinkkanen and Martin, the study of rational semigroups has grown in the past decade (see, for example, [13, 29, 30, 36, 39] as well as the references therein). These works generally focus, however, on semigroups that contain at least one non-Möbius map. Möbius groups, especially discrete ones, have been studied since the pioneering work of Fricke and Klein (see, for example, [2, 20, 21, 23]. This paper thus fills in a gap in the literature by focusing on Möbius semigroups. When convenient, however, we will treat rational semigroups in general.

A rational semigroup divides the Riemann sphere into two complementary sets, an open Fatou set and a closed Julia set with opposite dynamical properties. In order to define these sets we first recall the notion of a *normal family* of holomorphic maps. Let U be an open set in $\widehat{\mathbb{C}}$. A family of holomorphic maps $\mathcal{F}$ on U with values in $\widehat{\mathbb{C}}$ is normal when every sequence $\{h_n\} \in \mathcal{F}$ has a subsequence that converges uniformly on each compact set in U. We will make particular use of three types of normal families of Möbius transformations.

Type 0) Suppose $\mathcal{F}$ consists of isometries of the spherical metric,[4] so each $h \in \mathcal{F}$ is a spherical rotation of the form $h(z) = (az - \bar{b})/(bz + \bar{a})$ for some $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$. Then $\mathcal{F}$ is a normal family on $\widehat{\mathbb{C}}$.

Type 1) Suppose $\mathcal{F}$ stabilizes ∞, so each $h \in \mathcal{F}$ has the form $h(z) = az + b$ (while such maps are sometimes called *complex affine* we prefer the term *linear*). If $|a| = 1$ then $h(z)$ is an isometry of $\mathbb{C}$ with its usual metric. If $|a| = 1$ for all $h \in \mathcal{F}$ then $\mathcal{F}$ is a normal family on $\mathbb{C}$.

Type 2) Suppose that $\mathcal{F}$ stabilizes $\{0, \infty\}$. Each $h \in \mathcal{F}$ is *monomial*, that is $h(z) = az$ or $h(z) = a/z$ for some a. Then each $h(z)$ is an isometry of the metric $|dz|/|z|$ on $\mathbb{C}^*$ (i.e., for $\rho(z) = 1/|z|$ we have $h^*\rho(z) \stackrel{def}{=} \rho(h(z))|h'(z)| = \rho(z)$) and so $\mathcal{F}$ is a normal family on $\mathbb{C}^*$.

[4]For $z, w \in \mathbb{C}$, $2|z - w|(1 + |z|^2)^{-1/2}(1 + |w|^2)^{-1/2}$ is the chordal distance from z to w (see [2], p.22).

Let S be a rational semigroup. As in [16] we define the *Fatou set*[5] $F(S)$ as the union of all open sets $U \subset \widehat{\mathbb{C}}$ on which S forms a normal family. The *Julia set* $J(S)$ is defined as the complement of $F(S)$ in $\widehat{\mathbb{C}}$. When $S = \prec h \succ$ is cyclic we let $F(h) = F(S)$ and $J(h) = J(S)$.

Remark 2.1. *We note that semigroups which are conjugate by a Möbius map (or the complex conjugate of a Möbius map) transfer normality in the obvious way. Specifically, if h is such a map and S_1 and S_2 are rational semigroups such that $S_2 = hS_1h^{-1} = \{hsh^{-1} : s \in S_1\}$, then $F(S_2) = h(F(S_1))$ and $J(S_2) = h(J(S_1))$ (see* [3, 16]*).*

The dynamics near the Julia set exhibits sensitive dependence on initial conditions. On the Fatou set, however, nearby points have similar dynamic behavior For instance, when a Möbius semigroup S is conjugate in $\mathcal{M}$ to one of the 3 types of families discussed above, Remark 2.1 shows that $F(S)$ contains the region where S acts isometrically. This easily gives the following.

Proposition 2.2. *Let S be a Möbius semigroup.*
0) If S, or some semigroup conjugate to S in $\mathcal{M}$, preserves spherical distance then $F(S) = \widehat{\mathbb{C}}$, so $|J(S)|$[6] *$= 0$.*
1) If p is a neutral fixed point of each $s \in S$ then p is the only possible Julia point, so $|J(S)| \leq 1$.
2) If S stabilizes a two-point set $\{p, q\}$ then p and q are the only possible Julia points, so $|J(S)| \leq 2$.

The behavior of a Möbius semigroup near a Fatou point is quite restricted, as the following lemma shows. The proof uses the well-known fact that $\mathcal{M}$ is simply triply transitive on $\widehat{\mathbb{C}}$, that is any two distinct ordered triples of points of $\widehat{\mathbb{C}}$ are related by a unique Möbius transformation ([2], Theorem 4.1.1).

Lemma 2.3. *If D is an open set in $\widehat{\mathbb{C}}$ and $s_1, s_2, ... \in \mathcal{M}$ converge uniformly to h on compact subsets of D then either h is locally constant, with $|h(D)| \leq 2$, or $s_n \to s$ in $\mathcal{M}$ and $h = s|D$.*

Proof. If $|h(D)| \geq 3$ then we can choose $p_i \in D$, $i = 0, 1, 2$, with $h(p_i)$ distinct. Let $s, m_1, m_2, ...$ be the unique elements of $\mathcal{M}$ with $s(p_i) = h(p_i)$ and $m_n(s_n(p_i)) = h(p_i)$ for all i. Thus $m_n \to Id$, since $s_n(p_i) \to s(p_i)$ for all i, and $s = m_n s_n$ in $\mathcal{M}$ for all n, since they agree at 3 distinct points. Thus $s_n \to s$ in $\mathcal{M}$ so $s_n|D$ converges uniformly to $s|D$. $\square$

We will see that the Julia set of a rational semigroup can be described in terms of fixed points. Recall that at a fixed point $p = h(p) \in \widehat{\mathbb{C}}$ of a rational mapping $h(z)$ we call $\lambda = h'(p)$ the *multiplier* of h at p (with the usual convention when $p = \infty$). The fixed point p is a *sink* (or *attracting*) if $|\lambda| < 1$ and a *source* (or *repelling*) if $|\lambda| > 1$. When $|\lambda| = 1$ we say p is *neutral* (or *indifferent*).

Suppose a rational semigroup S is normal on open set U and some $s \in S$ has a fixed point $p \in U$. If p were repelling, then $|(s^n)'(p)| = |s'(p)|^n \to +\infty$ and so no subsequence of the iterates s^n could converge uniformly on any compact neighborhood of p. Hence p must either be attracting or neutral. Thus any source of any $s \in S$ must be a Julia point.

[5] Some authors use $N(S)$ for the Fatou set since it is the set of normality.

[6] $|X|$ denotes the cardinality of a set X.

We let *Source* S (respectively *Sink* S) be the smallest closed set containing all the sources (respectively sinks) of elements of S. Then we see that Source $S \subset J(S)$. Except for one special circumstance, mentioned in Proposition 2.2, equality must hold.

Theorem 2.4. *Let S be a rational semigroup. Either Source $S = J(S)$ or S is a Möbius semigroup such that some $p \in \widehat{\mathbb{C}}$ is a neutral fixed point of each $s \in S$ and $J(S) = \{p\}$.*

Proof. [16] proves that sources are dense in $J(S)$ whenever $|J(S)| \geq 3$ and S is not a Möbius semigroup. Their proof remains valid, however, when S is a Möbius semigroup and $|J(S)| \geq 3$. [31], p. 5, treats both cases.

Clearly if $J(S)$ is empty then Source S is empty as well. The remaining cases, when $J(S)$ consists of 1 or 2 points, are easily dealt with using Theorem 2.11 below. □

This powerful theorem reexpresses the Julia set, which was defined via complex analysis, in purely topological terms. For whether a fixed point p of $s \in S$ is attracting, repelling, or neutral depends only on the local topological dynamics of s near p.

Remark 2.5. *Theorem 2.11 for a Möbius semigroup S follows from the fact that any Julia point p that is not fixed by S must lie in Source S. We sketch an elementary proof of this fact (c.f.* [32]*).*

We may suppose $p = 0 \neq s(0)$, for some $s \in S$. As $0 \in J(S)$, the Arzela-Ascoli Theorem implies that there is no neighborhood of 0 on which S is equicontinuous as a family of maps to $\widehat{\mathbb{C}}$. So we may choose $f_n \in S$ and $z_n \in \mathbb{C}$, $z_n \to 0$, such that f_n expands the spherical metric at z_n by at least n. Passing to a subsequence, and replacing f_n by sf_n if need be, we may suppose that $|f_n(\infty)| > c$ for all n, where $c > 0$. We now show that f_n has a source near 0 for n large, so $0 \in$ Source S.

Let D be the closed disc centered at 0 of radius $c/2$. Let q_n be a spherical rotation with $q_n(\infty) = f_n(\infty)$. As $|q_n(\infty)| > c$, the disc $D_n = q_n^{-1}(D)$ lies in a bounded region in $\mathbb{C}$.

We factor $f_n = q_n r_n$ in $\mathcal{M}$, so r_n fixes ∞ and r_n expands the spherical metric at z_n by at least n. As $z_n \to 0$, $r_n^{-1}(z) = a_n z + b_n$ where $(a_n, b_n) \to (0, 0)$. So for n large, $r_n^{-1}(D_n)$ is a small disc near 0. Thus $f_n^{-1}(D) = r_n^{-1}(D_n) \subset$ Int D, so f_n^{-1} has a sink near 0.

A Möbius semigroup S determines an inverse semigroup $S^{-1} = \{h^{-1} : h \in S\}$. We denote its Julia and Fatou sets by

$$J'(S) = J(S^{-1}) \text{ and } F'(S) = F(S^{-1}),$$

respectively.

We can easily describe the Julia set of a Möbius semigroup S in terms of the action of S^{-1} on hyperbolic space.

Proposition 2.6. *Let S be a Möbius semigroup. Then for each $x \in H^3$, $J(S) = \overline{S^{-1}x} \cap \widehat{\mathbb{C}}$.*

Proof. Let $K = \overline{S^{-1}x} \cap \widehat{\mathbb{C}}$. We first show that K contains $J(S)$.

Consider first the special case when S has a unique Julia point $p \in \widehat{\mathbb{C}}$ that is a neutral fixed point of each $s \in S$. We may assume $p = \infty$. $S^{-1}x$ lies on the horosphere Σ through x with center p and the only accumulation point of Σ in $\widehat{\mathbb{C}}$ is p. Also S is unbounded in $\mathcal{M}$ (see Proposition 2.12) and so $S^{-1}x$ is unbounded in Σ. Thus $p \in K$ so $J(S) \subset K$.

For any S one easily sees that Source $S \subset K$. Combining Theorem 2.4 with the preceding paragraph shows that K contains $J(S)$ for all S.

If K does not equal $J(S)$ then some $k \in K$ is a Fatou point. Suppose D is an open disc containing k such that $\overline{D}$ lies in $F(S)$ and that $s_1, s_2, ... \in S$ with $s_n^{-1}(x) \to k$. Passing to a subsequence, we may assume $s_n|\overline{D}$ converges uniformly to $h : \overline{D} \to \widehat{\mathbb{C}}$. As $s_n^{-1}(x) \to k$, s_n does not converge in $\mathcal{M}$. By Lemma 2.3, h is constant. Say $h(\overline{D}) = \{q\}$.

Let H be the open halfspace of H^3 bounded by D. Then $s_n(H)$ is bounded by $s_n(D)$ and so it is located very near to q for large n. But for n large enough, H contains $s_n^{-1}(x)$ and so $x \in s_n(H)$. As $x \in H^3$ and $q \in \widehat{\mathbb{C}}$, we have $x \neq q$ so this is a contradiction. Thus $K = J(S)$. □

We now turn to the invariance properties of Julia and Fatou sets. We distinguish, as usual, three kinds of invariance. If h is a map of a set X into itself, a subset Y of X is:

i) forward invariant under h if $h(Y) \subset Y$;

ii) backward invariant under h if $h^{-1}(Y) \subset Y$;

iii) completely invariant under h if $h(Y) \subset Y$ and $h^{-1}(Y) \subset Y$.

Let $\mathcal{F}$ be a family of maps from X to X. Y is forward, backward, or completely invariant under $\mathcal{F}$ if it is forward, backward, or completely invariant under each $h \in \mathcal{F}$, respectively.

We will mainly be concerned with a closed nonempty set $X \subset \widehat{\mathbb{C}}$ and a rational semigroup S. When X is forward invariant under S and X is compact and nonempty we say that X is *S-invariant.*

The natural invariance properties of Julia and Fatou sets are as follows.

Proposition 2.7 ([16], p. 360)**.** *Let S be a rational semigroup. $F(S)$ is forward invariant under S and consequently $J(S)$ is backward invariant under S.*

Proposition 2.7 and Theorem 2.4 give the following.

Corollary 2.8. *Source S is backward invariant under S for any rational semigroup S.*

This is surprising since the set of sources of elements of S is not backward invariant, in general, but its closure Source S must be so.

Example 2.9. *Let $s_1(z) = 2z$, $s_2(z) = 2z - 1$, and $S = \prec s_1, s_2 \succ$. Note that the interval $(0, 1)$ is backward invariant under S. Although $q = 1$ is a source for s_2, $1/2 = s_1^{-1}(1)$ is not fixed by any $s \in \prec s_1, s_2 \succ$. For if $s(1/2) = 1/2$, we would have the contradiction $1/2 \in s^{-1}((0, 1)) \subset s_1^{-1}((0, 1)) \cup s_2^{-1}((0, 1)) = (0, 1/2) \cup (1/2, 1)$.*

For most rational semigroups S, neither $F(S)$ nor $J(S)$ is completely invariant. But there are exceptions, such as cyclic semigroups ([22], Invariance Lemma 4.1) and certain Möbius s emigroups (Section 7.3).

The following straightforward proposition regarding finitely generated semigroups has been noted by many people (c.f. [35], p.719).

Proposition 2.10. *For a rational semigroup $S = \prec h_1, ..., h_n \succ$, we have $J(S) = \cup_{i=1}^n h_i^{-1}(J(S))$ and $F(S) = \cap_{i=1}^n h_i^{-1}(F(S))$.*

We refer to the above condition on the set $J(S)$ as *backward self-similarity.* We also note that it implies a *partial* forward invariance in the sense that for any $z \in J(S)$, there exists some generator h_i such that $h_i(z) \in J(S)$.

A rational semigroup S is *elementary* when $|J(S)| \leq 2$ and otherwise *non-elementary.* In fact, as we now show, S is elementary whenever $J(S)$ is finite, in which case it must be a Möbius semigroup conjugate to one of three special types, as follows.

Theorem 2.11. *Let S be a rational semigroup such that $J(S)$ is finite. Then $S \subset \mathcal{M}$ and S is elementary. Further, S is conjugate in $\mathcal{M}$ to a semigroup S^* whose elements s^* are as follows, depending on $|J(S)|$,*
$|J(S)| = 2$: *monomials $s^*(z) = az^n$, with $n = \pm 1$,*
$|J(S)| = 1$: *linear maps $s^*(z) = az + b$, with $|a| \leq 1$, or*
$|J(S)| = 0$: *spherical rotations $s^*(z) = (az - \bar{b})/(bz + \bar{a})$, with $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$.*

Proof. For each $s \in S$, $J(s) \subset J(S)$ is finite. So s must be Möbius ([22], Lemma 4.5 and Corollary 4.11). Thus $S \subset \mathcal{M}$. As s is bijective and $J(S)$ is finite and backward invariant by s, s permutes $J(S)$.

First suppose $|J(S)| = n \geq 3$. Since $s^{-1}(J(S)) \subset J(S)$ and a Möbius transformation is determined by its values at any 3 points, $|S| \leq n(n-1)(n-2)$. As S is finite, S is a normal family on $\widehat{\mathbb{C}}$ and so $J(S)$ is empty. This contradiction proves S is elementary.

Next suppose $|J(S)| = 2$. Let S^* be the semigroup obtained by conjugating S by a Möbius map which takes $J(S)$ to $\{0, \infty\}$. Then each $s \in S^*$ stabilizes $\{0, \infty\}$ so s is of the form $z \mapsto az^n$.

Now suppose $|J(S)| = 1$. Let S^* be the semigroup obtained by conjugating S by a Möbius map which takes $J(S)$ to $\{\infty\}$. Each $s \in S^*$ fixes ∞ and so s is of the form $z \mapsto az + b$. Further, $|a| \leq 1$ else s would have a repelling fixed point in $\mathbb{C}$, thus contradicting the fact that $J(S^*) = \{\infty\}$.

The cases when $J(S)$ have 1 or 2 elements conclude our proof of Theorem 2.4.

Our final assertion, when $J(S)$ is empty, is just (1) $\Rightarrow$ (3) in the following proposition. $\square$

Proposition 2.12. *Let S be a Möbius semigroup and let G be the group generated by S. The following are equivalent:*

(1) *$J(S) = \emptyset$*
(2) *S is bounded, i.e., its closure $\overline{S} \subset \mathcal{M}$ is compact*
(3) *S fixes a point in H^3 and thus S is conjugate to a semigroup of spherical rotations*
(4) *$J(G) = \emptyset$*
(5) *G is bounded, i.e., its closure $\overline{G} \subset \mathcal{M}$ is compact*
(6) *G fixes a point in H^3 and thus G is conjugate to a group of spherical rotations*
(7) *Every nonidentity element of G is elliptic.*

Proof. (1) $\Rightarrow$ (2). Suppose $J(S)$ is empty, so S is a normal family on $\widehat{\mathbb{C}}$. Then every sequence in S has a subsequence that converges in $\mathcal{M}$ so S is bounded in $\mathcal{M}$.

(2) $\Rightarrow$ (3). Consider a bounded Möbius semigroup S. Choose a point $p \in H^3$ and consider $S(p) = \{s(p) | s \in S\}$, the S-orbit of p. $S(p)$ is bounded in H^3 since $\overline{S}$ is compact. Let $C(p)$ be the closed convex hull of $S(p)$ in H^3, that is $C(p)$ is the intersection of all hyperbolic halfspaces containing $S(p)$.

Since $S(p)$ is bounded in H^3, $C(p)$ is compact as well as convex. Moreover $C(p)$ has the same diameter as $S(p)$. For each $s \in S$ we have $s(S(p)) \subset S(p)$ and so $s(C(p)) \subset C(p)$. As $s \in S$ acts on the compact metric space $C(p)$ by an isometry, we have $s(C(p)) = C(p)$ ([24], Exer. 7 p. 182).

Clearly $C(p') \subset C(p)$ for all $p' \in C(p)$. Moreover, if $p' \in C(p)$ is not an extreme point of $C(p)$ then no point of $S(p')$ is an extreme point of $C(p)$. It follows that the diameter of $C(p')$ is strictly less than the diameter of $C(p)$.

We may choose $b \in C(p)$ so that the diameter of $C(b)$ is minimal. Then every point of $C(b)$ is an extreme point, so $C(b)$ consists of just one point. Thus S fixes b. By moving b to $(0,0,1)$ by a Möbius transformation we may conjugate S to a semigroup S^* of the required form (see [2], p. 63).

(3) $\Rightarrow$ (1). Clearly $J(S^*)$ is empty, since S^* consists only of isometries of the spherical metric. It follows that $J(S)$ is empty.

Thus (1)-(3) are equivalent. Since G is also a semigroup in its own right, the equivalence of (4)-(6) then follows. Clearly (3) is equivalent to (6) and (6) $\Rightarrow$ (7), so it suffices to show the following.

(7) $\Rightarrow$ (4). Suppose G fixes a point p in $\widehat{\mathbb{C}}$ that is a neutral fixed point of each $g \in G$. We may conjugate G so that $p = \infty$ and G is a group of transformations $az + b$ with $|a| = 1$. As G contains no translations, the commutator of any two elements of G is trivial. Thus G is abelian and any two nonidentity elements of G must fix the same point in $\mathbb{C}$. Thus G is bounded and so $\Lambda(G)$ is empty. Now (4) follows by Theorem 2.4, which was proven above. □

Remark 2.13. *(a) The equivalence of (6) and (7) in Proposition 2.12 is Theorem 4.3.7 of* [2]. *We have given a self-contained proof for the sake of completeness.*

(b) When S is bounded and $p \in H^3$ we set $d(p) = \dim C(p)$ and give $C(p)$ the measure defined by the hyperbolic $d(p)$-volume. The mean squared distance to points of $C(p)$ is a strictly convex function on H^3 whose unique minimum $b(p)$ is the barycenter, or center of mass, of $C(p)$ ([26]*, Theorem 1). As S preserves this measure on $C(p)$ and S preserves hyperbolic distance, S fixes $b(p)$. This gives another proof that* (2) $\Rightarrow$ (3)*.*

Example 2.14. *There is no condition for semigroups in Proposition 2.12 analogous to (7) because an unbounded Möbius semigroup S may consist entirely of elliptic elements. For example, let $S = \prec az, az+1 \succ$ where $|a| = 1$ and a is not a root of unity. Each $s \in S$ has the form $s(z) = a^n z + b$ for some $n > 0$, $b \in \mathbb{C}$, so s is elliptic. But the axes of az and $az+1$ do not meet in H^3. Thus S fixes no point in H^3 and so, since* (2) $\Rightarrow$ (3) *in Proposition 2.12, S is unbounded.*

Remarkably, all non-elementary rational semigroups have many features in common. This follows from Montel's Theorem, which is the major tool in the study of normal families.

Theorem 2.15 ([3], p. 57)**.** *(Montel) Let $\mathcal{F}$ be a family of meromorphic functions on an open set $U \subset \widehat{\mathbb{C}}$ with values in $\widehat{\mathbb{C}} \setminus \{a, b, c\}$ where $a, b, c \in \widehat{\mathbb{C}}$ are three distinct points. Then $\mathcal{F}$ is normal.*

We first use Montel's Theorem to quantify the sensitive dependence on initial conditions near $J(S)$.

Proposition 2.16 (Transitivity for semigroups)**.** *Let S be a rational semigroup. Suppose $z \in J(S)$ and let N be a neighborhood of z. Then the union U of the images $s(N)$ for $s \in S$ can omit at most two points of $\widehat{\mathbb{C}}$. If S is Möbius then U can omit at most one point of $\widehat{\mathbb{C}}$.*

It is easy to see that these results are optimal. For the cyclic semigroup S generated by z^2 and for any small neighborhood N of the Julia point $z = 1$ we have $U = \mathbb{C}^*$. For a semigroup of linear maps containing $h(z) = 2z$ and any small neighborhood of the Julia point $z = 0$ we have $U = \mathbb{C}$.

Proof. Replacing N by its interior, we may suppose that N is open. Note that $s(N) \subset U$ for all $s \in S$ but S is not normal on N since $z \notin F(S)$. By Montel's theorem, U omits at most two points of $\widehat{\mathbb{C}}$.

Suppose S is Möbius and U omits exactly two points p and q. Then the set $\{p, q\}$ must be stabilized by S, and so by Proposition 2.2, we have either $z = p$ or $z = q$. But then clearly for any $s \in S$ we would have $z = s^2(z) \in s^2(N) \subset U$ which contradicts the fact that U omits p and q. $\square$

When S is cyclic, the set U in Proposition 2.16 contains $J(S)$, as in [22], Transitivity Theorem 4.7. This is not so, however, for $S = \prec 2z, (z+1)/2 \succ$ with $z = 0 \in J(S)$. For as already noted, $U = \mathbb{C}$ for a small neighborhood N of 0 and so U omits the Julia point ∞.

The classification of elementary semigroups in Theorem 2.11 suggests that any backward invariant set $X \subset \widehat{\mathbb{C}}$ for a rational semigroup S should be quite special. We give three results concerning such an X, two of which use Montel's Theorem.

Proposition 2.17. *Let $X \subset \widehat{\mathbb{C}}$ be backward invariant for a rational semigroup S. If X is finite then X is completely invariant under S.*

Proof. Let $s \in S$. Since s maps $\widehat{\mathbb{C}}$ onto itself we must have $s(s^{-1}(X)) = X$ and so $|s^{-1}(X)| \geq |X|$. Since $s^{-1}(X) \subset X$, by hypothesis, we see that $s^{-1}(X) = X$. Thus $s(X) = s(s^{-1}(X)) = X$. $\square$

Proposition 2.18. *Let $X \subset \widehat{\mathbb{C}}$ be backward invariant for a rational semigroup S. Then $J(S) \subset \overline{X}$ or $|X| \leq 2$.*

Proof. Suppose $n = |X| \geq 3$. As X is backward invariant, so is its closure $\overline{X}$. Montel's Theorem applies to the open forward invariant set $U = \widehat{\mathbb{C}} \backslash \overline{X}$ and gives $U \subset F(S)$, so $J(S) \subset \overline{X}$. $\square$

Proposition 2.19. *Let $X \subset \widehat{\mathbb{C}}$ be backward invariant for a rational semigroup S. If X is finite and contains 3 or more points then S is Möbius and finite.*

Proof. Suppose $n = |X| \geq 3$. By Proposition 2.18, $J(S) \subset X$ so $J(S)$ is finite. By Proposition 2.11, each $s \in S$ is Möbius. The values of s^{-1} at three points of X determine s, so $|S| \leq n(n-1)(n-2)$. $\square$

In addition to the methods used in the given references, the following remark may be justified by the previous propositions. In particular, one can take X to be the derived set of $J(S)$ in Proposition 2.18 in order to show that $J(S)$ is perfect.

Remark 2.20. *(see* [16] *p.363,* [31] *p.765). Let S be a non-elementary rational semigroup. Then $J(S)$ is the smallest closed backward invariant set containing at least three points. Moreover $J(S)$ is perfect and thus uncountable.*

Suppose the S in this remark is Möbius. Propositions 2.2 and 2.17 show that any closed nonempty backward invariant set that does not contain $J(S)$ must contain exactly one point. In the following example ∞ is such a point, which happens to be a Fatou point. We also observe the existence of a closed set $X \neq \widehat{\mathbb{C}}$ strictly containing $J(S)$ that is backward self-similar.

Remark 2.21. *Let* $h_1(z) = 3z$, $h_2(z) = 3z - 2$, *and* $S = \prec h_1, h_2 \succ$. *Example 4.4 shows that* $J(S)$ *is the standard middle third Cantor set, yet* $X = \widehat{\mathbb{R}}$ *satisfies* $X = \cup_{i=1}^{2} h_i^{-1}(X)$. *Also* $\{\infty\}$ *is backward invariant but does not contain* $J(S)$.

With the aid of Theorem 2.11 we now see that each elementary S has a finite orbit in the 3-ball $H^3 \cup \widehat{\mathbb{C}}$.

Corollary 2.22. *An elementary rational semigroup* S *either fixes a point in* H^3 *or has an orbit in* $\widehat{\mathbb{C}}$ *with one or two points.*

Proof. Any such S must be a Möbius semigroup. Since $J(S)$ is backward invariant with $|J(S)| \leq 2$, it is completely invariant by Proposition 2.17. Thus when $J(S)$ is nonempty, any Julia point must have an orbit contained in $J(S)$. When $J(S)$ is empty, Theorem 2.11 shows that S fixes a point in H^3. □

3. Möbius groups

We recall the following definitions and results which can be found in [2], our main reference for Möbius groups. For every complex matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc \neq 0$ there is a corresponding $m(z) = (az + b)/(cz + d)$ in $\mathcal{M}$. We write $tr(M) = a + d$ and $det(M) = ad - bc$ for the trace and determinant of M and note that trace is homogeneous of degree 1 while determinant is homogeneous of degree 2. Since $m(z)$ determines M up to a scalar factor, we may unambiguously define $tr^2[m] = tr^2(M)/det(M)$. This invariant of $m(z)$ determines its dynamical properties, as follows.

Theorem 3.1 ([2], p. 67)**.** *Let* Id *denote the identity map on* $\widehat{\mathbb{C}}$ *and let* $m \in \mathcal{M}$, $m \neq Id$. *Then*

1. $tr^2[m] = 4$ *if and only if* m *is* <u>*parabolic*</u>, *i.e.,* m *has only one neutral fixed point and* m *is conjugate to the translation* $z \mapsto z + 1$.
2. $tr^2[m] \in [0, 4)$ *if and only if* m *is* <u>*elliptic*</u>, *i.e.,* m *has two neutral fixed points and* m *is conjugate to a rotation* $z \mapsto kz$ *with* $|k| = 1$.
3. $tr^2[m] \in (4, +\infty)$ *if and only if* m *is* <u>*hyperbolic*</u>, *i.e.,* m *has an attracting and a repelling fixed point and* m *is conjugate to* $z \mapsto kz$ *with* $k \in \mathbb{R}$ *and* $|k| > 1$.
4. $tr^2[m] \notin [0, +\infty)$ *if and only if* m *is* <u>*strictly loxodromic*</u>, *i.e.,* m *has an attracting and a repelling fixed point and* m *is conjugate to* $z \mapsto kz$ *with* $|k| > 1$ *and* $k \notin \mathbb{R}$.

Finally, a *loxodromic* map is any $m \in \mathcal{M}$ which is either hyperbolic or strictly loxodromic. Beware that this terminology is not universal! Some authors use loxodromic to mean strictly loxodromic and use hyperbolic to encompass all loxodromic maps. Others use hyperbolic to encompass both loxodromic maps and elliptic maps.

Let G be a Möbius group. We say G is an *elementary Möbius group* if it has a finite orbit in either $\widehat{\mathbb{C}}$ or H^3 and otherwise we say that G is a *non-elementary Möbius group* ([2], p.83).

Remark 3.2. *Corollary 2.22 (as well as Theorem 5.1.3 in* [2]*) shows that if the Möbius semigroup* G *is elementary then* G *is also an elementary Möbius group. Alas* G *may have a finite orbit in* $\widehat{\mathbb{C}}$ *and an infinite Julia set, as in Example 3.10, in which case the elementary*

Möbius group G is not elementary! As our main interest is in semigroups, we will avoid ambiguity by treating the phrases "elementary Möbius group" and "non-elementary Möbius group" as indivisible, just as if they were single words. With this convention "the Möbius group G is non-elementary must mean that $|J(G)| > 2$ (although we will avoid such confusing statements altogether).

The class of elementary Möbius groups contains all abelian subgroups of $\mathcal{M}$, all finite subgroups of $\mathcal{M}$ and the stabilizer[7] of each point in $\widehat{\mathbb{R}}^3$. For example, the groups generated by the non-elementary semigroups studied in Examples 3.10, 6.1, 6.3 and 6.4 are subgroups of the stabilizer of ∞ in $\mathcal{M}$. For a more complete discussion of elementary Möbius groups see [2, 27].

When G is a non-elementary Möbius group, the limit set (or unstable set) $\Lambda(G)$ is defined in [2], p. 97, as the closure of the set of all sinks and sources of elements of G. It follows from Theorem 2.4 that $\Lambda(G) = J(G)$. However [12] defines the limit set for *any* Möbius group G to be $\widehat{\mathbb{C}} \cap \overline{Gx}$, where x is any point in H^3. By Proposition 2.6, $\widehat{\mathbb{C}} \cap \overline{Gx} = J(G)$, so this coincides with the previous definition when G is a non-elementary Möbius group. We will adopt the latter definition of the limit set, so that $\Lambda(G) = J(G)$ for any Möbius group G. The ordinary set (or stable set) $\Omega(G)$ is defined as the open set in $\widehat{\mathbb{C}}$ complementary to $\Lambda(G)$ ([2], p. 97). Thus $\Omega(G) = F(G)$.

Recall that a Möbius group G is *discrete* if it is discrete as a subspace of the topological space $\mathcal{M}$ or, equivalently, if Id is an isolated point of G.

Two Möbius groups Γ and Γ' are *commensurable* if they share a finite index subgroup, that is if the intersection $\Gamma \cap \Gamma'$ has finite index both in Γ and in Γ' ([20], p. 170, or [12]). Commensurability is an equivalence relation on Möbius groups. If Γ' is commensurable with a non-elementary Möbius group Γ then Γ' is also a non-elementary Möbius group and $\Lambda(\Gamma') = \Lambda(\Gamma)$. If Γ and Γ' are commensurable then so are the closed groups $\overline{\Gamma}$ and $\overline{\Gamma'}$. Any group Γ' commensurable with a discrete group Γ is also discrete.

Each Möbius semigroup S generates a Möbius group G. If $h_i,\ i \in I$, are semigroup generators of S then $h_i,\ i \in I$, are also group generators of G. We are interested quite generally in the relation between the dynamics of S and the dynamics of G.

For example, we can compare and contrast the connectivity of Julia sets and limit sets.

Theorem 3.3. *Let G be a Möbius group generated by a non-elementary semigroup S. If $J(S)$ is connected then $\Lambda(G)$ is also connected.*

This theorem is based on the following lemma.

Lemma 3.4. *Assume that the Möbius semigroup $S = \prec h_i : i \in I \succ$ is non-elementary and that some set $K \subset J(S)$ meets $h_i^{-1}(K)$ for all i. If K is connected and $|K| > 1$ then $J(S)$ is connected.*

See Theorem 1 of [39] and Theorem 2.1 of [38] for similar results regarding semigroups which allow non-Möbius rational maps in the semigroup.

Proof of Lemma 3.4. As K is connected, K is contained in some connected component C of $J(S)$. C is closed in $J(S)$ and hence compact. As K is connected and $|K| > 1$, both K and C are infinite.

[7] Let G be a subgroup of $\mathcal{M}$ and let x be a point in $\widehat{\mathbb{R}}^3$. The *stabilizer* of x in G is the subgroup $G_x = \{g \in G : g(x) = x\}$.

Fix $i \in I$ and let $C_i = h_i^{-1}(C)$. Clearly C_i is connected. As $J(S)$ is backward invariant under S, $C_i \subset J(S)$. By our assumption that K meets $h_i^{-1}(K)$, C_i must meet C so $C \cup C_i$ is connected. As C is a component of $J(S)$, it follows that $C_i \subset C$.

As the h_i generate S, we see that C is backward invariant under S. As $C \subset J(S)$ is infinite and compact, Remark 2.20 shows that $C = J(S)$. Thus $J(S)$ is connected. □

Proof of Theorem 3.3. We take generators h_i, $i \in I$, for S and regard G as a semigroup with generators h_i, h_i^{-1}, $i \in I$. We will show that we may apply Lemma 3.4 to the semigroup G with $K = J(S)$. Note that $J(S) \subset J(G)$, since $S \subset G$. Also $|J(S)| \geq 3$, since S is non-elementary, and $J(S)$ is connected, by hypothesis.

Now fix $i \in I$. $h_i^{-1}(J(S)) \subset J(S)$, since $J(S)$ is backward invariant under S, so applying h_i to both sides we also get $J(S) \subset h_i(J(S))$. Hence both generators h_i and h_i^{-1} meet the conditions of Lemma 3.4. We deduce that $\Lambda(G) = J(G)$ is connected. □

Consider $S = \prec 2z \succ$, so $G = \langle 2z \rangle$. Then $J(G) = \{0, \infty\}$ is disconnected even though $J(S) = \{\infty\}$ is connected. This shows that one must rule out elementary Möbius groups in Theorem 3.3.

Note that the converse of Theorem 3.3 is false. For in Example 3.10 $J(S)$ is a Cantor set but $\Lambda(G)$ is connected.

Theorem 3.3 shows that when $\Lambda(G)$ is disconnected then $J(S)$ and $J'(S)$ are also disconnected. Since $J(S_1) \subset J(S_2)$ whenever $S_1 \subset S_2$, we obtain a similar, but simpler, result.

Proposition 3.5. *Let S be a Möbius semigroup and G the group generated by S. If $\Lambda(G)$ is totally disconnected, then $J(S)$ and $J'(S)$ are also totally disconnected.*

We now consider sensitive dependence on initial conditions near a limit point. Since a limit set is a Julia set, Proposition 2.16 implies

Corollary 3.6 (Transitivity for groups)**.** *Let G be a Möbius group. Suppose N is a neighborhood of $z \in \Lambda(G)$. Then the union U of the images of $g(N)$ for $g \in G$ contains all but at most one point of $\widehat{\mathbb{C}}$.*

If the Julia set $J(h)$ of a rational function h has an interior point then $J(h) = \widehat{\mathbb{C}}$, as in [22], Corollary 4.8. We will now show that limit sets behave similarly.

Corollary 3.7. *If the limit set $\Lambda(G)$ of a Möbius group G contains an interior point then $\Lambda(G) = \widehat{\mathbb{C}}$.*

Proof. Let $N \subset \Lambda(G)$ be nonempty and open. The union U of the forward images $g(N)$ is a subset of $\Lambda(G)$, by forward invariance. U is dense in $\widehat{\mathbb{C}}$ (by Corollary 3.6). Since $\Lambda(G)$ is a closed set, it follows that $\Lambda(G) = \widehat{\mathbb{C}}$. □

On the other hand, Example 6.1 below shows that $J(S) \neq \widehat{\mathbb{C}}$ may have nonempty interior. This shows that the forward invariance of the limit set is essential in the last corollary.

Non-elementary Möbius groups are necessarily complicated.

Theorem 3.8 ([2], p. 90)**.** *Every non-elementary Möbius group contains infinitely many loxodromic elements, no two of which have a common fixed point in $\widehat{\mathbb{C}}$.*

The limit set of a non-elementary Möbius group must also be complicated, as shown by the following consequence of Remark 2.20.

Theorem 3.9. *Let G be a Möbius group such that $|\Lambda(G)| \geq 3$. Then $\Lambda(G)$ is a perfect set and hence uncountable. Moreover each G-invariant set in $\widehat{\mathbb{C}}$ with three or more points contains $\Lambda(G)$.* [8]

In particular, $\Lambda(G)$ is the unique minimal G-invariant set, as shown in [2], p. 97, in the case that G is a non-elementary Möbius group.

Theorem 2.4 implies that every non-elementary Möbius semigroup S contains infinitely many loxodromic elements. When these all share a fixed point p, however, S must fix p. This implies that S generates an elementary Möbius group, as in the following example.

Example 3.10. *Let $S = \prec 3z, 3z-2 \succ$ so $S^{-1} = \prec z/3, \ (z+2)/3 \succ$ and the group generated by S is $G = \langle 3z, 3z-2 \rangle$. $J(S)$ is the Cantor middle-thirds set (see Example 4.4) which implies that S and G are non-elementary semigroups. But S^{-1} is a normal family in $\mathbb{C}$ so $J'(S) = \{\infty\}$. Thus S^{-1} is an elementary semigroup. Since G fixes ∞, G is an elementary Möbius group.*

We show $\Lambda(G) = \widehat{\mathbb{R}}$ as follows. First, we see that $\Lambda(G) \subset \widehat{\mathbb{R}}$ by Theorem 3.9 since $\widehat{\mathbb{R}}$ is completely invariant under G. If $\Lambda(G) \neq \widehat{\mathbb{R}}$, then there exist $a, b \in \mathbb{R}$ such that $a < b$, the interval $(a, b) \subset \mathbb{R} \setminus \Lambda(G)$ and $a \in \Lambda(G)$. Letting $p(z) = 3z$ and $q(z) = 3z - 2$, we see $a_n = p^{-n}(q^{-1}(p^{n+1}(a))) = a + 2/3^{n+1}$. By complete invariance each $a_n \in \Lambda(G)$ and $a_n \searrow a$. However, this contradicts the fact that (a, b) does not meet $\Lambda(G)$, and so the result follows.

It is also possible to characterize the stable set of a discrete Möbius group using the concept of *discontinuous group action.*

Definition 3.11. *A subgroup G of $\mathcal{M}$ acts discontinuously on a completely invariant open set $X \subset \widehat{\mathbb{C}}$ if for every compact subset K of X there are only finitely many $g \in G$ such that $g(K)$ meets K.*

When G acts discontinuously on X, this action does not have any sensitive dependence on initial condition. When G is a discrete non-elementary Möbius group, G acts discontinuously on $\Omega(G)$ and $\Omega(G)$ contains any completely invariant open set X on which G acts discontinuously (see [2] p. 99 and 104). This clearly contrasts with the sensitive dependence on initial conditions that holds near a limit set.

The limit set of a Möbius group G with $|\Lambda(G)| \geq 3$ is constrained by a dichotomy.

Corollary 3.12. *Let G be a Möbius group with $|\Lambda(G)| \geq 3$. Then $\Lambda(G)$ is either connected or else has uncountably many connected components.*

Note that the example $G = \langle 2z \rangle$ with $\Lambda(G) = \{0, \infty\}$ shows that the condition $|\Lambda(G)| \geq 3$ above is critical.

Proof. We imitate the proof of the corresponding result for the Julia set of a rational function of degree two or more ([22], p.47). Suppose $\Lambda(G)$ can be expressed as the union $A_0 \cup A_1$ of two disjoint, non-empty compact subsets. Since $\Lambda(G)$ is perfect by Theorem 3.9, we have $|A_0| = +\infty$ and $|A_1| = +\infty$.

We show that neither A_0 nor A_1 can be connected. Choose an open set U which meets A_0 but not A_1. Let $k \in G$ be a map with a repelling fixed point in $U \cap A_0$ (see Theorem 2.4).

[8] Recall, that an S-invariant set is by definition a non-empty, compact set which is forward invariant under S.

Since k is loxodromic and $|A_1| > 1$, we may choose n to be large enough so that $k^n(U)$ meets both A_0 and A_1. Set $h = k^n$. Since $\Lambda(G)$ is backward invariant, we must have that $h(A_0)$ meets both A_0 and A_1. We can therefore use the forward invariance of $\Lambda(G)$ to express A_0 as the disjoint union of non-empty compact subsets $A_{00} = A_0 \cap h^{-1}(A_0)$ and $A_{01} = A_0 \cap h^{-1}(A_1)$.

We have just shown the following.

Lemma 3.13. *Let G be a Möbius group with $|\Lambda(G)| \geq 3$. When $\Lambda(G)$ is disconnected, each nonempty closed and open subspace of $\Lambda(G)$ is also disconnected.*

Using this lemma, we can similarly express A_1 as a disjoint union of non-empty compact subsets A_{10} and A_{11}. Again applying this technique we can express any of the sets A_{00}, A_{01}, A_{10} and A_{11} as the disjoint union of non-empty compact subsets. In particular, for any sequence $\alpha_1 \dots \alpha_k$ where each α_j is 0 or 1 we can construct a non-empty compact subset $A_{\alpha_1 \dots \alpha_k}$ such that for any of the uncountable infinite sequences $\alpha_1 \dots \alpha_k \dots$ the infinite decreasing intersection

$$\cap_k A_{\alpha_1 \dots \alpha_k}$$

is non-empty and contains at least one connected component of $\Lambda(G)$. Since each such intersection is disjoint from any other such intersection the result follows. □

The preceding proof shows that the Cantor set $\{0,1\}^{\mathbb{N}}$ is a quotient space of $\Lambda(G)$ when this limit set is disconnected and $|\Lambda(G)| \geq 3$. More is true, however.

Proposition 3.14. *When $\Lambda(G)$ is disconnected and $|\Lambda(G)| \geq 3$ there is a natural map $\pi : \Lambda(G) \to Q$, where Q is a Cantor set, π is onto, and each fiber $\pi^{-1}(q)$, $q \in Q$, is connected.*

Proof. Let X be any topological space and let I be the set of all open and closed subspaces $A \subset X$. For each such A let χ_A be the characteristic function of A. In combination, these characteristic functions define a mapping χ from X to the product space $\{0,1\}^I$. The image of this map is $Q(X)$, the space of *quasicomponents* of X, and we have a natural projection $\pi : X \to Q(X)$ that maps each $x \in X$ to its quasicomponent.

When X is a compact metric space, I is countable and so $Q(X)$ is compact, metric, and totally disconnected. For such X, every quasicomponent is a component ([24] Section 5-1, Exer. 4) so the fibers of π are just the components of X.

When $X = \Lambda(G)$ is disconnected, $|Q(X)| > 1$. By Lemma 3.13, $Q(X)$ has no isolated point. Thus $Q(X)$ is a Cantor set. □

Corollary 3.12 does not necessarily hold in the more general setting of rational semigroups. We record this and other such differences in the following table which compares the dynamics of different classes of rational semigroups S. The first column shows three potential properties of the Julia and Fatou sets. The second column refers to both the dynamics of a rational function of degree two or more (i.e., $S = \prec h \succ$ with $\deg(h) \geq 2$) and the dynamics of a Möbius group S with $|J(S)| \geq 3$ (i.e., $S = \prec h_i, h_i^{-1} : i \in I \succ$ with each h_i Möbius). The third column refers to the dynamics of a rational semigroup S with no restrictions on the degree of its generators.

The results in the table for rational functions can be found in [22], and the results for Möbius groups and rational semigroups can be found in this paper as indicated (see also [16]).

<table>
<tr><td></td><td>rational function; Möbius group</td><td>rational semigroup</td></tr>
<tr><td>$J(S)$ and $F(S)$ are completely invariant</td><td rowspan="3">yes[11]</td><td>$F(S)$ is forward invariant; $J(S)$ is backward invariant[10]</td></tr>
<tr><td>$J(S)$ is connected or has uncountably many connected components</td><td rowspan="2">no[12]</td></tr>
<tr><td>$J(S)$ has empty interior or $J(S) = \widehat{\mathbb{C}}$</td></tr>
</table>

We now study when the inclusion $J(S) \subset \Lambda(G)$ is an equality.

Proposition 3.15. *Let S be a non-elementary Möbius semigroup such that $J(S)$ is completely invariant. Then $J(S) = \Lambda(G)$ where G is the group generated by S.*

For Theorem 3.9 gives the opposite inclusion $\Lambda(G) \subset J(S)$.

Beware that $J(S)$ may be completely invariant with $J(S) \neq J'(S)$, as illustrated in the following example.

Example 3.16. *Let q_n be a dense sequence of real numbers and for each $n \in \mathbb{N}$ let s_n be a linear map with real coefficients with source q_n. Let $S = \prec s_n : n \in \mathbb{N} \succ$. Theorem 3.9 gives $\Lambda(G) \subset \widehat{\mathbb{R}}$. Since $q_n \in J(S)$ for all n, $\widehat{\mathbb{R}} \subset J(S)$. Thus $J(S) = \Lambda(G) = \widehat{\mathbb{R}}$ and so $J(S)$ is completely invariant. But S^{-1} is normal on $\mathbb{C}$ and so $J'(S) = \{\infty\} \neq J(S)$.*

4. Contracting iterated function systems

Let S be a rational semigroup and $K \subset \widehat{\mathbb{C}}$ an S-invariant set equipped with a metric d, consistent with the topology on K. We say that S is an IFS (iterated function system) on (K, d). S *uniformly contracts* (K, d) if there exists a constant c, $0 < c < 1$, such that for each $s \in S$ we have $s^*d \leq cd$ (that is, $d(s(z), s(w)) \leq cd(z, w)$ for all $z, w \in K$). When S uniformly contracts (K, d) we say that S is a CIFS (contracting iterated function system) on (K, d) or, less precisely, a CIFS on K.

We define the *attractor* $\mathcal{A} = \mathcal{A}(K, S)$ as the closure of the set of all fixed points in K of the maps $s \in S$. We suppress the dependence on S and K when there is no chance for confusion.

For example consider the quadratic family $S_c = \prec z^2 + c \succ$. The set $\{\infty\}$ is an attractor for all $c \in \mathbb{C}$. For small c, S_c uniformly contracts the disc $K = \Delta(0, 1/2)$ in the Euclidean metric, so S_c has multiple attractors.

The situation simplifies for Möbius semigroups, however.

Proposition 4.1. *Suppose S is a Möbius semigroup and K is an S-invariant set that is uniformly contracted by S. Either K is a fixed point of S or $\mathcal{A}(K, S) = J'(S)$.*

Proof. We assume $|K| > 1$ and show $\mathcal{A}(K, S) = J'(S)$.

Suppose $s \in S$ is loxodromic with sink p and source q. Then for $z \in K, z \neq q$, the sequence $s^n(z), n \geq 0$, lies in K and converges to p. Thus $p \in K$, as K is compact. But $q \notin K$ as s contracts K and both p and q are fixed by s,

[10]See Proposition 2.7 and Example 3.10.

[11]See Proposition 2.7 and Corollaries 3.12 and 3.7.

[12]See Examples 6.1, 6.3, and 6.4.

Suppose $s \in S$ is parabolic with fixed point p. Then $s^n(z) \to p$, for any $z \in K$, so $p \in K$.

Suppose $s \in S$ is elliptic. For some sequence $n_k \to \infty$, s^{n_k} converges uniformly to Id on $\widehat{\mathbb{C}}$. For distinct points $z, w \in K$ this implies that the distance from $s^{n_k}z$ to $s^{n_k}w$ does not tend to 0, contradicting the assumption that S uniformly contracts K. Thus no $s \in S$ is elliptic.

All told, we see that each fixed point in K is either a sink or a parabolic fixed point, hence a point in $J'(S)$. Thus $\mathcal{A}(K,S) \subset J'(S)$. Since each sink lies in K we see that Sink $S \subset \mathcal{A}(K,S)$. By applying Theorem 2.4 to S^{-1}, we see that either $\mathcal{A}(K,S) =$ Sink $S =$ Source $S^{-1} = J(S^{-1}) = J'(S)$ or some $p \in \widehat{\mathbb{C}}$ is a neutral fixed point of each $s \in S$.

Suppose such a p exists. Then as S contains no elliptic elements, each $s \in S$ is parabolic with fixed point p. In this case $J'(S) = \{p\}$, by Proposition 2.2. But the parabolic fixed point p lies in K and hence lies in $\mathcal{A}(K,S)$. So one still has $\mathcal{A}(K,S) = J'(S)$ in this case. □

As mentioned in the introduction of this paper, we are interested in the properties of $\mathcal{A}$ in relation to the *random dynamics* (or random walk) when $S = \prec h_1, \dots, h_N \succ$ is a finitely generated CIFS. Specifically, it is known that $\mathcal{A}$ is the ω-limit set[12] of the random walk given by the following description: starting at a point x_0 in K, randomly select (we may for simplicity assume a uniform probability) a number ι_1 from $\{1, \dots, n\}$ and define $x_1 = h_{\iota_1}(x_0)$. Then randomly pick ι_2 from $\{1, \dots, n\}$ and define $x_2 = h_{\iota_2}x_1 = h_{\iota_2}h_{\iota_1}(x_0)$. Continue in this way to generate an orbit x_j for $j = 0, 1, \dots$. With probability one, the ω-limit set of the orbit x_j is exactly $\mathcal{A}$. We should mention that there exists a unique stationary Borel measure μ for this process whose support is exactly $\mathcal{A}$ and that an ergodic theorem holds for μ-almost all orbits. See [17, 1, 9] or [28] for details and precise statements of all the above results.

We define the *compact image* $S(N)$ of a compact set $N \subset \widehat{\mathbb{C}}$ as the smallest compact set that contains each image $s(N)$, $s \in S$. Compact image commutes with finite unions in the sense that if $N_1, ..., N_k$ are compact with union N then $S(N)$ is the union of $S(N_1), ..., S(N_k)$. When S is finitely generated and N is S-invariant then $S(N)$ is the union of the images $h(N)$ where h runs over a finite set of generators of S.

Let $S = \prec h_1, \dots, h_n \succ$ be a CIFS on K, as above. One can construct the attractor $\mathcal{A}$ iteratively and characterize it as a fixed point, as follows.

Theorem 4.2 ([17], p. 724)**.** *Consider the space of all non-empty compact subsets $D \subset K$, endowed with the Hausdorff metric. Then each sequence $D, S(D), S^2(D) = S(S(D)), \dots$ converges to $\mathcal{A}$. In particular, for any open set $U \subset \widehat{\mathbb{C}}$ containing $\mathcal{A}$ there exists $N > 0$ such that $S^k(D) \subset U$ for all $k \geq N$. Moreover $\mathcal{A}$ is the unique nonempty compact set in K such that*

$$\mathcal{A} = S(\mathcal{A}) = \cup_{i=1}^{n} h_i(\mathcal{A}). \tag{4.1}$$

We refer to (4.1) as the *self-similarity* property of $\mathcal{A}$ since it expresses $\mathcal{A}$ is the union of small (contracted) copies of itself $h_i(\mathcal{A})$. Beware that the assumption of uniform contraction on K is critical here, as illustrated by Remark 2.21 (using the contracting inverse semigroup S^{-1} instead of the semigroup S given there).

Example 4.3 (Linear contractions)**.** *Consider $S = \prec h_1, \dots, h_n \succ$ where each h_i is a linear contraction, i.e., for each $i = 1, \dots, n$ we have $h_i(z) = a_i z + b_i$ where $|a_i| < 1$. In this case,*

[12] The ω-limit set of a sequence x_j is the set of all y such that there exists a subsequence $x_{j_k} \to y$.

it is not hard to see that for R sufficiently large, the set $K = \overline{\Delta(0,R)}$ is forward invariant under each h_i, and therefore under S. Since these maps are uniformly contracting with respect to the Euclidean metric (with contraction coefficient $c = \max_{i=1,\dots,n}|a_i|$), we have that S is a CIFS with an attractor $\mathcal{A} \subset K$. We also note that for any compact set $D \subset \mathbb{C}$, we have $S^k(D) \to \mathcal{A}$ (since one can choose R large enough that $D \subset K$). It follows that $\mathcal{A}$ is independent of R, for R sufficiently large.

Under the conditions of Proposition 4.1, with S finitely generated, Theorem 4.1 implies that any self-similar S-invariant set $B \subset K$ must equal $J'(S)$. This allows us to calculate many Julia sets.

Example 4.4. *Consider $S = \prec h_1, h_2 \succ$ for $h_1(z) = 3z$ and $h_2(z) = 3z - 2$. The inverse semigroup S^{-1} is a CIFS on, say, $K = \overline{\Delta(0,10)}$. Since the middle third Cantor set C satisfies $C = \cup_{i=1}^2 h_i^{-1}(C)$, we conclude $J(S) = \mathcal{A}(S^{-1}) = C$.*

5. Thick attractors

Let S be a finitely generated Möbius semigroup that is a CIFS on some compact set K^*. When the interior of K^* contains $\mathcal{A} = \mathcal{A}(K^*, S)$ we say this attractor $\mathcal{A}$ is *thick*.[13] As a thick attractor for S attracts the orbits of all points in the large set K^*, we see that there are many seed points whose random orbit will generate a good picture of $\mathcal{A}$.

Thick attractors arise whenever certain simple dynamical conditions are met.

Theorem 5.1. *Let K and K' be nonempty disjoint compact sets in $\widehat{\mathbb{C}}$, not both singleton sets, and let $h_1, \dots, h_n$, $n \geq 1$, be loxodromic Möbius transformations such that for each j, $h_j(K) \subset K$ and $K' \subset h_j(K')$. Let S be the semigroup generated by the h_j, so the S-invariant set K is disjoint from the S^{-1}-invariant set K'. Then S uniformly contracts K and $\mathcal{A}(K,S) = J'(S) =$ Sink S is a thick attractor.*

The following example illustrates the importance of the conditions put on $|K|$ and $|K'|$ above.

Example 5.2. *Let $S = \prec 2z \succ, K = \{0\}$, and $K' = \{\infty\}$. Here $J'(S) = \{\infty\} =$ Sink S, but since it is not a subset of K, it cannot be $\mathcal{A}(K,S)$.*

Although Example 5.2 shows how the conclusion of Theorem 5.1 can fail when $|K| = |K'| = 1$, this exceptional case can be fully analyzed. Indeed, suppose without loss of generality that $K = \{0\}$ and $K' = \{\infty\}$. Then both 0 and ∞ are fixed by S. Thus each generator $h_j \in S$ is of the form $z \mapsto a_j z$. Since, as noted above, each h_j is then an isometry on $\mathbb{C}^*$ endowed with the metric $|dz|/|z|$ we have that $\mathbb{C}^* \subset F(S)$. If $\max|a_j| < 1$, then $J(S) = \{\infty\}$ and $K = \{0\} = J'(S)$ is a thick attractor (using $K^* = \overline{\Delta(0,r)}$ for any $r > 0$). If $\min|a_j| > 1$, then $J(S) = \{0\}$ and $K' = \{\infty\} = J'(S)$ is a thick attractor (using $K^* = \widehat{\mathbb{C}} \setminus \Delta(0,r)$ for any $r > 0$). If $\min|a_j| < 1$ and $\max|a_j| > 1$, then S does not have a thick attractor and $\{0, \infty\} = J'(S) = J(S)$.

Proof of Theorem 5.1. Since h_1 has no neutral fixed point, Theorem 2.4 shows that Sink $S =$ Source $S^{-1} = J(S^{-1}) = J'(S)$.

We first prove the theorem in the cases where K or K' has at most two points.

[13] Beware that S may also be a CIFS on another set K with the same attractor $\mathcal{A}$ for which Int K does not contain $\mathcal{A}$.

Consider the case $|K'| = 2$. We may suppose, by conjugating S in $\mathcal{M}$, that $K' = \{0, \infty\}$. As K' is backward invariant by h_1 and as h_1 is loxodromic, $h_1(z) = az$ for some a with $|a| \neq 1$. Now K is nonempty so we may choose $z \in K \subset \mathbb{C}^*$. The sequence $z, h_1(z), h_1^2(z), ...$ lies in K and converges to either the origin (if $|a| < 1$) or to ∞ (if $|a| > 1$), contradicting the fact that K is closed and disjoint from K'. Thus $|K'| \neq 2$. Likewise one can show $|K| \neq 2$.

Now suppose that $|K'| = 1$ and $|K| > 1$. We may suppose, by conjugating S in $\mathcal{M}$, that $K' = \{\infty\}$ so $h_j(z) = a_j z + b_j$ for some a_j, b_j. As h_j is loxodromic, $h_j(K) \subset K$, $K \subset \mathbb{C}$ is compact, and $|K| > 1$ we must have $|a_j| < 1$ (else there would exist a point in K which is not the repelling fixed point of h_j and such a point would iterate under h_j to ∞). Thus S uniformly contracts $\mathbb{C}$, with its usual distance function and with contraction constant equal to the maximum of $|a_1|, ..., |a_n|$, and so S uniformly contracts K. When $s \in S$, s has a source at ∞ so the only fixed point of s in K is its sink. Thus Sink $S = \mathcal{A}(K, S)$. Consider the compact set K^* of all complex numbers whose distance from K is at most 1. Then S is also a CIFS on K^* with $\mathcal{A}(K^*, S) = \mathcal{A}(K, S) \subset K \subset K^*$, so this attractor is thick. Thus the theorem holds when $|K'| = 1$ and $|K| > 1$.

When $|K| = 1$ and $|K'| > 1$ we may, in a similar fashion, suppose $K = \{\infty\}$ so that $h_j(z) = a_j z + b_j$ for some a_j, b_j with $|a_j| > 1$. Then each $s \in S$ is loxodromic with sink at ∞. It is trivial that S uniformly contracts K and the rest of the theorem is clear in this case.

For the rest of the proof we shall assume K and K' have at least 3 points each. We need three lemmas in order to show that S uniformly contracts K.

Lemma 5.3. *Each $s \in S$ is loxodromic with sink in K and source in K'. Moreover $s(K)$ is strictly contained in K and $s(K')$ strictly contains K'.*

Proof. Say $s \in S$ is loxodromic with sink p and source q. As $|K| > 1$ we can choose $z \in K, z \neq q$. As $s(K) \subset K$, the sequence $z, s(z), s^2(z), ...$ approaches p. As K is compact, $p \in K$. Likewise $q \in K'$. As K and K' are disjoint, we see that $q \notin K$. Iterating s^{-1}, we see that $s^{-1}(K)$ is not a subset of K, so $s(K)$ is strictly contained in K. Likewise $s(K')$ strictly contains K'.

In particular $h_j(K)$ is strictly contained in K for all j. This implies $s(K)$ is strictly contained in K for all $s \in S$. Likewise $s^{-1}(K')$ is strictly contained in K', so K' is strictly contained in $s(K')$.

Suppose s is elliptic. After conjugating s, we may suppose that s preserves spherical distance on $\widehat{\mathbb{C}}$. Then s is an isometry of the compact metric space K, so $s(K) = K$ ([24], Exer. 7 p. 182). This contradicts the strict inclusion we just established, so s is not elliptic.

Suppose s is parabolic with fixed point p. Then $s(K) \subset K$ and K is compact and nonempty. If $z \in K$ then the orbit $z, s(z), s^2(z), ...$ approaches p and so $p \in K$. Reasoning likewise with s^{-1}, we see that $p \in K'$, contradicting our assumption that K and K' are disjoint. Thus s is not parabolic

The only remaining possibility is that s is loxodromic. $\square$

Let U be the complement of K' in $\widehat{\mathbb{C}}$ and let V be a component of U. Then V is open and connected and its complement has at least three points so V has a unique hyperbolic metric. This is a complete Riemannian metric of curvature -1 compatible with the complex structure on V. We let d_V denote the distance function on V defined by this hyperbolic metric. We will use the various d_V to define via (5.2) a distance function d on K and prove that S uniformly contracts (K, d).

As $K \subset U$ is compact and the components of U are disjoint and open, K meets only finitely many components $U_1, ..., U_m$ of U. We have $m \geq 1$ since K is nonempty. We let $d_j = d_{U_j}$ be the hyperbolic distance function on U_j.

Since K' is backward invariant under S, U is forward invariant under S. Thus S acts on the finite index set $F = \{1, ..., m\}$ so that $s(U_j) \subset U_{s(j)}$ for all $j \in F$ and $s \in S$. For $s(U_j) \subset U$ is connected so $s(U_j) \subset V$ for some component V of U. As $K \cap U_j$ is nonempty and $s(K \cap U_j) \subset K \cap V$, we must have $V = U_k$ for some $k \in F$. As the U_i's are disjoint, k is uniquely determined by j and s and we may define $s(j) = k$.

The following result is certainly consistent with S being uniformly contracting on K.

Lemma 5.4. *There is a $\lambda > 0$ such that for every $s \in S$ expressed as a word of length at least λ in the generators $h_1, ..., h_n$ one has $s(K) \subset U_k$ for some $k = k(s)$.*

Proof. We claim that each $s \in S$ fixes a unique point of F. Recall from Lemma 5.3 that s is loxodromic with sink p, say, in K. As K is covered by the U_j, there is a $j \in F$ with $p \in U_j$ which implies $s(j) = j$. If $k \in F$ and $s(k) = k$ then $s(U_k) \subset U_k$. Thus $p \in \overline{U_k}$. As the U_j are disjoint and open, $k = j$. Thus j is the unique fixed point of s in F.

Consider now the product action of S on $F^2 = F \times F$. By the uniqueness result just proven, any fixed point in F^2 of any element of S must lie on the diagonal $D \subset F^2$. Let $l = m^2 - m = |F^2| - |D|$ and let $s = h_{j_l} ... h_{j_1} \in S$ be a product of l generators h_j.

We claim that $s(F^2) \subset D$. To see this we define $s_i = h_{j_i} ... h_{j_1}$ for $0 \leq i \leq l$, with the understanding that $s_0 = Id$. For $(j, k) \in F^2$ the pairs $s_i(j, k) \in F^2$, $0 \leq i \leq l$, are either distinct or not. If these pairs are distinct then our choice of l implies that $s_i(j, k) \in D$ for some i. If these pairs $s_i(j, k)$ are not distinct then there exist $i < i'$ with $s_i(j, k) = s_{i'}(j, k)$. But $s_i(j, k)$ is fixed by $h_{j_{i'}} ... h_{j_{i+1}} \in S$ and so $s_i(j, k) \in D$. Either way, since D is S -invariant and $l \geq i$ we have $s(j, k) = s_l(j, k) \in D$. This proves our claim.

Now let $k = s(1)$ and pick $j \in F$. We have shown that $s(j, 1) \in D$, so we have $s(j) = k$. Thus $s(U_j) \subset U_k$. As the U_j, $j \in F$, cover K, we find that $s(K) \subset U_k$ as well. □

The preceding proof shows that we may take $\lambda = m^2 - m$, but in a given example a much smaller λ may suffice.

Now define the open set U_K to be the union of all components of U that meet K, so $U_K = U_1 \cup ... \cup U_m$. We define a *generalized* distance function δ on U_K by defining $\delta(z, w) = \infty$ when z, w lie in different components of U_K and $\delta(z, w) = d_j(z, w)$ when $z, w \in U_j$. For $l \geq \lambda$, we define a distance function δ_l on U_K by setting $\delta_l(z, w)$ equal to the maximum value of $\delta(s(z), s(w))$ over all $s \in S$ expressible as a word of length l in the generators $h_1, ..., h_n$. Note that our choice of λ assures that $\delta_l(z, w)$ is finite, and we leave it to the reader to verify that δ_l is a distance function. We surely have $\delta_l(h_j(z), h_j(w)) \leq \delta_{l+1}(z, w)$ for all $z, w \in U_K$, since sh_j is expressible as a word of length $l + 1$ whenever s is expressible as a word of length l. In a more convenient notation, we have

$$h_j^* \delta_l \leq \delta_{l+1}. \tag{5.1}$$

Lemma 5.5. *There is a $\mu > 0$ such that for every $j \in F$ and each $s \in S$ expressible as a word of length at least μ in the generators $h_1, ..., h_n$, $s(U_j)$ is strictly contained in $U_{s(j)}$.*

Proof. Fix $s \in S$ and let $f \in F$ be the unique fixed point of s and let p and q be the sink and source of s, respectively. We have seen that $p \in U_f \cap K$. As U_f is an open neighborhood of p, if $s^{-1}(U_f) \subset (U_f)$ we would find that U_f contains $\widehat{\mathbb{C}}\backslash\{q\}$, contradicting the fact that the complement of U contains at least three points. Thus $s(U_f)$ is strictly contained in U_f.

Suppose $s = h_{j_m}...h_{j_1} \in S$ is a product of m generators h_j. Let $s_i = h_{j_i}...h_{j_1}$ for $i = 1, ..., m$. Given $j \in F$ we define $j_0, ..., j_m$ in F so that $j_0 = j$ and $s_i(U_j) \subset U_{j_i}$. Since $|F| = m$ we must have $j_a = j_b = f$ for some $a < b$ and some $\ell \in F$. Then $s^* = s_b s_a^{-1} \in S$ and $s^*(\ell) = \ell$. Hence from above $s^*(U_\ell)$ is strictly contained in U_ℓ. It follows that $s(U_j)$ is strictly contained in U_{j_m}. □

The preceding proof shows that we may take $\mu = m$, but in a given example a much smaller μ may suffice.

Fix a generator $h = h_i$ of S. Suppose $j, k \in F$ and $h(U_j) \subset U_k$. As h is holomorphic, a theorem of Pick shows that h cannot increase hyperbolic distance. That is $d_k(h(z), h(w)) \leq d_j(z, w)$ for all $z, w \in U_j$, or, more concisely, $h^*d_k \leq d_j$. Moreover Pick shows that if $h(U_j)$ is *strictly* contained in U_k then h is uniformly contracting on each compact set $X \subset U_j$, so there is a contraction constant $c = c(X, i, j) < 1$ such that $h^*d_k \leq cd_j$ as distance functions on X ([22], Theorem 2.11).

Let $c(K) < 1$ be the maximum of $c(K \cap U_j, i, j)$ over all i and j such that $h_i(U_j)$ is strictly contained in some U_k (note that this does not necessarily happen for all i, j). If $j, k \in F$ with $h_i(U_j)$ strictly contained in U_k then $h_i^*d_k \leq c(K)d_j$ as distance functions on $K \cap U_j$. By our choice of μ, we see that for every $s = h_{j_\mu}...h_{j_1} \in S$ expressed as a product of μ generators and for every $j \in F$ we have $s^*d_k \leq c(K)d_j$ as distance functions on $K \cap U_j$, where $k = k(s, j)$. By our choice of λ, we find $\delta_{l+\mu} \leq c(K)\delta_l$ as distance functions on K for all $l \geq \lambda$.

Let $a = a(K) = c(K)^{1/\mu} < 1$. Define the distance function d on U_K by

$$d = a^{\mu-1}\delta_\lambda + a^{\mu-2}\delta_{\lambda+1} + ... + a\delta_{\lambda+\mu-2} + \delta_{\lambda+\mu-1}. \tag{5.2}$$

Using (5.1) together with $\delta_{\lambda+\mu} \leq c(K)\delta_\lambda = a^\mu \delta_\lambda$ on K, we see that on K we have

$$\begin{aligned} h_j^* d &= h_j^*(a^{\mu-1}\delta_\lambda + a^{\mu-2}\delta_{\lambda+1} + ... + a\delta_{\lambda+\mu-2} + \delta_{\lambda+\mu-1}) \\ &\leq a^{\mu-1}\delta_{\lambda+1} + a^{\mu-2}\delta_{\lambda+2} + ... + a\delta_{\lambda+\mu-1} + \delta_{\lambda+\mu} \\ &\leq a^{\mu-1}\delta_{\lambda+1} + a^{\mu-2}\delta_{\lambda+2} + ... + a\delta_{\lambda+\mu-1} + a^\mu\delta_\lambda = ad. \end{aligned}$$

As $a < 1$, S uniformly contracts (K, d). Now Proposition 4.1 shows that $\mathcal{A}(K, S) = J'(S)$.

It only remains to check that the attractor $J'(S)$ is thick. Let $K^* \subset U_K$ be the compact unit neighborhood of K in the hyperbolic metric, that is, $z \in K^*$ when $z \in U_K$ and $d_K(z, w) \leq 1$ for some $w \in K$. By the theorem of Pick K^* is forward invariant under S, and so we may repeat the above argument, but with K replaced by K^*, to show that S uniformly contracts (K^*, d). But Sink $S \subset K$ lies in the interior of K^* so Sink S is thick. □

Note that the preceding proof simplifies greatly when $m = 1$, that is when K lies in a single component $V \subset U$ or, equivalently, when K' does not separate K in $\widehat{\mathbb{C}}$. In this case S uniformly contracts (K, d_V) and so the last two lemmas are not needed at all. In Example 6.2 we show that $K' = J(S)$ may separate $K = J'(S)$, so $J'(S) \subset F(S)$ is a thick attractor that meets two or more components of $F(S)$. Hence the above generality is required for our purposes.

Let S be a finitely generated Möbius semigroup. We define an *S-block* to be a nontrivial compact set $N \subset \widehat{\mathbb{C}}$ whose interior contains $S(N)$ (by nontrivial we mean that neither N nor its complement is empty).

Example 5.6. *Recall from the introduction that our motivation is to study $S_\beta = \prec f_\beta, g_\beta \succ$ where $f_\beta(z) = \beta + 1/z$ and $g_\beta(z) = -\beta + 1/z$ (see also Section 7.1). We now give a block for S_β when $|\Re\beta - \Im\beta| > 2$. Let A be the segment from 1 to $-i$ and let*[14] *$E = A \cup 1/A \cup -A \cup -1/A$, so E is a convex curve in $\mathbb{C}$. By construction, $E = 1/E$. Let N be the compact region in $\widehat{\mathbb{C}}$ containing ∞ and bounded by E. Then $1/N \pm \beta$ is the bounded region with boundary $E \pm \beta$ and so N is a block for S_β. The set $\{\overline{z} | z \in N\}$ is a block for S_β when $|\Re\beta + \Im\beta| > 2$. Combining these two cases, we see that S_β has a block whenever $|\Re\beta| + |\Im\beta| > 2$.*

The term "block" is suggested by the "isolating blocks" which Conley used to study invariant sets of flows. Conley's work was based on Smale's pioneering study of hyperbolic attractors for smooth flows, in which forward invariant compact regions played a prominent role.

Blocks give a practical way to detect thick attractors.

Theorem 5.7. *For a finitely generated Möbius semigroup S the following are equivalent:*
1) S has a thick attractor $J'(S)$
2) there is an S-block
3) S^{-1} has a thick attractor $J(S)$
4) there is an S^{-1}-block
5) $J'(S)$ is disjoint from $J(S)$ and the generators of S are loxodromic.

Proof. We show 1) implies 2). Let S be a CIFS on (K, d) whose attractor $\mathcal{A}$ lies in Int K. Let U be the open ϵ-neighborhood of $\mathcal{A}$ in (K, d), where ϵ is chosen small enough that U lies in Int K, and let $N = \overline{U}$. Then since S uniformly contracts (K, d) and N lies in the closed ϵ-neighborhood of the S-invariant set $\mathcal{A}$, $S(N) \subset U$. Thus N is an S-block.

Now suppose 2) holds. Let N be an S-block and U its interior. Since N is compact and $\widehat{\mathbb{C}} \setminus N$ is nonempty, S is a normal family on U by Montel's Theorem. Thus $U \subset F(S)$. Since N is forward invariant under S, it is backward invariant under S^{-1}, which implies $J'(S) = J(S^{-1}) \subset N$ by Lemma 2.18. Since $S(N)$ is also an S-block, $J'(S) \subset S(N) \subset U \subset F(S)$ and so $J(S)$ is disjoint from $J'(S)$. Let $s \in S$. Since $s(N) \subset U$, we see that s is neither elliptic nor the identity. The map s is also not parabolic since its fixed point would have to be in both $J(s) \subset J(S)$ and $J(s^{-1}) \subset J'(S)$. Thus s is loxodromic and 5) holds.

Now suppose 5) holds. Letting $K = J'(S)$ and $K' = J(S)$ in Theorem 5.1, we see that if $|J'(S) \cup J(S)| \geq 3$, then 1) must follow. When $|J'(S)| = 1 = |J(S)|$ we quickly see from the discussion immediately following Example 5.2 that using $K = J'(S)$ and $K' = J(S)$ we must have $J'(S)$ is a thick attractor for S. Thus 1), 2), and 5) are equivalent, which by replacing S by S^{-1} quickly shows that 5) is equivalent to 3) and 4). □

Remark 5.8. *We can see directly that 2) implies 4). Let N be an S-block with interior U and let $N' = \widehat{\mathbb{C}} \setminus U$. Then N' is an S^{-1}-block since for $h \in S$ we have $N \subset h^{-1}(U)$ so $h^{-1}(N') \subset \widehat{\mathbb{C}} \setminus N \subset$ Int N'.*

Remark 5.9. *The existence of a block for S restricts the dynamics of nearby semigroups. For suppose N is an S-block and let V be an open set such that $S(N) \subset V$ and $\overline{V} \subset$ Int N. The Möbius transformations $m(z)$ with $m(N) \subset V$ define an open neighborhood Σ of S in $\mathcal{M}$. Σ is a Möbius semigroup and N is a Σ-block. For any semigroup $S^* \subset \Sigma$ it follows that*

[14]For a set $X \subset \widehat{\mathbb{C}}$ we use the notation $-X = \{-z : z \in X\}$, $X^{-1} = 1/X = \{1/z : z \in X\}$, $X + \alpha = \{z + \alpha : z \in X\}$ and $\alpha X = \{\alpha z : z \in X\}$, for any $\alpha \in \mathbb{C}$.

N is also an S^*-block. This implies that in a continuously parametrized family of Möbius semigroups, the condition that N be a block defines an open set of parameters. Letting N vary, we see that the existence of a block also defines an open set of parameters. In this sense, blocks (and hence thick attractors) are a stable phenomenon.

For a fixed finitely generated Möbius semigroup S, an S-block N can be used to calculate Julia sets and Fatou sets. We define N^{-n} for $n > 0$ as the set of all points $z \in \widehat{\mathbb{C}}$ such that $s(z) \in N$ for all $s \in S$ that are expressible as a product of n generators. Clearly N^{-n} is compact. N^{-n} is nontrivial as it contains the sink but not the source of any generator of S. We also define $N^0 = N$ and $N^n = S^n(N)$ for $n > 0$. Then for all integers n, $S(N^n) \subset N^{n+1}$ and N^{n+1} lies in the interior of N^n.

We now describe the limiting behavior of these nested compact sets N^n.

Proposition 5.10. *Let N be an S-block, where S is a finitely generated Möbius semigroup. Then $J'(S) = \cap\, N^n$ and $F(S) = \cup\, N^n$, $n \in \mathbb{Z}$.*

Proof. Let K be the intersection of all the N^n and let $\mathcal{O}$ be the union of all the N^n. Then $\mathcal{O}$ is open, since it is the union of all the Int N^n. We let $K' = \widehat{\mathbb{C}} - \mathcal{O}$. Since each N^n is strictly contained in $\widehat{\mathbb{C}}$ and the sequence Int N^n is decreasing, K' is compact and nonempty. Clearly K' and K are disjoint. Since each N^n is S-invariant, K is S-invariant and K' is S^{-1}-invariant.

Moreover $S(K) = K$. For if $x \in K$ then for each m we have $x \in s_1...s_m(N)$ with each s_i a generator of S. By the pigeonhole principle, some generator must occur infinitely often as s_1, which implies that $x \in s_1(K)$.

By Theorem 4.2 we see that $K = \mathcal{A}(S, N)$. By Theorem 5.1, with K replaced by N, we see that $\mathcal{A}(S, N) = J'(S)$ and so $K = J'(S)$.

Likewise $S'(K') = K'$ and $K' = \mathcal{A}(S', N') = J(S)$, where N' is the block of Remark 5.8. □

Remark 5.11. *For S as in Proposition 5.10, the dynamics of S on $F(S)$ becomes trivial when we ignore the dynamics of S on $J'(S)$. More precisely, let $X = F(S)/J'(S)$ with the quotient topology and denote the singular point $J'(S)/J'(S) \in X$ by p. X is locally compact and metrizable. S acts naturally on X since both $F(S)$ and $J'(S)$ are forward invariant. Fix an integer n. The sets $S^k(N^n)/J'(S) \subset X, k = 1, 2, 3, ...$ are nested neighborhoods of p. Since $S^k(N^n) \subset S^{k+n}$, the intersection of these neighborhoods is p. Thus the family of mappings $s : N^n/J'(S) \to N^n/J'(S)$, $s \in S$, converges uniformly to the constant map p. But each compact $K \subset F(S)$ is contained in Int N^n for some n, since these open sets form a nested family whose union is $F(S)$. Thus the family of mappings $s : X \to X$, $s \in S$, converges to the constant map p, uniformly on each compact subset of X.*

On the other hand, Theorem 2.4 shows that any open set that meets $J(S)$ contains a source of some $s \in S$, whose S-orbit is not attracted to $J'(S)$. In this sense, $F(S)$ is the basin of the attractor $J'(S)$.

In fact $J'(S)$ can be calculated from the orbit of any Fatou point z, as follows by setting $Z = \{z\}$ in this result.

Corollary 5.12. *Let S be a Möbius semigroup generated by a finite number of loxodromic maps such that $J'(S) \subset F(S)$. For each compact subset $Z \subset F(S)$ the sequence $S^n(Z)$ converges to $J'(S)$ in the Hausdorff metric on compact subsets of $\widehat{\mathbb{C}}$.*

Proof. By Theorem 5.7, there is an S-block N. By Proposition 5.10, $Z \subset N^n$, for some n, and so $S^k(Z) \subset N^{n+k}$ for all $k > 0$. By Theorem 5.1, there is a compact neighborhood K of $J'(S)$ that is uniformly contracted by S. We can choose k so that Int K contains N^{n+k}. Now apply Theorem 4.2 with $D = S^k(Z) \subset K$. We see that $S^n(Z),\ n \geq k$, converges to $J'(S)$ in the Hausdorff metric on compact subsets of (K, d). Let d_S denote the spherical distance function on $\widehat{\mathbb{C}}$. Since d_S and d define the same topology on K, we also have convergence in the Hausdorff metric on compact subsets of (K, d_S). Thus $S^n(Z),\ n \geq 0$, converges to $J'(S)$ in the Hausdorff metric on compact subsets of $(\widehat{\mathbb{C}}, d_S)$. □

Remark 5.13. *To conclude, we note that the existence of a block implies that the long term states of a randomly generated orbit must accumulate on $J'(S)$, provided the initial condition is in $F(S)$. Moreover the ω-limit set of such a random orbit is all of $J'(S)$ (with probability one). Hence, plotting the points of such a random orbit (and discarding the first 50 or so as transient) approximates points in $J'(S)$.*

6. Some Julia sets of Möbius semigroups

In this section we illustrate various possible properties of the Julia set of a finitely generated non-elementary Möbius semigroup. We particulary present examples that contrast these Julia sets with the Julia sets of rational maps of degree ≥ 2 or with the limit sets of Möbius groups.

We first show that a Julia set can have interior without being all of $\widehat{\mathbb{C}}$, contrary to the case for limit sets (Corollary 3.7) and for Julia sets of rational functions of degree greater than or equal to two ([22], p. 58).

Example 6.1 (F neither empty nor dense)**.** *Define S to be the semigroup generated by $h_{\pm}(z) = (\pm b - iz)/a$, where $0 < a < 1$ and $b = 1 - a^2$. Then $S^{-1} = \prec iaz \pm ib \succ$ is a CIFS on K for any large disc K with the Euclidean metric, as $|ia| < 1$. Let R be the closed rectangular region with corners $\pm a \pm i$. Then $h_+^{-1}(R) \cup h_-^{-1}(R) = R$ if $a \geq 1/\sqrt{2}$. For such a, Theorem 4.2 implies that $\mathcal{A}(S^{-1}) = R$. By Proposition 4.1 we see that $J(S) = R$. Thus $J(S) \neq \widehat{\mathbb{C}}$ and Int $J(S)$ is nonempty, as desired.*

Furthermore, $J'(S) = \{\infty\}$ since S^{-1} is normal on $\mathbb{C}$, and the group G generated by S has $\Lambda(G) = \widehat{\mathbb{C}}$, by Corollary 3.7, since $R = J(S) \subset \Lambda(G)$.

Corollary 7.22 (with $\beta^2 \notin \mathbb{R}$ and G_β not discrete) gives many 2-generator Möbius semigroups whose Julia sets equal $\widehat{\mathbb{C}}$.

We now show that $J(S)$ may separate $J'(S)$, meaning that $J'(S)$ may lie in the Fatou set $F(S)$ but meet more than one such Fatou component.

Example 6.2 (J separates J')**.** *Choose a closed triangular region $T \subset \mathbb{C}$ such that the open triangle U whose vertices are the midpoints of T contains 0. Label the vertices of T as c_i and consider the doubling dilation $h_i(z) = 2z - c_i$ centered at c_i for $i = 1, 2, 3$. Let N be a closed r-neighborhood of $T \backslash U$ for r positive and small, so that $0 \notin N$ but $Int(h_i(N))$ contains N for all i. Let S_T be the expanding linear semigroup generated by $h_i,\ i = 1, 2, 3$. Then $J'(S_T) = \{\infty\}$ and, since T is backward invariant under S_T, $J(S_T) \subset T$ is bounded. Indeed, as in Example 4.4 we see that $J(S_T) \subset N$ is a Sierpinski triangle and U is a bounded component of $F(S_T)$. Thus $J(S_T)$ separates 0 from ∞.*

Let $h(z) = b/(z-a)$, so $h^{-1}(z) = a + b/z$. Choose b very small so that $h^{-1}(N)$ is small and then choose a so that $h^{-1}(N)$ is contained in Int N. Let S be the Möbius semigroup generated by h and S_T. Then $J'(S_T) \subset J'(S)$ so $\infty \in J'(S)$ and $0 = h(\infty) \in J'(S)$.

By construction Int N contains $S^{-1}(N)$ and so N is an S^{-1}-block. By Theorem 5.7 $J'(S) \subset F(S)$. Since $J(S)$ contains $J(S_T)$ it follows that $J(S)$ separates the points 0 and ∞ of $J'(S)$.

By adding k appropriate loxodromic generators to S_T, with sources in distinct components of $F(S_T)$, we can likewise construct a semigroup S as in Theorem 5.1 in which $J'(S)$ meets more than k Fatou components.

Recall from Corollary 3.12 that a disconnected limit set with three or more points has uncountably many components and that the same holds for the Julia set of a rational map of degree ≥ 2. The next two examples show that the situation is not so simple for Möbius semigroups.

Example 6.3 (J with $n > 1$ components)**.** *Let J be the union of $n > 1$ disjoint closed line segments $J_1, J_2, \ldots, J_n$ in $\mathbb{R}$. Choose m linear maps a_j such thavt $\cup_{j=1}^m a_j(J) = [0,1]$ (depending on the number and configuration of the J_k this may require a fair number of maps a_j, but it can always be done). Now, for each $k = 1, \ldots, n$, let b_k be a linear map whose image of $[0,1]$ is exactly J_k. Let $S = \prec b_k \circ a_j : j = 1, \ldots, m; k = 1, \ldots, n \succ$ and note that the generators are all linear contractions on $\mathbb{C}$ since each $b_k \circ a_j$ maps J_k strictly inside itself. Since $J = \cup_{k=1}^n b_k([0,1]) = \cup_{k=1}^n b_k(\cup_{j=1}^m a_j(J)) = \cup_{k=1}^n \cup_{j=1}^m b_k \circ a_j(J)$, we have by Theorem 4.2 that J is the attractor set of the contracting iterated function system S. By Proposition 4.1, $J(S^{-1}) = J$ has exactly n components.*

In this example $J(S) = \{\infty\}$ since S is normal on $\mathbb{C}$. We can also calculate $\Lambda(G)$ for the group G generated by S. $J = J(S^{-1}) \subset \Lambda(G)$ and Theorem 2.4 implies that some $h \in S^{-1}$ has a repelling fixed point in J_1 that is not an endpoint of J_1. Forward invariance shows that $\mathbb{R} = \cup_{j=1}^\infty h^j(J) \subset \Lambda(G)$. Since $\widehat{\mathbb{R}}$ is completely invariant under G, however, Theorem 3.9 implies that $\Lambda(G) = \widehat{\mathbb{R}}$.

We note that Example 6.3 can be adjusted to generate a Julia set with exactly two components using 4 generators by using $J_1 = [2, 2.49]$, $J_2 = [2.51, 3], a_1(z) = z - 2$, and $a_2(z) = 0.1 + (z-2)/2$. This approach can also be adjusted to produce n components of $J(S)$ with $2n$ generators (just select intervals J_k to be subintervals of [2, 3] with small gaps in between and use a_1 and a slight change to the constant term in a_2). However, also see Example 6.4.

For any $n \in \mathbb{N}$, there are examples of polynomial semigroups whose Julia set has exactly n components (such a semigroup may also be chosen so all its elements have degree ≥ 2 and so that its planar postcritical set is bounded). In the first such examples, the semigroup was generated by $2n$ elements [38]. However, 4 generators are sufficient for any n [33].

The following example shows for any $n \in \mathbb{N}$, there is a 5 generator Möbius semigroup whose Julia set has exactly n components. Likewise 5 generators can produce a countably infinite number of Julia components.

Example 6.4 (J with $\aleph_0$ components)**.** *Let $J_1 = [2, 2.49]$ and $J_2 = [2.51, 3]$ and label the gap between them $H = (2.49, 2.51)$. Now consider the linear maps (as in Example 6.3) chosen so that $b_1([0,1]) = J_1, b_2([0,1]) = J_2, a_1(z) = z - 2$, and $a_2(z) = 0.1 + (z-2)/2$. Thus for $S = \prec b_k \circ a_j \succ$ we have $\mathcal{A}(S) = J_1 \cup J_2$.*

Set $h(z) = 0.02(z - 2.49) + 2.49$ and note that $h(3) < 2.51$. Thus $J_2, h(J_2), h^2(J_2), \ldots$ are all disjoint intervals. Let $S^* = \prec h, b_k \circ a_j \succ$. We claim that $J^* = J_1 \cup J_2 \cup \cup_{n=1}^{\infty} h^n(J_2)$, which has exactly $\aleph_0$ components, is the attractor set of S^*.

Since each map a_j is linear and maps both J_1 and J_2 into $[0,1]$, each map a_j must also map the gap H into $[0,1]$. Hence each composite map $b_k \circ a_j$ not only maps $J = J_1 \cup J_2$ into itself, it also maps H into J. Since $\cup_{n=1}^{\infty} h^n(J_2) \subset H$ and $\mathcal{A}(S) = J_1 \cup J_2$ is forward invariant under S, we then have that J^* is forward invariant under each $b_k \circ a_j$. Clearly from the definition of h we see that J^* is also forward invariant under h. Thus $\mathcal{A}(S^*) \subset J^*$. Since $S \subset S^*$ implies $J_1 \cup J_2 = \mathcal{A}(S) \subset \mathcal{A}(S^*)$ and $\mathcal{A}(S^*)$ is forward invariant under h, we then see that $J^* \subset \mathcal{A}(S^*)$. Hence $\mathcal{A}(S^*) = J^*$. Thus, by Proposition 4.1, we see that the inverse semigroup of S^* has Julia set equal to J^*.

We again note that, as in Example 6.3, $J(S^*) = \{\infty\}$ and $\Lambda(G) = \widehat{\mathbb{R}}$, where G is the group generated by S^*.

If the above map h is chosen to instead have its sink in the open interval $(2, 2.49)$, then some $h^k(J_2)$ would meet J_1 and thus the attractor set J^* would have a finite number of components. One could then assure that we have exactly n components of J^* by adjusting h carefully. This constructs a Julia set with exactly n components out of only 5 Möbius generators.

7. Caruso semigroups

The previous sections have set up a framework for the study of Möbius semigroups. In this section, we explore the dynamics of a particular family of Möbius semigroups S_β, $\beta \in \mathbb{C}^*$. We study the topological properties of the Julia set of S_β and the limit set of the group G_β it generates. We will especially examine the extreme cases where the Julia sets of S_β and S_β^{-1} are equal or disjoint.

7.1. **Preliminaries.** For a fixed nonzero complex number β we let $f(z) = \beta + 1/z$ and $g(z) = -\beta + 1/z$. We define the Möbius group $G_\beta = \langle f, g \rangle$. the semigroup $S_\beta = \prec f, g \succ$, and its inverse semigroup $S_\beta^{-1} = \prec f^{-1}, g^{-1} \succ$.

Note that $f^{-1}(z) = 1/(z - \beta)$ and $g^{-1}(z) = 1/(z + \beta)$. It follows that $fg^{-1}(z) = z + 2\beta$. Thus the only point in $H^3 \cup \widehat{\mathbb{C}}$ with a finite orbit under fg^{-1} is $\infty \in \widehat{\mathbb{C}}$. As $f(\infty) \neq \infty$, G_β has no finite orbit in $H^3 \cup \widehat{\mathbb{C}}$. By definition, this shows that G_β is a non-elementary Möbius group.

As S_β (respectively $S_\beta{}'$) generates G_β, it has no finite orbit in $H^3 \cup \widehat{\mathbb{C}}$. Corollary 2.22 now shows that S_β (respectively $S_\beta{}'$) is non-elementary.

We denote the Julia set of S_β by $J_\beta = J(S_\beta)$, the Julia set of S_β^{-1} by $J'_\beta = J(S_\beta^{-1})$, and the limit set of G_β by $\Lambda_\beta = \Lambda(G_\beta)$. Similarly, we denote the Fatou set of S_β by $F_\beta = F(S_\beta)$, the Fatou set of S_β^{-1} by $F'_\beta = F(S_\beta^{-1})$ and the stable set of G_β by $\Omega_\beta = \Omega(G_\beta)$.

From Remark 2.20 and Theorem 2.4 we obtain the following.

Proposition 7.1. *For each $\beta \in \mathbb{C}^*$, S_β and S_β^{-1} are non-elementary and G_β is a non-elementary Möbius group. The sources of S_β, S_β^{-1} and G_β are dense in J_β, J'_β and Λ_β, respectively. Each of these sets is perfect and uncountable.*

We now introduce various transformations. We define three involutions[15] of $\widehat{\mathbb{C}}$, $m(z) = -z$, $r(z) = 1/z$, and $k(z) = \overline{z}$, where m, r, and k denote *minus, reciprocal*, and *conjugate*, respectively. We let $i(z) = iz$, so $i^2(z) = m(z)$. We define the *translation* $t(z) = t_\beta(z) = z+\beta$ and let $u = u_\beta = rtr$, so t and u are parabolic and $u(z) = z/(\beta z + 1)$ fixes the origin. Note that $f = tr$ and $g = t^{-1}r$.

We will need the following Möbius groups later on. Define $H_\beta = \langle t, r\rangle$ so G_β is a subgroup of H_β of index at most two. Let $P_\beta = \langle t, u\rangle$ so P_β is a parabolically generated subgroup of H_β of index at most two. A superscript + will be used to denote the groups obtained by adjoining m, so ${G_\beta}^+ = \langle f, g, m\rangle$, $H_\beta^+ = \langle t, r, m\rangle$ and $P_\beta^+ = \langle t, u, m\rangle$. All these groups are commensurable, so each has limit set Λ_β and ordinary set Ω_β and each is discrete when any is discrete.

Define a *Koebe group* to be a 2-generator non-elementary Möbius group such that the commutator of the generators is parabolic and one generator has order two (see [41] Fig. 8, or the groups $G_2[\tau]$ in [25]). G_β^+ is Koebe since it is generated by f and m, m has order 2, and $fmf^{-1}m^{-1} = fg^{-1}$ is parabolic.

Remark 7.2. *Conversely, every Koebe group is conjugate to G_β^+ for some nonzero β. For one may assume that the order two generator is m and that the other generator is $h(z) = (az+b)/(cz+d)$ where $bc \neq 0$ and where at least one of a, d is nonzero (since m, h generate a non-elementary Möbius group). In order that $hmh^{-1}m^{-1}$ be parabolic one must have $ad = 0$ and so h is conjugate to an f_β by a Möbius transformation kz or k/z that commutes with m.*

We represent f and g by the matrices $M(\beta)$ and $M(-\beta)$ where

$$M(\beta) = \begin{pmatrix} \beta & 1 \\ 1 & 0 \end{pmatrix}$$

and use them to calculate $tr^2[f] = tr^2[g] = -\beta^2$. Thus by Theorem 3.1 f and g have the same type and are classified as a function of the parameter β. Specifically, f and g (as well as f^{-1} and g^{-1}) are

- *parabolic* if and only if $\beta = \pm 2i$,
- *elliptic* if and only if β is purely imaginary with $|\beta| < 2$,
- *hyperbolic* if and only if β is purely imaginary with $|\beta| > 2$,
- *strictly loxodromic* for any other value of β.

The study of J_β, J'_β and Λ_β is simplified by noticing that the functions f, g and their inverses are conjugate to each other. This holds as two Möbius transformations u, v, neither the identity, are conjugate if and only if $tr^2[u] = tr^2[v]$ ([2],p.66). Through simple computations one can make these conjugacies explicit.

Proposition 7.3. *We have the following conjugacies:*

(1) $f \circ m = m \circ g$,
(2) $f \circ r = r \circ g^{-1}$, *and*
(3) $f \circ mr = mr \circ f^{-1}$.

We summarize these results (along with some direct consequences) in Figure 2 where we write $w \leftarrow h \rightarrow v$ if w and v are conjugate via an involution h.

[15] A map h is an involution if $h^2 = Id$, i.e., $h = h^{-1}$.

$$\begin{tikzcd}
f \arrow[rr, leftrightarrow, "-z"] \arrow[dd, leftrightarrow, "1/z"'] & & g \arrow[dd, leftrightarrow, "1/z"] \\
& -1/z \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\
g^{-1} \arrow[rr, leftrightarrow, "-z"] & & f^{-1}
\end{tikzcd}$$

FIGURE 2. This diagram shows the conjugacies between the elements f, g, f^{-1} and g^{-1} under the involutions $1/z, -z$ and $-1/z$.

Proposition 7.3 describes some of the connections between the elements of S_β, S_β^{-1}, and G_β. Together with Remark 2.1, it explains a large amount of symmetry in the structure of J_β, J'_β and Λ_β, which we state as follows. See Figure 3.

Theorem 7.4. *J_β, J'_β, and Λ_β satisfy the following properties*[16]*:*

(1) $J_\beta = -J_\beta$ *and* $J'_\beta = -J'_\beta$
(2) $J_\beta = 1/J'_\beta$
(3) $J_\beta = f^{-1}(J_\beta) \cup g^{-1}(J_\beta) = (J_\beta - \beta)^{-1} \cup (J_\beta + \beta)^{-1}$
(4) $J'_\beta = f(J'_\beta) \cup g(J'_\beta) = (J_\beta + \beta) \cup (J_\beta - \beta)$
(5) $\Lambda_\beta = -\Lambda_\beta$
(6) $\Lambda_\beta = 1/\Lambda_\beta$
(7) $\Lambda_\beta = \Lambda_\beta + \beta$

Proof. Proposition 7.3(1) shows that $S_\beta = mS_\beta m^{-1}$ and so by Remark 2.1, we conclude $J_\beta = m(J_\beta) = -J_\beta$. Similarly, the rest of (1) and (2), (5) and (6) can be shown. Parts (3) and (4) are an application of backward self-similarity (Proposition 2.10) to the semigroups S_β and S_β^{-1} (for the second equality in (4) we also made use of (2)). Part (7) uses (6) and Proposition 2.7 to compute $\Lambda_\beta = f(\Lambda_\beta) = 1/\Lambda_\beta + \beta = \Lambda_\beta + \beta$. $\square$

Corollary 7.5. *For all values of $\beta \neq 0$, we have $0, \infty \in \Lambda_\beta$. Also, if 0 or ∞ lie in J_β or J'_β then both 0 and ∞ lie in both J_β and J'_β.*

Note that by Theorems 8.1(3) and 8.2(3), it is possible for $J_\beta \cap J'_\beta \neq \emptyset$ without containing ∞.

Proof. Parts 6 and 7 of Theorem 7.4 imply $0, \infty \in \Lambda_\beta$. Part 4 of Theorem 7.4 shows that $\infty \in J_\beta$ if and only if $\infty \in J'_\beta$. Part 2 of Theorem 7.4 shows both $0 \in J_\beta$ if and only if $\infty \in J'_\beta$, and $\infty \in J_\beta$ if and only if $0 \in J'_\beta$. The result then follows. $\square$

The relations so far encountered do not exhaust the set of symmetries of S_β, S_β^{-1} and G_β. We focus now on the symmetries that originate in the parameter β-plane with respect to the maps $m(z) = -z$, $k(z) = \overline{z}$, and $i(z) = iz$. For convenience in describing the symmetry

[16]Recall footnote 14.

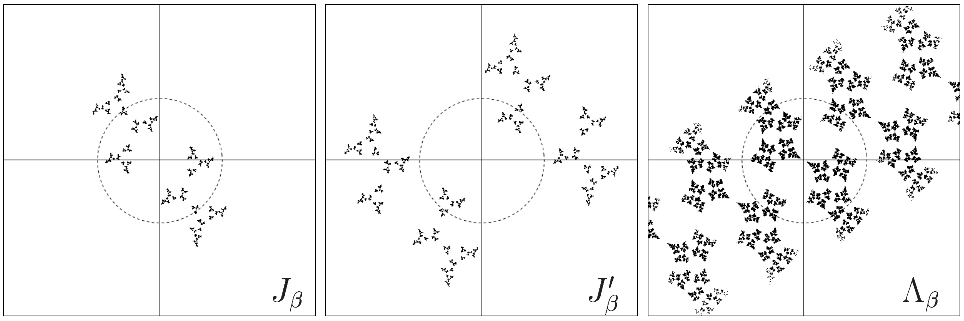


FIGURE 3. Symmetries and relations in dynamical plane. From left to right we have J_β, J'_β and Λ_β for $\beta = 1.3e^{i\pi/6}$. It is very easy to spot the symmetries described in Theorem 7.4. The unit circle (dashed) is shown for reference. These pictures were generated by choosing a seed at random, computing one million random iterates in its orbit and dropping the first hundred. See Remark 5.13.

under $i(z)$ we extend S_β to a semigroup ${S_\beta}^+ = \prec m, f_\beta, g_\beta \succ = \prec m, f_\beta \succ$, recalling that $mf_\beta m = g_\beta$.

Using some simple equations in $\mathcal{M}$ we can understand how S_β and G_β (or ${S_\beta}^+$ and G_β^+) transform under m, k, and i. In the following proposition we use subscripts to make explicit the dependence of the maps $f = f_\beta, g = g_\beta$ on the parameter β.

Proposition 7.6. *(1)* $f_{-\beta} = g_\beta$, $g_{-\beta} = f_\beta$, *so* $S_{-\beta} = S_\beta$, $G_{-\beta} = G_\beta$.
(2) $f_{\overline{\beta}} = kf_\beta k^{-1}$, $g_{\overline{\beta}} = kg_\beta k^{-1}$, *so* $S_{\overline{\beta}} = kS_\beta k^{-1}$, $G_{\overline{\beta}} = kG_\beta k^{-1}$.
(3)$g_{i\beta} = i(g_\beta m)i^{-1}, f_{i\beta} = i(f_\beta m)i^{-1}, m = imi^{-1}$, *so* $S_{i\beta}^+ = i{S_\beta}^+ i^{-1}$, $G_{i\beta}^+ = iG_\beta^+ i^{-1}$.

This proposition explains the behavior of the sets J_β and Λ_β under the symmetries m, k, i of the parameter β-plane.

Theorem 7.7. *The Julia set J_β and the limit set Λ_β have the following relations:*

(1) $J_\beta = J_{-\beta}$ *and* $\Lambda_\beta = \Lambda_{-\beta}$,
(2) $k(J_\beta) = J_{\bar\beta}$ *and* $k(\Lambda_\beta) = \Lambda_{\overline{\beta}}$,
(3) $iJ_\beta = J_{i\beta}$ *and* $i\Lambda_\beta = \Lambda_{i\beta}$.

Proof. By Proposition 7.6(1), we have $S_\beta = S_{-\beta}$ and $G_\beta = G_{-\beta}$ and so (1) follows.

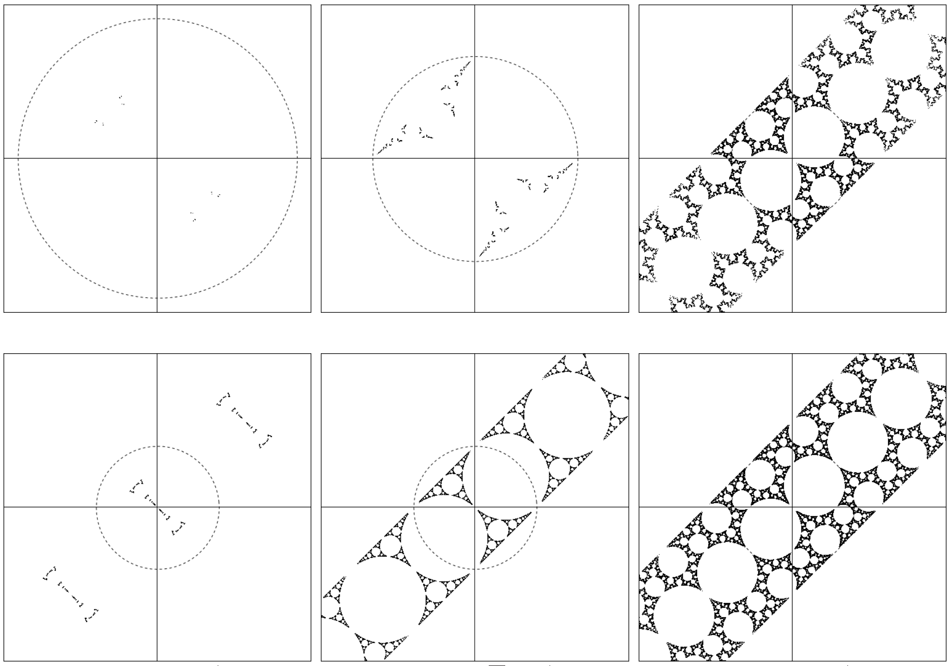
Proposition 7.6(2) shows that for $k(z) = \bar z$, we have $S_{\bar\beta} = kS_\beta k^{-1}$ and $G_{\bar\beta} = kG_\beta k^{-1}$ and so (2) follows from Remark 2.1.

Likewise Proposition 7.6(3) implies $J(S_{i\beta}^+) = iJ({S_\beta}^+)$ and $\Lambda(G_{i\beta}^+) = i\Lambda(G_\beta^+)$. Thus to prove (3) it suffices to show $J({S_\beta}^+) = J_\beta$ and $\Lambda(G_\beta^+) = \Lambda_\beta$.

Since $S_\beta = mS_\beta m^{-1}$ by Proposition 7.3(1), we have $m^{-1}(J_\beta) = J_\beta$ by Remark 2.1. Thus J_β is backward invariant by m and by S_β and so it is backward invariant by ${S_\beta}^+$. As S_β is non-elementary, its extension ${S_\beta}^+$ is likewise non-elementary. Thus Remark 2.20 applies to ${S_\beta}^+$ and implies that $J({S_\beta}^+) \subset J_\beta$. But the other inclusion is clear since $S_\beta \subset {S_\beta}^+$. Thus $J({S_\beta}^+) = J_\beta$.

In a similar way one shows that $\Lambda(G_\beta^+) = \Lambda_\beta$. □

The symmetries of J_β and Λ_β under the substitution of $i\beta$ or $\overline{\beta}$ for β allow us to focus on values of β in the sector $0 \le \arg\beta \le \pi/4$. Consider the two rays that bound this sector.



$\beta = 2e^{i\pi/4}$ $\beta = \sqrt{2}e^{i\pi/4}$ $\beta = e^{i\pi/4}$

FIGURE 4. Top row: the Julia sets J_β for different values of β with argument $\pi/4$ as in Theorem 8.2. On the left ($\beta = 2e^{i\pi/4}$), we barely see the Cantor set J_β inside the unit circle. In the middle ($\beta = \sqrt{2}e^{i\pi/4}$), the Cantor set J_β intersects the unit circle at the points $\pm 1, \pm i$. On the right ($\beta = e^{i\pi/4}$), we have a gasket Julia set that is both forward and backward invariant. Bottom row: the limit sets Λ_β for the same values of β. In the middle, we see the Apollonian gasket. These pictures were generated by choosing a seed at random, computing one million random iterates in its orbit and dropping the first hundred. See Remark 5.13.

When $\arg\beta = 0$, each set $J_\beta, J'_\beta, \Lambda_\beta$ is contained in the real line. When $\beta = re^{i\pi/4}$ with $r \in \mathbb{R}$, however, the diagonal line $\arg\theta = \pi/4$ is a line of symmetry of each of these sets. For instance, since $\beta = i\overline{\beta}$, Theorem 7.7 implies that $ik(J_\beta) = iJ_{\overline{\beta}} = J_{i\overline{\beta}} = J_\beta$. See Figure 4.

7.2. **Connected J_β.** In this section we investigate the consequences of a connected Julia set J_β. We already know from Lemma 3.3 that Λ_β must be connected when J_β is connected. We also have the following result.

Theorem 7.8. *Suppose J_β is connected. Then J_β meets J'_β.*

The proof is based on 2 lemmas, the first of which relates J'_β and F_β.

Lemma 7.9. *When $\infty \in F_\beta$, J'_β meets the unbounded component V of F_β. When $0 \in F_\beta$, J'_β meets $\overline{U}$ where U is the component of 0 in F_β*

Proof. Suppose $\infty \in F_\beta$. Then J_β is bounded, nonempty, and closed and so there is a point $z \in J_\beta$ of maximum modulus. By the triangle inequality, $2|z| \leq |z+\beta| + |z-\beta|$. Since the three points $z,\ z+\beta,\ z-\beta$ are distinct and collinear, they cannot all have the same modulus. Thus for some choice of sign we have $|z| < |z \pm \beta|$ and so $z \pm \beta \in V$. But since $z \in J_\beta$, Theorem 7.4(4) shows that $z \pm \beta \in J'_\beta$. This proves the first statement.

For the second statement, suppose $0 \in F_\beta$ and let U be the component of 0 in F_β. Since $J'_\beta = 1/J_\beta$, we see $\infty \in F'_\beta$. Let V' be the unbounded component of F'_β, and note that $U = 1/V'$. We now show J_β meets $\overline{V'}$, thus giving $J'_\beta = 1/J_\beta$ meets $\overline{U} = 1/\overline{V'}$ as desired. Under our assumption, J'_β is bounded, closed, and nonempty. We choose a point $w^* \in J'_\beta$ that maximizes $\Im(w/\beta) = \mathrm{dist}(w, \beta\mathbb{R})/|\beta|$ over all $w \in J'_\beta$. Hence J'_β is contained in the closed half plane H containing $\beta\mathbb{R}$ having boundary line $\beta\mathbb{R} + w^*$, and so the complement of H lies in V'. Then $w^*, w^* + \beta, w^* - \beta \in \beta\mathbb{R} + w^* \subset \overline{V'}$. But by Theorem 7.4(4), we can choose the sign so that $w^* \pm \beta$ lies in J_β as well. □

The second lemma is entirely topological.

Lemma 7.10. *Let K be a compact nonempty connected subset of $\mathbb{C}^*$. If $-K = K$ then K separates 0 from ∞ in $\widehat{\mathbb{C}}$.*

Proof. Let W be the open ϵ-neighborhood of K in $\mathbb{C}$, where ϵ is less than the distance from K to the origin. We first show that W separates 0 from ∞ in $\widehat{\mathbb{C}}$.

Clearly W is open, $-W = W$, and W is connected and nonempty. Thus we may choose a smooth path $\gamma(t) \in W$, $0 \le t \le 1$, such that $\gamma(1) = -\gamma(0)$. The variation in $\arg(\gamma(t))$ between $t = 0$ and $t = 1$ is $n\pi$, for some odd integer n. Concatenating $\gamma(t)$ with the path[17] $-\gamma(t)$ gives a closed loop $\delta(t) \in W$, where $\delta(t) = \gamma(2t)$ for $0 \le t \le 1/2$ and $\delta(t) = -\gamma(2t-1)$ for $1/2 \le t \le 1$. The variation in $\arg(\delta(t))$ between $t = 0$ and $t = 1$ is $2n\pi$. Since $n \neq 0$ (as n is odd) each path $c(t)$ in $\widehat{\mathbb{C}}$ from 0 to ∞ must pass through W.

As $\epsilon \to 0$ we see, since K is compact, that $c(t)$ must also pass through K. □

From these lemmas we can derive the following.

Proposition 7.11. *For $\beta \in \mathbb{C}^*$, J'_β is not contained in any simply connected component of F_β.*

Proof. Suppose to the contrary that U is a simply connected component of F_β and $J'_\beta \subset U$. Then J'_β and J_β are disjoint so, by Corollary 7.5, 0 and ∞ lie in $F_\beta \cap F'_\beta$. Using both parts of Lemma 7.9, we see that 0 and ∞ lie in U.

Since $J'_\beta = -J'_\beta \subset -U$, U meets $-U$. Since $-F_\beta = F_\beta$, $-U$ is a component of F_β. Thus $-U = U$.

Note that the compact set $K = \widehat{\mathbb{C}} \backslash U$ is nonempty, since $J_\beta \subset K$. Since $-U = U$ we have $-K = K$. Moreover K is connected since U is simply connected. This contradicts Lemma 7.10. □

Proof of Theorem 7.8. Suppose to the contrary that J_β is connected and disjoint from J'_β. Then $J'_\beta \subset F_\beta$. By Theorem 7.4(2), $J'_\beta = 1/J_\beta$. Thus J'_β is connected so J'_β lies in some component U of F_β. Since J_β is connected, however, U must be simply connected. This contradicts Proposition 7.11. □

Theorems 5.1 and 7.8 and Corollary 7.5 give necessary conditions for the existence of a thick attractor.

[17] Note that in this context $-\gamma(t)$ do not denote the reverse path of $\gamma(t)$, but rather the reflection of γ through the origin.

Corollary 7.12. *If S_β has a thick attractor then J_β and J'_β are disjoint and each is disconnected and bounded.*

The disconnectedness in this corollary is special to the Caruso family of semigroups S_β. For instance, Example 6.1 gives a non-elementary finitely generated Möbius semigroup with a connected thick attractor.

7.3. **When does J_β equal Λ_β?** By Proposition 2.7, the Julia set of any rational semigroup S is backward invariant. Examples 3.10 and 6.1 show that the Julia set is not necessarily S-invariant. This lack of invariance occurs for many Caruso semigroups as well, including those considered in Example 5.6.

But when J_β is S_β-invariant, Propositions 3.15 and 7.1 show that $J_\beta = \Lambda_\beta$. In many cases this enables us to calculate J_β. Accordingly we will study conditions under which J_β is S_β-invariant. We begin with a special property of the family S_β.

Proposition 7.13. *For all values of β, if any two of the three sets J_β, J'_β, and Λ_β are equal, then $J_\beta = J'_\beta = \Lambda_\beta$.*

Proof. If $J_\beta = J'_\beta$ then J_β is S-invariant and so $J_\beta = \Lambda_\beta$. If $J_\beta = \Lambda_\beta$ then ${J_\beta}' = 1/J_\beta = 1/\Lambda_\beta = \Lambda_\beta$ by Theorem 7.4. Likewise $J'_\beta = \Lambda_\beta$ implies $J_\beta = \Lambda_\beta$. □

By Example 3.16 this proposition does not hold for Möbius semigroups in general. But we can even weaken the hypotheses of Proposition 7.13 with the same conclusion.

Proposition 7.14. *If $J'_\beta \subset J_\beta$ or $J_\beta \subset J'_\beta$, then $J'_\beta = J_\beta = \Lambda_\beta$.*

Proof. Suppose $J'_\beta \subset J_\beta$ and let $z \in J_\beta$. Then since $J_\beta = 1/J'_\beta$ by Theorem 7.4, we have $1/z \in J'_\beta \subset J_\beta$ by hypothesis. But again using $J_\beta = 1/J'_\beta$, we conclude that $z \in J'_\beta$. Thus $J_\beta = J'_\beta$, from which Proposition 7.13 gives the desired conclusion.

A similar argument applies when $J_\beta \subset J'_\beta$. □

We now give a different condition under which $J_\beta = \Lambda_\beta$.

Proposition 7.15. *If $Id \in \overline{S_\beta}$ then $\overline{S_\beta} = \overline{G_\beta}$ and $J_\beta = \Lambda_\beta$.*

Proof. Recall that by Proposition 7.3(1), the involution $m(z) = -z$ satisfies $mfm^{-1} = g$ and $mgm^{-1} = f$. Thus $mS_\beta m^{-1} = S_\beta$.

By assumption there is a sequence $s_j \in S_\beta$ such that $s_j \to Id$. We may assume that $s_j \neq f$ and $s_j \neq g$ for all j, since $f \neq Id \neq g$. Each s_j can be written as a word in f and g of length at least two. Using the symmetry provided by conjugation by m, we may suppose that each s_j begins with the letter f. We factor $s_j = fs'_j$ with $s'_j \in S$. As $s'_j \to f^{-1}$ we find that $f^{-1} \in \overline{S_\beta}$. Conjugating by m we see that $g^{-1} \in \overline{S_\beta}$ as well.

Since $\overline{S_\beta}$ is a semigroup that contains the semigroup generators f, g, f^{-1}, g^{-1} of G_β, we find $G_\beta \subset \overline{S_\beta}$. Thus $\overline{G_\beta} \subset \overline{S_\beta}$. As the opposite inclusion is trivial, our first conclusion holds.

It is clear that any family $\mathcal{F} \subset \mathcal{M}$ is normal on an open set $U \subset \widehat{\mathbb{C}}$ if and only if its closure is normal on U. Thus for any Möbius semigroup S, both S and $\overline{S}$ have the same Fatou set and hence the same Julia set.

This gives $J_\beta = J(S_\beta) = J(\overline{S_\beta}) = J(\overline{G_\beta}) = J(G_\beta) = \Lambda_\beta$. □

Remark 7.16. *By a similar proof, $J(S) = \Lambda(G)$ whenever S is a non-elementary Möbius semigroup that generates G and $\overline{S}$ contains a generating set for S^{-1}. The symmetries of S_β enabled us to simplify this condition on $\overline{S}$.*

Proposition 7.15 immediately gives the following.

Theorem 7.17. *If S_β contains an elliptic element then S_β is dense in G_β and $J_\beta = \Lambda_\beta$.*

Example 7.18. *The word f has $tr^2[f] = -\beta^2$. For $\beta = ix$, $x \in (-2,2)$, we see $0 \leq tr^2[f] < 4$ and so the preceding theorem gives $J_\beta = \Lambda_\beta$. By the symmetry in Theorem 7.7(3), we also have $J_\beta = \Lambda_\beta$ when $\beta \in (-2,2)$.*

Example 7.19. *The word f^2g^2 has $tr^2[f^2g^2] = (2+\beta^4)^2$. If $\beta = (1 \pm i)x$ with $-1 < x < 1$, then f^2g^2 is elliptic, and again $J_\beta = \Lambda_\beta$.*

For instance, $\beta = \sqrt{i}$ fits into this last family of examples with $x = 1/\sqrt{2}$. The corresponding set $J_\beta = \Lambda_\beta$ is the gasket presented in Figure 4.

7.4. **The closure of G_β.** Whenever G_β is not discrete we will compute $\overline{G_\beta} \subset \mathcal{M}$ and use this to calculate the limit set Λ_β. We recall that $\mathcal{M}$ is a Lie group of real dimension 6. It is also connected. $\overline{G_\beta}$ is a closed subgroup of $\mathcal{M}$ and hence a Lie subgroup of some dimension d_β.

We define $\mathcal{M}_{\mathbb{R}}$, the *real Möbius group*, to be the stabilizer in $\mathcal{M}$ of the real circle $\widehat{\mathbb{R}} \subset \widehat{\mathbb{C}}$. This is the 3-dimensional subgroup of $\mathcal{M}$ consisting of transformations $m(z) = (az+b)/(cz+d)$ with real a, b, c, d. When β is real, both G_β and its closure lie in $\mathcal{M}_{\mathbb{R}}$. $\mathcal{M}_{\mathbb{R}}$ is a Lie subgroup of $\mathcal{M}$ of dimension 3 with two connected components, distinguished by the sign of the nonzero real number $ad - bc$.

Likewise we define $\mathcal{M}_{i\mathbb{R}}$, the *imaginary Möbius group*, to the the stabilizer of the imaginary circle $i\widehat{\mathbb{R}} \subset \widehat{\mathbb{C}}$, so $\mathcal{M}_{i\mathbb{R}} = i\mathcal{M}_{\mathbb{R}}i^{-1}$ where $i(z) = iz$. This is also a Lie subgroup of $\mathcal{M}$ of dimension 3 with two connected components. When $\beta \in i\mathbb{R}$, both G_β and its closure lie in $\mathcal{M}_{i\mathbb{R}}$.

Theorem 7.20. *(Density) Either G_β is discrete or it is dense in*
1) $\mathcal{M}_{\mathbb{R}}$, with β real,
2) $\mathcal{M}_{i\mathbb{R}}$, with β imaginary, or
3) $\mathcal{M}$, with β neither real nor imaginary.

Proof. In each case, it is enough to see that G_β meets each component of the indicated Lie group and that d_β is at least as large as the dimension of that group.

It is easiest to show G_β meets the required components. Since $\mathcal{M}$ is connected, there is no difficulty when β is neither real nor imaginary. Say β is real. The matrices $M(\beta)$ and $M(-\beta)$ that represent the generators of G_β have determinant -1, and so G_β meets the non-identity component of $\mathcal{M}_{\mathbb{R}}$. The argument is similar for imaginary β.

Assume G_β is not discrete. We only need to show that $d_\beta \geq 6$ whenever β^2 is not real and that $d_\beta \geq 3$ for all β. For these estimates we need to study the Lie algebra L of $\mathcal{M}$.

We can identify L as usual with the space of all 2×2 complex matrices of trace 0. The exponential map $\exp : L \to \mathcal{M}$ sends $l \in L$ to the Möbius transformation defined by the invertible 2×2 matrix e^l. For every parabolic Möbius tranformation p there is a unique $\nu_p \in L$ such that $\nu_p^2 = 0$ and $p = \exp \nu_p$. We call ν_p the *nilpotent logarithm* of p. We call the linear map $n_p : L \to L$ given by $n_p(x) = [\nu_p, x]$ the *nilpotent derivation* of p.

Lemma 7.21. *Let G be a Möbius group normalized by a parabolic element $p \in \mathcal{M}$, i.e., $pGp^{-1} = G$. Then n_p stabilizes the Lie subalgebra $L(\overline{G}) \subset L$ and $n_p^3 = 0$.*

Proof. Let $\nu = \nu_p$ and $n = n_p$. As $\nu^2 = 0$, $n^2(x) = -2\nu x\nu$ and $n^3(x) = 0$ for all x. Thus $n^3 = 0$.

By Lie theory, $D = e^n : L \to L$ is the derivative at $Id \in \mathcal{M}$ of the group automorphism of $\mathcal{M}$ that sends m to $\exp(\nu)m\exp(-\nu) = pmp^{-1}$. This automorphism maps G to G and hence maps $\overline{G}$ to $\overline{G}$. Thus D stabilizes $L(\overline{G})$.

Since $n^3 = 0$, $D = e^n = I + n + n^2/2$. Squaring both sides of $n + n^2/2 = D - I$ and using $n^3 = 0$ yields $n^2 = (D-I)^2$, which then is used to solve for $n = (D-I) - (D-I)^2/2$. Since D stabilizes $L(\overline{G})$, n must also do so. □

By Proposition 7.3, we see r normalizes G_β. Hence, the parabolic elements $t = fr$ and $u = rf$ also normalize G_β. By the lemma, n_t and n_u each stabilize the Lie algebra of G_β, which we denote $L_\beta \subset L$.

Let e_{ij} denote the 2×2 complex matrix whose (i,j) entry is 1 and whose other entries are 0, where $1 \le i, j \le 2$. L has a basis over $\mathbb{C}$ consisting of the matrices e_{12}, e_{21} and $e_{11} - e_{22}$. We find $\nu_t = \beta e_{12}$ and $\nu_u = \beta e_{21}$.

We now estimate d_β. Since G_β is not discrete, $L_\beta \neq 0$. Since $n_t^3 = 0$, $n_t(L_\beta) \subset L_\beta$, and $L_\beta \neq 0$, there is a nonzero element in $\ker(n_t) \cap L_\beta$. This element commutes with ν_t and so it must have the form ae_{12} for some nonzero a. By considering n_u we likewise find that $be_{21} \in L_\beta$ for some nonzero b. Since L_β is a Lie subalgebra of L, L_β is closed under bracket. Thus $ab(e_{11} - e_{22}) = [ae_{12}, be_{21}] \in L_\beta$. This shows $d_\beta \ge 3$.

Now $n_u n_t(ae_{21}) = a'e_{21}$, where $a' = 2\beta^2 a$. When β^2 is not real, this shows L_β contains the complex line spanned by e_{21}. Likewise, by considering $n_t n_u(ae_{12})$, L_β contains the complex line spanned by e_{12}. Then L_β contains the bracket of these two complex lines and so $d_\beta \ge 6$. □

Theorem 7.20 has simple consequences for Λ_β, since $\Lambda_\beta = \Lambda(\overline{G_\beta})$.

Corollary 7.22. *When G_β is not discrete, as when $|\beta| < 1$, then $\Lambda_\beta = \widehat{\mathbb{R}}$ when β is real, $\Lambda_\beta = i\widehat{\mathbb{R}}$ when β is imaginary, and $\Lambda_\beta = \widehat{\mathbb{C}}$ when β^2 is not real.*

Proof. When $0 < |\beta| < 1$, Shimizu's Lemma ([21] II.C.5) implies that the group $H_\beta = \langle t, r\rangle$ is not discrete. Since G_β is a subgroup of H_β of index at most two, G_β is not discrete either. □

8. Some J_β and Λ_β

We will determine whether or not G_β is discrete and find Λ_β and J_β whenever β^2 is real or imaginary. The discreteness criteria and the limit sets are found using results of Hecke (when β^2 is real) and of Klimenko and Kopteva (when β^2 is imaginary).

We summarize our results in two related theorems and devote this section to their proofs.

Theorem 8.1. *Suppose β^2 is real.*
1) G_β is discrete for $|\beta| \ge 2$ and for $|\beta| = 2\cos\pi/k$, $k \ge 3$, but for no other β [15]
2) <u>For $|\beta| > 2$</u>: Λ_β is a Cantor set in $\beta\widehat{\mathbb{R}}$. J_β and $J_\beta{}'$ are disjoint Cantor sets in Λ_β. $J_\beta{}'$ is a thick attractor for S_β.
3) <u>For $\beta = \pm 2$ or $\pm 2i$</u>: $\Lambda_\beta = \beta\widehat{\mathbb{R}}$. J_β and $J_\beta{}'$ are Cantor sets in Λ_β with $J_\beta \cap J_\beta{}' = \{\beta/2, -\beta/2\}$. S_β uniformly contracts $J_\beta{}'$ but S_β has no thick attractor.
4) <u>For $|\beta| < 2$</u>: $J_\beta = J_\beta{}' = \Lambda_\beta = \beta\widehat{\mathbb{R}}$ and S_β has no attractor.

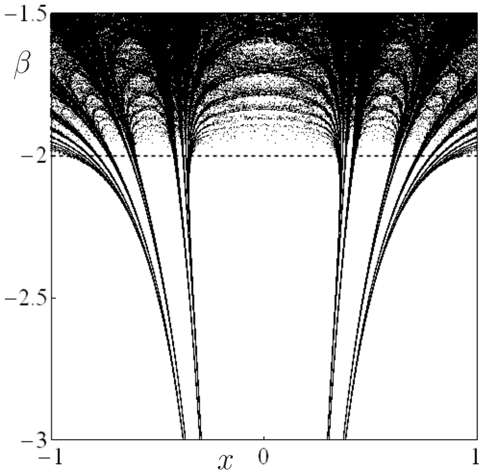


FIGURE 5. The picture shows the Julia set of S_β when $-3 \le \beta \le -1.5$. For each real value of β, the Julia set lies in the real axis. The Cantor sets are the horizontal sections of the curtains. The computer has a hard time filling the Julia sets of the semigroups when $-2 < \beta < -1.5$, where $J_\beta = \widehat{\mathbb{R}}$. See Remark 5.13.

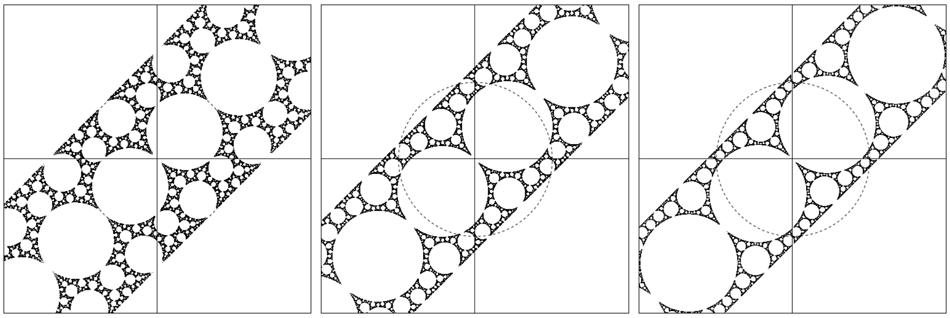


FIGURE 6. Three gasket Julia sets that correspond to $\beta = \sqrt{2i\cos(\pi/k)}$ for $k = 3, 4$ and 5, respectively. These Julia sets are completely invariant. See Remark 5.13.

See Figure 5.

Theorem 8.2. *Suppose β^2 is imaginary.*
1) G_β is discrete for $|\beta^2| \ge 2$ and for $|\beta^2| = 2\cos\pi/k,\ k \ge 3$, but for no other β [18]
2) For $|\beta^2| > 2$: Λ_β is a Cantor set in $\widehat{\mathbb{C}}$. J_β and $J_\beta{}'$ are disjoint Cantor sets in Λ_β. $J_\beta{}'$ is a thick attractor for S_β.
3) For $\beta = \pm 1 \pm i$: Λ_β is a gasket. J_β and $J_\beta{}'$ are Cantor sets in Λ_β with $J_\beta \cap J_\beta{}' = \{\pm 1, \pm i\}$. S_β uniformly contracts $J_\beta{}'$ but S_β has no thick attractor.
4) For $|\beta^2| < 2$: $J_\beta = J_\beta{}' = \Lambda_\beta$ and S_β has no attractor. $\Lambda_\beta = \widehat{\mathbb{C}}$ unless $|\beta^2| = 2\cos\pi/k,\ k \ge 3$, in which case Λ_β is a gasket.

See Figures 6 and 4.

We recall the reflection group case of Poincare's Polyhedron Theorem ([21], [27], [10]). An elliptic Möbius transformation is called *primitive* if it defines a rotation by an angle $2\pi/k$ for some integer $k \geq 2$.

Theorem 8.3. *Suppose $\mathcal{P}$ is a convex polyhedron in the compact ball $H^3 \cup \widehat{\mathbb{C}}$ bounded by finitely many planes. Let Γ be the group of hyperbolic isometries generated by the reflections ρ_a in these planes. Γ is discrete if and only if all the dihedral angles of $\mathcal{P} \cap H^3$ are submultiples of π or, equivalently, when the product $\rho_a \rho_b$ of any two generating reflections that fix an edge of $\mathcal{P}$ is either loxodromic, parabolic, or primitive elliptic of some order $n_{ab} \geq 2$. Suppose Γ is discrete. Γ has a presentation with generators r_a and relations $r_a^2 = 1$ and $(r_a r_b)^{n_{ab}} = 1$. Also the interior of $\mathcal{P} \cap \widehat{\mathbb{C}}$ is a fundamental domain for the action of Γ on $\Omega(\Gamma)$.*

Next we define a reflection group Γ_β whenever β^2 is real or imaginary. Let $\mu = \beta/\overline{\beta}$ so $\mu = \pm 1$ when β^2 is real and $\mu = \pm i$ when β^2 is imaginary. Let $\rho(z) = \mu \overline{z}$. We define $\sigma = \rho r$ if β is real, $\sigma = \rho r m$ if β is imaginary, and $\sigma = \rho u$ if β^2 is imaginary. The following lemma is straightforward and its proof is omitted.

Lemma 8.4. *ρ is the reflection that fixes $\beta\widehat{\mathbb{R}}$, ρm is the reflection that fixes $i\beta\widehat{\mathbb{R}}$, and $\rho t m$ is the reflection that fixes $i\beta\widehat{\mathbb{R}} + \beta/2$. σ is the reflection whose fixed circle is $|z| = 1$ if β^2 is real and $|\beta z + 1| = 1$ if β^2 is imaginary.*

We define $\Gamma_\beta = \langle \rho, \rho m, \rho t m, \sigma \rangle$. The fixed circles of these generators bound fixed planes in H^3 that bound a convex polygon $\mathcal{P}_\beta$ that contains ∞ (this specifies $\mathcal{P}_\beta$ when β^2 is imaginary but there are two congruent choices if β^2 is real). All the dihedral angles of $\mathcal{P}_\beta$ are right angles except for the dihedral angle made by the fixed planes of $\rho t m$ and σ, if these two planes intersect at all. By Theorem 8.3 Γ_β is discrete if and only if $\rho t m \sigma$ is either loxodromic, parabolic, or primitive elliptic of order $k \geq 3$. Now $\Gamma_\beta \cap \mathcal{M} = \langle m, tm, \rho\sigma \rangle$ is either H_β^+, if β^2 is real, or P_β^+, if β^2 is imaginary. Thus $\Gamma_\beta \cap \mathcal{M}$ is commensurable with G_β.

Note that, by calculation, we have $tr^2[\rho t m \sigma]$ equals $\beta^2, -\beta^2$, or $-\beta^4$ in the respective cases that $\beta \in \mathbb{R}, \beta \in i\mathbb{R}$, or $\beta^2 \in i\mathbb{R}$. By Theorem 4.3.11 in [2], an elliptic map m of order k satisfies $tr^2[m] = 4\cos^2(\pi/k)$ if and only if m is a rotation of angle $\pm 2\pi/k$. Parts (1) of Theorems 8.1 and 8.2 then follow (using Theorem 3.1).

Theorem 8.3 also determines the ordinary sets and hence the limit sets, as given in the above theorems. When β^2 is imaginary, for instance, and $\rho t m \sigma$ is primitive elliptic of order $k \geq 3$, $\mathcal{P}_\beta \cap \widehat{\mathbb{C}}$ consists of a bounded triangle T_B and an unbounded triangle T_U that are related by the involution rm, since conjugation by rm interchanges ρ with ρm and $\rho t m$ with σ. The sides of T_B are fixed by $\rho, \rho t m$, and σ. Its vertex angles are 0 at the vertex 0, $\pi/2$ at the vertex $\beta/2$, and π/k at its 3rd vertex. The triangle reflection group $\langle \rho, \rho t m, \sigma \rangle$ is Fuchsian (there is a Möbius change of variables taking T_B to a hyperbolic triangle in the upper halfplane with the same angles) and its ordinary set contributes a disc to Ω_β. In this way one sees that Ω_β is an infinite union of discs and so Λ_β is a gasket.

When $\rho t m \sigma$ is loxodromic, on the other hand, there is a gap between the fixed planes of $\rho t m$ and σ and so Λ_β is a Cantor set.

It remains to determine J_β and $J_\beta{}'$ in each case. Recall from Example 5.6 that S_β has a block when $|\Re\beta| + |\Im\beta| > 2$. Theorem 5.7 shows then that J_β and $J_\beta{}'$ are disjoint and that $J_\beta{}'$ is an attractor for S_β, which proves part 2) of each theorem.

Let $W = (\rho t m \sigma)^2$. Now $\rho t m \sigma$ is fm, f, or fmf as β is real, β is imaginary, or β^2 is imaginary. So W is respectively fg, f^2, or $fg^2 f$, hence an element of S_β. Suppose $\rho t m \sigma$ is elliptic. Then W is elliptic so $J_\beta = \Lambda_\beta$, by Theorem 7.17, and $J_\beta{}' = \Lambda_\beta$, by Proposition 7.13. Sources are dense in $J_\beta{}'$ by Theorem 2.4, so S_β does not uniformly contract $J_\beta{}'$. By Proposition 4.1, any S-invariant set uniformly contracted by S_β is a point. But G_β is a non-elementary Möbius group and so S_β has no fixed point. Thus S_β has no attractor. We now calculate Λ_β, using Corollary 7.22 when G_β is not discrete and using Theorem 8.3 as above when G_β is discrete. This proves part 4) of each theorem.

Assume β is as in part 3) of Theorem 8.2. Then $W = fg^2f$ is parabolic so S_β has no thick attractor by Lemma 5.3.

The set $\{\pm 1, \pm i\}$ consists of the fixed points of the parabolic elements $fg^2f, g^2f^2, gf^2g, f^2g^2$ of S_β and so it is contained in $J_\beta \cap J_\beta{}'$. Let $N_\beta = N$ if $\beta^2 = 2i$ and $N_\beta = k(N)$ if $\beta^2 = -2i$, where N is as in Example 5.6. Then N_β is S_β-invariant so $J_\beta{}' \subset N_\beta$. It follows that $J_\beta \subset 1/N_\beta$ and so $J_\beta \cap J_\beta{}'$ is contained in ∂N_β. But $J_\beta{}' \subset S_\beta{}^2(N_\beta)$ and $S_\beta{}^2(N_\beta) \cap \partial N_\beta = \{\pm 1, \pm i\}$. This shows that $J_\beta \cap J_\beta{}' = \{\pm 1, \pm i\}$.

Let $K_\beta = \cap S_\beta{}^n N_\beta$. Each component of K_β is a point since S_β contracts the hyperbolic metric on Int N_β by the Theorem of Pick and since $\cap W^n(N_\beta)$ is a point (recalling $W = fg^2f$ is parabolic). Thus if we set $h_0 = f$ and $h_1 = g$ there is a homeomorphism $h : K_\beta \to \{0,1\}^{\mathbb{N}}$ such that $h(z) = (i_1, i_2, ...)$ if $z \in h_{i_1}h_{i_2}...h_{i_n}(N_\beta)$, for each $n \geq 1$. Each periodic sequence in $\{0,1\}^{\mathbb{N}}$ is $h(z)$ for z the sink of some loxodromic element of S_β or $z \in \{\pm 1, \pm i\}$. Thus sinks are dense in K_β and so $K_\beta \subset J_\beta{}'$ by Theorem 2.4. But K_β is S_β-invariant and so $J_\beta{}' \subset K_\beta$ by Remark 2.20. Thus $J_\beta{}' = K_\beta$ is a Cantor set, so $J_\beta = 1/J_\beta{}'$ is also a Cantor set.

We define the *inverse 2-shift* to be the semigroup $\prec j_0, j_1 \succ$ on $\{0,1\}^{\mathbb{N}}$ where $j_a(i_1, i_2, ...) = (a, i_1, i_2, ...)$. Let d be the usual metric on $\{0,1\}^{\mathbb{N}}$, so $d(x,y) = 2^{-N}$ if $x_N \neq y_N$ but $x_n = y_n$ for all $n < N$. Then $d(i_a(x), i_a(y)) = d(x,y)/2$ so the inverse 2-shift is a CIFS.

Under the homeomorphism h, we can identify the action of S_β on $J_\beta{}'$ with the inverse 2-shift. Thus if we give $J_\beta{}'$ the metric h^*d, S_β uniformly contracts $J_\beta{}'$.

This proves part 3) of Theorem 8.2. Part 3) of Theorem 8.1 can be proven in similar fashion, so both theorems are proven.

Remark 8.5. *As in the preceding proof, the action of S_β on $J_\beta{}'$ can be identified with the inverse 2-shift for all β with $|\Re\beta| + |\Im\beta| \geq 2$. Thus S_β uniformly contracts $J_\beta{}'$ for all such β.*

References

[1] Michael F. Barnsley, John H. Elton, and Douglas P. Hardin. Recurrent iterated function systems. *Constr. Approx.*, 5(1):3–31, 1989.

[2] Alan F. Beardon. *The geometry of discrete groups*, volume 91 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1983.

[3] Alan F. Beardon. *Iteration of rational functions*, volume 132 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[4] Rainer Brück. Geometric properties of Julia sets of the composition of polynomials of the form $z^2 + c_n$. *Pacific J. Math.*, 198(2):347–372, 2001.

[5] Rainer Brück, Matthias Büger, and Stefan Reitz. Random iterations of polynomials of the form $z^2 + c_n$: connectedness of Julia sets. *Ergodic Theory Dynam. Systems*, 19(5):1221–1231, 1999.

[6] Matthias Büger. Self-similarity of Julia sets of the composition of polynomials. *Ergodic Theory Dynam. Systems*, 17(6):1289–1297, 1997.

[7] Matthias Büger. On the composition of polynomials of the form $z^2 + c_n$. *Math. Ann.*, 310(4):661–683, 1998.

[8] Horacio A. Caruso and Sebastian M. Marotta. Sequences of complex numbers resembling the fibonacci series. *Revista Ciências Exatas e Naturais*, 2(1):49–59, 2000.

[9] John H. Elton. An ergodic theorem for iterated maps. *Ergodic Theory Dynam. Systems*, 7(4):481–488, 1987.

[10] David Epstein, Carlo Petronio. An exposition of Poincare'e polyhedron theorem. *Enseign. Math. (2)*, 40:113-170, 1994.

[11] John Erik Fornæss and Nessim Sibony. Random iterations of rational functions. *Ergodic Theory Dynam. Systems*, 11(4):687–708, 1991.

[12] Leon Greenberg. Commensurable groups of Moebius transformations. In *Discontinuous Groups and Riemann surfaces*, volume 79 of *Annals of Mathematics Studies*, 227-237. Princeton University Press, Princeton, 1974.

[13] Tatsunori Harada. The dynamics of nearly abelian polynomial semigroups at infinity. *Proc. Japan Acad. Ser. A Math. Sci.*, 73(3):36–37, 1997.

[14] Brian Hayes. The vibonacci numbers. *American Scientist*, 87(4):296–301, 1999.

[15] Erich Hecke. *Dirichlet series*. Planographed lecture notes, Princeton Institute for Advanced Study, Edwards Brothers, Ann Arbor, 1938.

[16] A. Hinkkanen and G. J. Martin. The dynamics of semigroups of rational functions I. *Proc. London Math. Soc. (3)*, 73(2):358–384, 1996.

[17] John E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.

[18] Elena Klimenko and Natalia Kopteva. Discrete $\mathcal{RP}$ groups with a parabolic generator. *Sibirsk. Mat. Zh.*, 46(6):1324-1334, 2005.

[19] Linda Keen and Caroline Series. The Riley slice of Schottky space. *Proc. London Math. Soc. (3)*, 69(1):72–90, 1994.

[20] Albert Marden. *Outer circles: an introduction to hyperbolic 3-manifolds*. Cambridge University Press, Cambridge, 2007.

[21] Bernard Maskit. *Kleinian groups*, volume 287 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1988.

[22] John Milnor. *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006.

[23] David Mumford, Caroline Series, and David Wright. *Indra's pearls: the vision of Felix Klein*. Cambridge University Press, New York, 2002.

[24] James R. Munkres. *Topology: a first course*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.

[25] Jouni Parkkonen. The outside of the Teichmuller space of punctured tori in Maskit's embedding. *Ann. Acad. Sci. Fenn. Math.* 24(2):305-342, 1999.

[26] Xavier Pennec. Intrinsic statistics on Riemannian manifolds: basic tools for geometric measurements. *J. Math. Imaging Vision* 25(1):127-154, 2006.

[27] John G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.

[28] Rich Stankewitz. *Completely invariant Julia sets of rational semigroups*. Ph.D. Thesis. University of Illinois, 1998.

[29] Rich Stankewitz. Completely invariant Julia sets of polynomial semigroups. *Proc. Amer. Math. Soc.*, 127(10):2889–2898, 1999.

[30] Rich Stankewitz. Uniformly perfect sets, rational semigroups, Kleinian groups and IFS's. *Proc. Amer. Math. Soc.*, 128(9):2569–2575, 2000.

[31] Rich Stankewitz. Density of repelling fixed points in the Julia set of a rational or entire semigroup. *J. Difference Equ. Appl.*, 16(5):763 – 771, 2010.

[32] Rich Stankewitz. Density of repelling fixed points in the Julia set of a rational or entire semigroup, II. Preprint, Ball State University, 2010.

[33] Rich Stankewitz and Hiroki Sumi. Dynamical properties and structure of julia sets of postcritically bounded polynomial semigroups. To appear in Transactions of the American Mathematical Society.

[34] Hiroki Sumi. Dynamics of postcritically bounded polynomial semigroups III: classification of semi-hyperbolic semigroups and random Julia sets which are Jordan curves but not quasicircles. *Ergodic Theory Dynam. Systems*, 30(6), 1869–1902, 2011.

[35] Hiroki Sumi. On dynamics of hyperbolic rational semigroups. *J. Math. Kyoto Univ.*, 37(4):717–733, 1997.
[36] Hiroki Sumi. Dynamics of sub-hyperbolic and semi-hyperbolic rational semigroups and skew products. *Ergodic Theory Dynam. Systems*, 21(2):563–603, 2001.
[37] Hiroki Sumi. Random dynamics of polynomials and devil's staircase-like functions in the complex plane. *Appl. Math. Comput.*, 187:489–500, 2007.
[38] Hiroki Sumi. Dynamics of postcritically bounded polynomials semigroups I: Connected components of the julia sets. *Discrete Cont. Dynam. Syst.* 29(3):1205–1244, 2011.
[39] Yeshun Sun and Chung-Chun Yang. On the connectivity of the Julia set of a finitely generated rational semigroup. *Proc. Amer. Math. Soc.*, 130(1):49–52 (electronic), 2001.
[40] Divakar Viswanath. *Lyapunov Exponents from Random Fibonacci Sequences to the Lorenz Equations.* Ph.D. Thesis. Cornell University, 1998.
[41] David J. Wright. Searching for the cusp. In *Spaces of Kleinian groups and hyperbolic 3-manifolds*, volume 329 of *London Math. Soc. Lecture Notes*, 301-336. Cambridge University Press, Cambridge, 2005.

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