

**Introduction to Quasiconformal Mappings in the
plane with an application to quasiconformal
surgery**

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1. Introduction

A large portion of this paper is from the Masters thesis of the first author, written under the direction of the second author. It is presented here, not as original work, but as a systematic introduction to the topic at hand. We attempt to present a fairly self-contained paper from which an interested reader could learn many of the aspects needed to understand the background necessary in the statement and proof of the Fundamental Theorem of Quasiconformal Surgery.

This paper will introduce a special kind of complex homeomorphism called a quasiconformal map. Quasiconformal maps have many applications in areas of heat conduction, electrostatic potential, and fluid flow. They are also a valuable tool in the field of complex dynamics. We follow [3] in that we will limit our discussion to the context of the extended complex plane. The study of quasiconformal maps, however, may be extended to higher dimensions.

Quasiconformal maps are generalizations of the well known conformal maps (and it will be shown that every conformal map is a quasiconformal map). Conformal maps impose a very strong condition on the differential (approximating linear map), whereas quasiconformal maps relax this condition considerably. However, quasiconformal maps still retain many aspects (or parallel aspects) of conformal maps. As such these maps become a more flexible tool for such procedures as we will see in the proof of the Fundamental Theorem of Quasiconformal Surgery. This theorem will allow one to glue together, in some sense, two different analytic maps – something prohibited, in the usual sense, by the Identity Theorem (Principle of Analytic Continuation).

The reader is assumed to have a basic understanding of general topology, complex analytic functions, measure theory, and integration theory. Wherever there is an exception, there will be detailed explanation and proof, along with precise references for those results whose proofs are “beyond the scope” of the paper.

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2. Notation

We follow Lehto and Virtanen [3] and state that throughout this paper *the plane* will be taken to mean the extended complex plane, denoted by $\overline{\mathbb{C}}$, which is homeomorphic to the Riemann Sphere. And we will call the usual Euclidean plane the *finite plane* and denote this by \mathbb{C} . We call a subset of the plane a domain if it is both open and connected. Also, all functions will be taken to be complex valued unless otherwise indicated.

We will use the following notation: $\Delta(z, r)$ is the Euclidean disk of radius r and center z , $C(z, r)$ is the Euclidean circle of radius r and center z , $Ann(z_0; r, R) = \{z : r < |z - z_0| < R\}$, $\partial\Omega$ and $\overline{\Omega}$ refer to the boundary and closure, respectively, of the set Ω taken in the topology of $\overline{\mathbb{C}}$.

Let U be an open subset of the plane. We will say a complex valued function on U given by $f = u + iv$ where $u = \operatorname{Re}(f)$ and $v = \operatorname{Im}(f)$ is in the class $C^1(U)$ if the partial derivatives $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$ both exist and are continuous on U . Also, when the necessary derivatives of f exist, we will set

$$(2.1) \quad \partial f = f_z = \frac{1}{2}(f_x - if_y) \quad \text{and} \quad \bar{\partial} f = f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

For this paper we will be using a “little-o” notation. So we will make the following definition.

DEFINITION 2.1. *Let f and g be complex valued functions defined in a neighborhood U of z_0 with $g(z) \neq 0$ for all $z \in U \setminus \{z_0\}$. We say $f = o(g)$ as $z \rightarrow z_0$ if $\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = 0$. If no confusion can arise, the “ $z \rightarrow z_0$ ” will be omitted when the choice of z_0 is clear.*

The following are properties that will be used in what follows.

LEMMA 2.2. *Consider functions $f, f_1, f_2, g, h,$ and $h_1,$ defined in a neighborhood of z_0 .*

a) *If $f_1 = o(g)$ and $f_2 = o(g)$, then $f_1 + f_2 = o(g)$.*

b) *If $f_1 = o(g)$ and $f_2 = o(g)$, then $f_1 f_2 = o(g^2)$.*

c) *Suppose*

i) $f = f_1 + o(g)$ with $\limsup_{z \rightarrow z_0} |f(z)| < +\infty,$

ii) $h = h_1 + o(g)$ with $\limsup_{z \rightarrow z_0} |h_1(z)| < +\infty.$

Then $fh = f_1 h_1 + o(g)$.

d) *Suppose*

i) $f = f_1 + o(g),$

ii) $h = h_1 + o(g)$ with both $h(z) \neq 0$ and $h_1(z) \neq 0$ for all $z \neq z_0,$

iii) $\limsup_{z \rightarrow z_0} \frac{1}{|h(z)|} < +\infty, \limsup_{z \rightarrow z_0} |f(z)| < +\infty,$ and $\limsup_{z \rightarrow z_0} |g(z)| < +\infty.$

Then $\frac{f}{h} = \frac{f_1}{h_1} + o(g)$.

e) *If $f = o(g)$, then*

i) $\bar{f} = o(\bar{g});$

ii) $|f| = o(|g|);$

iii) $f = o(|g|);$ and

iv) $f = o(\bar{g}).$

PROOF. Parts (a), (b), and (e) follow immediately from the definition. Part (c) follows from the following equalities:

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{f(z)h(z) - f_1(z)h_1(z)}{g(z)} &= \lim_{z \rightarrow z_0} \frac{f(z)h(z) - f(z)h_1(z) + f(z)h_1(z) - f_1(z)h_1(z)}{g(z)} \\ &= \lim_{z \rightarrow z_0} f(z) \frac{h(z) - h_1(z)}{g(z)} + \lim_{z \rightarrow z_0} h_1(z) \frac{f(z) - f_1(z)}{g(z)} = 0, \end{aligned}$$

which shows that $fh - f_1 h_1 = o(g)$ and thus $fh = f_1 h_1 + o(g)$.

Part (d) is shown as follows:

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{\frac{f(z) - f_1(z)}{h(z)} - \frac{f_1(z)}{h_1(z)}}{g(z)} &= \lim_{z \rightarrow z_0} \frac{f(z)h_1(z) - h(z)f_1(z)}{h(z)h_1(z)g(z)} \\ &= \lim_{z \rightarrow z_0} \frac{f(z)h_1(z) - f_1(z)h_1(z) + f_1(z)h_1(z) - h(z)f_1(z)}{h(z)h_1(z)g(z)} \\ &= \lim_{z \rightarrow z_0} \frac{1}{h(z)} \cdot \frac{f(z) - f_1(z)}{g(z)} + \lim_{z \rightarrow z_0} \frac{f_1(z)}{h(z)h_1(z)} \cdot \frac{h_1(z) - h(z)}{g(z)} = 0, \end{aligned}$$

and thus $\frac{f(z)}{h(z)} = \frac{f_1(z)}{h_1(z)} + o(g)$.

Note that we justify that $\lim_{z \rightarrow z_0} \frac{f_1(z)}{h(z)h_1(z)} \cdot \frac{h_1(z) - h(z)}{g(z)} = 0$ from the following argument. Since $f = f_1 + o(g)$, $\frac{f(z) - f_1(z)}{g(z)} \rightarrow 0$ as $z \rightarrow z_0$ by definition. But we have that $\limsup_{z \rightarrow z_0} |g(z)| < +\infty$ and so $f(z) - f_1(z) \rightarrow 0$ as $z \rightarrow z_0$. But since $\limsup_{z \rightarrow z_0} |f(z)| < +\infty$, we must also have that $\limsup_{z \rightarrow z_0} |f_1(z)| < +\infty$. We can similarly show $\limsup_{z \rightarrow z_0} \frac{1}{|h_1(z)|} < +\infty$ from the hypotheses $\limsup_{z \rightarrow z_0} \frac{1}{|h(z)|} < +\infty$ and $h = h_1 + o(g)$. \square

3. Differentiability in the Real Sense

In what follows we will sometimes substitute without mention the two-dimensional real vector $z = \begin{pmatrix} x \\ y \end{pmatrix}$ for $z = x + iy$.

DEFINITION 3.1. *Let $f : \Omega_1 \rightarrow \Omega_2$ be a function between finite plane domains. We say that f is differentiable in the real sense (also called \mathbb{R} -differentiable or real differentiable) at $z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ when there exists a matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are such that*

$$f(z) = f(z_0) + \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + E(z)$$

for all $z \in \Omega_1$ and the function $E(z)$ satisfies $\frac{|E(z)|}{|z - z_0|} \rightarrow 0$ as $z \rightarrow z_0$, i.e., $E(z) = o(z - z_0)$.

Let f be given as in Definition 3.1. Then using i simply as a place holder, we get

$$(3.1) \quad f(z) = f(z_0) + \alpha(x - x_0) + \beta(y - y_0) + i\gamma(x - x_0) + i\delta(y - y_0) + E(z).$$

By definition then, we have

$$\begin{aligned} f_x(z_0) &= \lim_{x \rightarrow x_0} \frac{f(x + iy_0) - f(x_0 + iy_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{f(x_0 + iy_0) + \alpha(x - x_0) + \beta(y_0 - y_0) + i\gamma(x - x_0) + i\delta(y_0 - y_0) + E(x + iy_0) - f(x_0 + iy_0)}{x - x_0} \\ &= \alpha + i\gamma + \lim_{x \rightarrow x_0} \frac{E(x + iy_0)}{x - x_0} = \alpha + i\gamma. \end{aligned}$$

Note that $\lim_{x \rightarrow x_0} \frac{E(x + iy_0)}{x - x_0} = 0$ since it is $\lim_{z \rightarrow z_0} \frac{E(z)}{z - z_0} = 0$, letting z approach z_0 along the line $z = x + iy_0$.

Similarly, $f_y(z_0) = \beta + i\delta$ and so by (3.1) we see

$$(3.2) \quad f(z) = f(z_0) + f_x(z_0)(x - x_0) + f_y(z_0)(y - y_0) + E(z).$$

Note that writing $f = u + iv$ where $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$, we then have

$$u_x(z_0) = \alpha, \quad u_y(z_0) = \beta, \quad v_x(z_0) = \gamma, \quad \text{and} \quad v_y(z_0) = \delta.$$

Now, using the fact that if $z = x + iy$, then

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i},$$

we see from (3.1) that:

$$\begin{aligned} f(z) &= f(z_0) + \frac{\alpha}{2}(z + \bar{z} - z_0 - \bar{z}_0) + \frac{\beta}{2i}(z - \bar{z} - z_0 + \bar{z}_0) \\ &\quad + i\frac{\gamma}{2}(z + \bar{z} - z_0 - \bar{z}_0) + \frac{i\delta}{2i}(z - \bar{z} - z_0 + \bar{z}_0) + E(z) \\ &= f(z_0) + \left(\frac{\alpha}{2} + \frac{\beta}{2i} + \frac{i\gamma}{2} + \frac{\delta}{2}\right)(z - z_0) + \left(\frac{\alpha}{2} - \frac{\beta}{2i} + \frac{i\gamma}{2} - \frac{\delta}{2}\right)(\bar{z} - \bar{z}_0) + E(z) \\ &= f(z_0) + \partial f(z_0)(z - z_0) + \bar{\partial} f(z_0)(\bar{z} - \bar{z}_0) + E(z) \quad \text{by (2.1)}. \end{aligned}$$

We have thus demonstrated the following theorem.

THEOREM 3.2. *If a function f is \mathbb{R} -differentiable at z_0 , then it has partial derivatives at z_0 and can be written in the form*

$$(3.3) \quad f(z) = f(z_0) + A(z - z_0) + B(\bar{z} - \bar{z}_0) + E(z)$$

where $A = \partial f(z_0)$ (the z -coefficient) and $B = \bar{\partial} f(z_0)$ (the \bar{z} -coefficient) are both complex and $E(z) = o(z - z_0)$.

The following theorem is a well known result indicating conditions which imply a function is \mathbb{R} -differentiable.

THEOREM 3.3 ([5], p. 100). *Suppose that a function $f = u + iv$ is defined in an open subset U of the finite plane and that the partial derivatives u_x , u_y , v_x , and v_y exist everywhere in U . If each of these partial derivatives is continuous at the point z_0 of U , then f is differentiable in the real sense at z_0 .*

Equation (3.3) says that a function $f(z)$ that is differentiable in the real sense at z_0 can be well approximated near z_0 by a linear function of the form

$$L(z) = f(z_0) + A(z - z_0) + B(\bar{z} - \bar{z}_0).$$

Note that L is just a translation ($z \mapsto z - z_0$), followed by $\tilde{L}(z) = Az + B\bar{z}$, followed by another translation ($z \mapsto z + f(z_0)$). Thus in order to analyze $L(z)$ (and hence locally analyze any real-differentiable function) we first consider $\tilde{L}(z) = Az + B\bar{z}$. If we let A_1, A_2 be the real and imaginary parts of A and B_1, B_2 be the real and imaginary parts of B we get

$$(3.4) \quad \tilde{L}(z) = (A_1 + iA_2)(x + iy) + (B_1 + iB_2)(x - iy)$$

$$(3.5) \quad = (A_1 + B_1)x + (B_2 - A_2)y + i[(A_2 + B_2)x + (A_1 - B_1)y].$$

We note that in matrix form we see from equation (3.5) that $z \mapsto \tilde{L}(z)$ is written

$$(3.6) \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{bmatrix} A_1 + B_1 & B_2 - A_2 \\ A_2 + B_2 & A_1 - B_1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence we immediately see that if a map f has form (3.3) with $E(z) = o(z - z_0)$, then f is real differentiable at z_0 (as given in Definition 3.1) with $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} A_1 + B_1 & B_2 - A_2 \\ A_2 + B_2 & A_1 - B_1 \end{bmatrix}$. We also note that the *Jacobian* of the map f at z_0 is then given by

$$(3.7) \quad J_f(z_0) = \det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = |A|^2 - |B|^2 = |\partial f(z_0)|^2 - |\bar{\partial} f(z_0)|^2.$$

We now discuss the concept of a total differential of a function f . The reader may reference [6], p.55-59 for the details of this matter. For a function f differentiable in the real sense at z_0 , the total differential is given by $df_{z_0}(z) = \partial f(z_0)z + \bar{\partial} f(z_0)\bar{z}$. Thus $df_{z_0}(z)$ is a linear function. When we write $df_{z_0}(z) \neq 0$ we mean that the function is not identically the zero function, i.e., not both $\partial f(z_0)$ and $\bar{\partial} f(z_0)$ are zero.

DEFINITION 3.4 ([3], p. 9). *A (not necessarily differentiable) homeomorphism $f : \Omega_1 \rightarrow \Omega_2$ between arbitrary point sets in the plane is sense-preserving if f preserves the orientation of the boundary of every Jordan domain D such that $\bar{D} \subseteq \Omega_1$.*

REMARK 3.5. *If a function f is sense-preserving, then its inverse f^{-1} is sense-preserving. Also if both f and g are sense-preserving, then $f \circ g$ is also sense-preserving.*

We refer the reader to the Orientation Theorem and subsequent discussion in [3], p. 9-10 which gives the following. Let $f : \Omega_1 \rightarrow \Omega_2$ be a homeomorphism between domains in $\bar{\mathbb{C}}$. If $J_f(z_0) > 0$ at some $z_0 \in \Omega_1$, then f is sense-preserving. Conversely, if $J_f(z_0) < 0$ at some $z_0 \in \Omega_1$, then f is sense-reversing. Thus we see that a sense-preserving map must satisfy $J_f(z) \geq 0$ for all $z \in \Omega_1$. We note that it is possible to have $J_f(z_0) = 0$ at some z_0 for a sense-preserving map, as is the case for $z_0 \in \mathbb{R}$ with the map $(x, y) \mapsto (x, y^3)$ defined on all of \mathbb{C} .

Note: When $df_{z_0}(z) \neq 0$ and f is sense-preserving we must have $|\partial f(z_0)| > 0$ and $|\partial f(z_0)| \geq |\bar{\partial} f(z_0)|$.

Returning to our analysis of \mathbb{R} -differentiable functions and hence functions of the form $\tilde{L}(z) = Az + B\bar{z}$, we can use basic linear algebra and (3.7) to obtain the following.

THEOREM 3.6. *The function $\tilde{L}(z) = Az + B\bar{z}$ from the finite plane to itself is bijective if and only if $|A| \neq |B|$. Furthermore, if $|A| \neq |B|$, then \tilde{L} is sense-preserving if and only if $|A| > |B|$.*

Let us take a closer look at the maps of the form $\tilde{L}(z)$ by first considering, for $0 < \mu < 1$, the maps $L_\mu(z) = z + \mu\bar{z} = x + iy + \mu x - i\mu y = (1 + \mu)x + i(1 - \mu)y$.

We see that there is an expansion in the x -direction and a compression in the y -direction under L_μ . We see then that a circle of radius r centered at the origin given by $(x, y) = (r \cos \theta, r \sin \theta)$ maps to an ellipse centered at the origin since $(x, y) = (r \cos \theta, r \sin \theta) \mapsto (\tilde{x}, \tilde{y}) = ((1 + \mu)r \cos \theta, (1 - \mu)r \sin \theta)$. Thus the image of the circle is given by $\left[\frac{\tilde{x}}{(1 + \mu)r}\right]^2 + \left[\frac{\tilde{y}}{(1 - \mu)r}\right]^2 = 1$, an ellipse with major axis $r(1 + \mu)$ and minor axis $r(1 - \mu)$.

Our main interest is in sense preserving maps and so let us now consider more general maps of the form $\tilde{L}(z) = Az + B\bar{z}$ with $|A| > |B|$. Writing $\mu = \frac{B}{A}$, $A = |A|e^{i\alpha}$, $\mu = |\mu|e^{i\beta}$ and $z = re^{i\theta}$, we have

$$(3.8) \quad \tilde{L}(z) = A \left(z + \frac{B}{A} \bar{z} \right) = |A|e^{i\alpha} (re^{i\theta} + |\mu|e^{i\beta}re^{-i\theta})$$

$$(3.9) \quad = |A|e^{i(\alpha + \frac{\beta}{2})} \left(re^{i(\theta - \frac{\beta}{2})} + |\mu|re^{-i(\theta - \frac{\beta}{2})} \right).$$

Thus if we let $g(z) = |A|e^{i(\alpha + \frac{\beta}{2})} \cdot z$ and $h(z) = e^{-i\frac{\beta}{2}} \cdot z$, we then have that $\tilde{L} = g \circ L_\mu \circ h$. But g and h are simply rotations and therefore

$$(3.10) \quad \tilde{L} = g \circ L_{|\mu|} \circ h$$

also maps circles to ellipses (see Figure 1) for which the ratio of the major to minor axis is $\frac{1+|\mu|}{1-|\mu|}$.

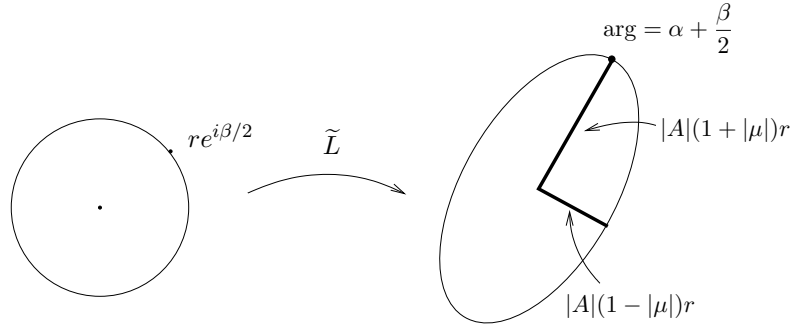


FIGURE 1

Next we consider a general \mathbb{R} -differentiable function $f : \Omega_1 \rightarrow \Omega_2$ which is expressed $f(z) = f(z_0) + \partial f(z_0)(z - z_0) + \bar{\partial} f(z_0)(\bar{z} - \bar{z}_0) + E(z)$ with $E(z) = o(z - z_0)$.

REMARK 3.7. *Since a function f which is real differentiable at z_0 is well approximated near z_0 by the linear function $L(z) = f(z_0) + \partial f(z_0)(z - z_0) + \bar{\partial} f(z_0)(\bar{z} - \bar{z}_0)$ we say that if $|\partial f(z_0)| > |\bar{\partial} f(z_0)|$, then f maps infinitesimal circles centered at z_0 to infinitesimal ellipses centered at $f(z_0)$. We also note that the ratio of the major axis to the minor axis of the infinitesimal ellipse is*

$$\frac{1 + |\mu|}{1 - |\mu|} = \frac{1 + \left| \frac{\bar{\partial} f}{\partial f} \right|}{1 - \left| \frac{\bar{\partial} f}{\partial f} \right|} = \frac{|\partial f| + |\bar{\partial} f|}{|\partial f| - |\bar{\partial} f|}.$$

This ratio is, of course, intimately connected with the directional derivative, which we state now.

DEFINITION 3.8. *The directional derivative of f at z_0 in the direction of θ is*

$$\partial_\theta f(z_0) = \lim_{r \rightarrow 0} \frac{f(z_0 + re^{i\theta}) - f(z_0)}{r}.$$

By examining the stretching a \mathbb{R} -differentiable map exhibits locally, as in Figure 1, the following lemma is clear. Nevertheless, a formal proof will follow.

LEMMA 3.9. *Let Ω_1 and Ω_2 be domains in $\bar{\mathbb{C}}$. If $f : \Omega_1 \rightarrow \Omega_2$ is sense preserving and real differentiable at z_0 , then*

$$\max_\theta |\partial_\theta f(z_0)| = |\partial f(z_0)| + |\bar{\partial} f(z_0)| \text{ and } \min_\theta |\partial_\theta f(z_0)| = |\partial f(z_0)| - |\bar{\partial} f(z_0)|.$$

PROOF. Letting $z = z_0 + re^{i\theta}$ we see that

$$\begin{aligned} \partial_\theta f(z_0) &= \lim_{r \rightarrow 0} \frac{f(z_0) + \partial f(z_0)(re^{i\theta}) + \bar{\partial} f(z_0)(re^{-i\theta}) + E(z) - f(z_0)}{r} \\ &= \partial f(z_0)e^{i\theta} + \bar{\partial} f(z_0)e^{-i\theta} + \lim_{z \rightarrow z_0} \frac{E(z)}{|z - z_0|} = \partial f(z_0)e^{i\theta} + \bar{\partial} f(z_0)e^{-i\theta}. \end{aligned}$$

Let $\theta_0 = \frac{\gamma - \alpha}{2}$ where $\alpha = \arg(\partial f(z_0))$ and $\gamma = \arg(\bar{\partial} f(z_0))$. Then one can easily show that $\max_\theta |\partial_\theta f(z_0)| = |\partial f(z_0)e^{i\theta_0} + \bar{\partial} f(z_0)e^{-i\theta_0}| = |\partial f(z_0)| + |\bar{\partial} f(z_0)|$.

Similarly, if we set $\theta_1 = \frac{\gamma - \alpha - \pi}{2}$, we get $\min_\theta |\partial_\theta f(z_0)| = |\partial f(z_0)e^{i\theta_1} + \bar{\partial} f(z_0)e^{-i\theta_1}| = |\partial f(z_0)| - |\bar{\partial} f(z_0)|$, noting that $|\partial f(z_0)| \geq |\bar{\partial} f(z_0)|$ since f is sense-preserving. \square

4. Tentative Definition of Quasiconformal Maps

We now wish to give a ‘‘tentative’’ definition for quasiconformal maps. This definition relates to the local stretching displayed by quasiconformal maps. For a differentiable map to be quasiconformal, there must be a bound to this stretching in one direction versus any other direction. First we must make a preliminary definition.

DEFINITION 4.1. *Let f be a function that is real-differentiable at z_0 such that $\partial f(z_0) \neq 0$. We call the ratio*

$$\mu_f(z_0) = \frac{\bar{\partial} f(z_0)}{\partial f(z_0)}$$

the complex dilatation of f at z_0 .

DEFINITION 4.2. *Let f be a sense-preserving function that is real-differentiable at z_0 such that $J_f(z_0) \neq 0$ (thus $|\mu_f(z_0)| \neq 1$). We set*

$$\tilde{K}_f(z_0) = \frac{1 + |\mu_f(z_0)|}{1 - |\mu_f(z_0)|} = \frac{|\partial f(z_0)| + |\bar{\partial} f(z_0)|}{|\partial f(z_0)| - |\bar{\partial} f(z_0)|} = \frac{\max_\theta |\partial_\theta f(z_0)|}{\min_\theta |\partial_\theta f(z_0)|}.$$

According to the discussion at the end of the last section $\tilde{K}_f(z_0)$ is the ratio of the maximum stretch to the minimum stretch of an infinitesimal circle around z_0 under f .

DEFINITION 4.3. *[Tentative Definition] Let $f : \Omega_1 \rightarrow \Omega_2$ be a real-differentiable sense-preserving bijection between plane domains with $J_f > 0$ on Ω_1 . For any $K \geq 1$, we say f is K -quasiconformal if $\tilde{K}_f = \sup_{z \in \Omega_1} \tilde{K}_f(z) \leq K$.*

REMARK 4.4. *Definition 4.3 requires f to be real-differentiable everywhere in Ω_1 , but in the following pages we will not impose such a strict requirement. It is for this reason that Definition 4.3 is our “tentative” definition.*

5. Conformal Maps

In this paper we will rely on many facts from the theory of conformal mappings of the complex plane. Let us now recall the definition of a conformal map.

DEFINITION 5.1. *Let Ω_1 and Ω_2 be open subsets of the plane $\overline{\mathbb{C}}$. We call a function $f : \Omega_1 \rightarrow \Omega_2$ conformal if f is bijective and analytic.*

Remark: Other sources may define a map to be conformal on its domain when the complex derivative both exists and is nonzero at each point of its domain. This amounts to f being analytic and locally bijective. For example, the map $f(z) = e^z$ defined on the finite plane \mathbb{C} is such a map, but it is not conformal by Definition 5.1.

Note that f is complex differentiable at $z_0 \in \mathbb{C}$ with $f(z_0) \in \mathbb{C}$, exactly when (see [5], p. 63) we can express $f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0)$. Thus, if f is conformal at z_0 , we then have that f is \mathbb{R} -differentiable with $\bar{\partial}f(z_0) = 0$, $\partial f(z_0) = f'(z_0) \neq 0$ and $\mu_f(z_0) = 0$. Here we say conformal $f : \Omega_1 \rightarrow \Omega_2$ maps infinitesimal circles centered at z_0 to infinitesimal circles centered at $f(z_0)$, since f would be well approximated near z_0 by $L(z) = f(z_0) + f'(z_0)(z - z_0)$ which maps such circles to such circles. It is clear then that a conformal map f is 1-quasiconformal according to our Tentative Definition 4.3 since $\tilde{K}_f(z) \equiv 1$.

REMARK 5.2. *When f is injective and real-differentiable in a domain Ω_1 with $\partial f(z)$ never vanishing and $\mu_f(z) \equiv 0$, then $\bar{\partial}f(z) \equiv 0$. Therefore f would be conformal onto its range and $\partial f = f'$.*

6. Geometric Definition of Quasiconformal Maps

In this section we will introduce the notion a quadrilateral in the plane $\overline{\mathbb{C}}$ and introduce the concept of a “conformal module” of a quadrilateral. This will help give us a precise definition of a quasiconformal map. The geometric definition will be more flexible than the “tentative” definition in Definition 4.3 since it will not require differentiability. However, as we will see in Section 8, such quasiconformal maps will turn out to be real-differentiable almost everywhere.

DEFINITION 6.1. *A quadrilateral $Q(z_1, z_2, z_3, z_4)$ is a Jordan domain in $\overline{\mathbb{C}}$ with four distinct points on ∂Q , the boundary of Q , with a positive cyclic order. We will call these four points the vertices of Q . We will also denote a quadrilateral more briefly by $Q(\vec{z})$, where $\vec{z} = (z_1, z_2, z_3, z_4)$, or simply by Q when the boundary points are either unimportant or understood.*

DEFINITION 6.2. *For $a > 0$, define $R(a) = Q(0, a, a+i, i)$ where Q is a rectangle with vertices in the usual sense at 0 , a , $a+i$, and i .*

THEOREM 6.3. *Let $Q(z_1, z_2, z_3, z_4)$ be a quadrilateral in the plane $\overline{\mathbb{C}}$. Then there exists a unique real number $a > 0$ and a conformal map f mapping Q onto $R(a)$ such that the map f , (extended to ∂Q as given by the Caratheodory-Osgood Theorem), satisfies $f(z_1) = 0$, $f(z_2) = a$, $f(z_3) = a+i$, and $f(z_4) = i$. We may sometimes denote this correspondence of vertices by $f(z_1, z_2, z_3, z_4) = (0, a, a+i, i)$.*

PROOF. By the Riemann Mapping Theorem there exists a conformal map $w : Q(z_1, z_2, z_3, z_4) \rightarrow \{z : \text{Im} z > 0\}$. By the Caratheodory-Osgood Theorem we can extend w to a homeomorphism from \overline{Q} to $\{z : \text{Im} z \geq 0\} \cup \{\infty\}$ such that, without loss of generality, $w(z_1, z_2, z_3, z_4) = (\alpha, \beta, \gamma, \infty)$. Now we can apply the Schwarz-Christoffel Formula ([6], p. 236) which guarantees the existence of a conformal map $g : \{z : \text{Im} z \geq 0\} \rightarrow R$ where R is some rectangle in the plane such that α, β, γ , and ∞ are mapped to the vertices of R . Now we can use a map h which simply rotates, translates, and dilates (hence h is conformal) R to some $R(a)$. Therefore $h \circ g \circ w(Q(z_1, z_2, z_3, z_4)) = R(a)$ and thus existence has been shown.

To show uniqueness, suppose that $f_1 : Q \rightarrow R(a')$ and $f_2 : Q \rightarrow R(a)$ are as in the statement of the theorem. Then the map $f_2 \circ f_1^{-1} : R(a') \rightarrow R(a)$ is conformal and fixes i and 0 . So we can repeatedly apply the Schwarz Reflection Principle to extend $f_2 \circ f_1^{-1} : R(a') \rightarrow R(a)$ to be conformal from the finite plane \mathbb{C} to itself. Therefore $f_2 \circ f_1^{-1}$ must have the form $f_2 \circ f_1^{-1}(z) = cz + d$ where c and d are complex number with $c \neq 0$ (see [5], p. 388). But since $f_2 \circ f_1^{-1}$ fixes i and 0 , we must have that $f_2 \circ f_1^{-1}$ is the identity map. Hence $a = a'$. \square

DEFINITION 6.4. We define the conformal module of a quadrilateral $Q(z_1, z_2, z_3, z_4)$ to be $M(Q(z_1, z_2, z_3, z_4)) = a$ where $a > 0$ is as in Theorem 6.3.

REMARK 6.5. If R is a rectangle with vertices (in the usual sense) z_1, z_2, z_3 , and z_4 , then $a = M(R(z_1, z_2, z_3, z_4)) = \frac{|z_2 - z_1|}{|z_3 - z_2|}$, i.e., the ratio of (oriented) side lengths of R . This holds since a linear map (rotation, dilation, translation) can conformally map R onto $R(a)$ while preserving this ratio.

The next lemma will display the effect on the modulus of a quadrilateral after having the vertices reordered or having the quadrilateral transformed by a conformal map.

LEMMA 6.6. Let $Q(z_1, z_2, z_3, z_4)$ be a quadrilateral in the plane. Then:

$$a) M(Q(z_2, z_3, z_4, z_1)) = \frac{1}{M(Q(z_1, z_2, z_3, z_4))}, \text{ and}$$

b) M is conformally invariant. That is, if f is conformal on a domain containing \overline{Q} , then $M(f(Q)(f(z_1, z_2, z_3, z_4))) = M(Q(z_1, z_2, z_3, z_4))$.

PROOF. a) Let f_1 be the conformal map, as in Theorem 6.3, that defines the conformal module a of Q . Set $f_2(z) = -iz + ia$ and $f_3(z) = \frac{z}{a}$. Then, since the composition of conformal maps is conformal (see Figure 2 below), we see that $f_3 \circ f_2 \circ f_1 : Q(z_2, z_3, z_4, z_1) \rightarrow R(\frac{1}{a})$ satisfies Theorem 6.3 above, thus showing $M(Q(z_2, z_3, z_4, z_1)) = \frac{1}{a} = \frac{1}{M(Q(z_1, z_2, z_3, z_4))}$.

b) Let $f : \Omega_1 \rightarrow \Omega_2$ be a conformal map with $\overline{Q} \subseteq \Omega_1$. Let a denote the conformal module of Q and let $g : Q \rightarrow R(a)$ be the corresponding conformal map as in Theorem 6.3. The composition $g \circ f^{-1} : f(Q) \rightarrow R(a)$ satisfies Theorem 6.3 and shows $M(f(Q)(f(z_1, z_2, z_3, z_4))) = a$. Thus M is conformally invariant. \square

We will now set the stage to develop another useful way of computing and estimating the conformal module of a quadrilateral. The vertices of a quadrilateral $Q(z_1, z_2, z_3, z_4)$ divide its boundary into four Jordan arcs (see Figure 3). We will denote these Jordan arcs (z_1z_2) , (z_2z_3) , (z_3z_4) , and (z_4z_1) .

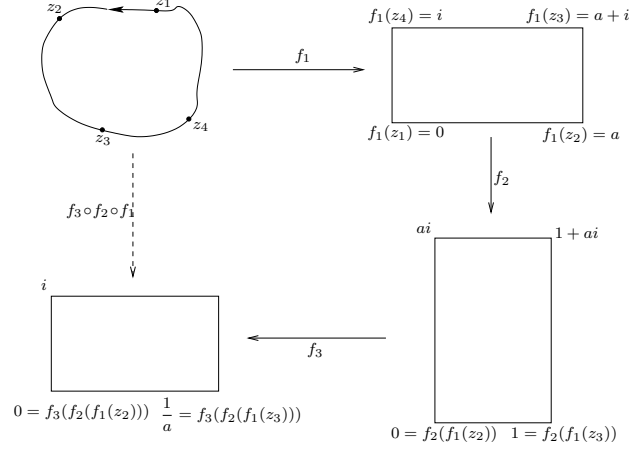


FIGURE 2

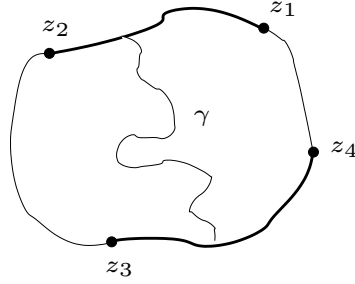


FIGURE 3

Let $\Gamma(Q) = \{\text{locally rectifiable Jordan arcs in } Q \text{ which join } (z_1 z_2) \text{ to } (z_3 z_4)\}$, and let $\mathcal{A}(Q) = \{\text{Borel measurable } \rho : Q \rightarrow [0, \infty) \text{ such that } \int_{\gamma} \rho(z)|dz| \geq 1 \text{ for all } \gamma \in \Gamma(Q)\}$. We will call such $\rho \in \mathcal{A}(Q)$ *admissible* (for Q). We will also let $\ell_{\rho}(\gamma) = \int_{\gamma} \rho(z)|dz|$ and call this the ρ -length of γ . When $\rho = 1$ we simply write $\ell(\gamma)$, the usual Euclidean length. We call $A_{\rho}(Q) = \iint_Q \rho^2(z) dx dy$ the ρ -area of Q . When $\rho = 1$ we write $A(Q)$ (or $\text{area}(Q)$), the usual Euclidean area.

DEFINITION 6.7. We define $M_1(Q(\vec{z})) = \inf_{\rho \in \mathcal{A}(Q)} \iint_Q \rho^2(z) dx dy$.

LEMMA 6.8. The function M_1 is conformally invariant, i.e., $M_1(f(Q)(f(z_1, z_2, z_3, z_4))) = M_1(Q(z_1, z_2, z_3, z_4))$ for any quadrilateral Q and any conformal f defined on a domain which contains \bar{Q} .

PROOF. Let $f : Q_1 \rightarrow Q_2$ be conformal. Take $\rho_2 \in \mathcal{A}(Q_2)$. Set $\rho_1(z) = \rho_2 \circ f(z) |f'(z)|$ for $z \in Q_1$. Choose any $\gamma_1 \in \Gamma(Q_1)$ and set $\gamma_2 = f \circ \gamma_1$. Note that $\gamma_2 \in \Gamma(Q_2)$. Then, letting γ_2 be a curve parametrized over the interval $[b, c]$ we have that $1 \leq \int_{\gamma_2} \rho_2(z)|dz| = \int_b^c \rho_2[\gamma_2(t)] |\dot{\gamma}_2(t)| dt = \int_b^c \rho_2\{f[\gamma_1(t)]\} |f'[\gamma_1(t)]| |\dot{\gamma}_1(t)| dt$

$= \int_b^c \rho_1[\gamma_1(t)]|\dot{\gamma}_1(t)|dt = \int_{\gamma_1} \rho_1(z)|dz|$. Therefore $\rho_1 \in \mathcal{A}(Q_1)$. We recall that $J_f = |f'|^2$ from equation (3.7) and so we apply the change of variable formula (see [5], p. 427) to get $M_1(Q_1) \leq A_{\rho_1}(Q_1) = \iint_{Q_1} [\rho_1(z)]^2 dx dy = \iint_{Q_1} \{\rho_2[f(z)]\}^2 |f'(z)|^2 dx dy = \iint_{Q_2} [\rho_2(w)]^2 du dv = A_{\rho_2}(Q_2)$.

Therefore, taking infimum over the right hand side, we get $M_1(Q_1) \leq M_1(Q_2)$. By the same argument using f^{-1} we obtain $M_1(Q_1) \geq M_1(Q_2)$. Therefore equality holds. \square

THEOREM 6.9. *For every quadrilateral Q in the plane $M_1(Q) = M(Q)$.*

PROOF. We will first show that $M_1(R(a)) = a = M(R(a))$. Let $\rho \in \mathcal{A}(R(a))$. Then for fixed $0 < x < a$, invoking the Cauchy-Schwarz inequality (see Appendix I) we have $1 \leq \ell_\rho(\gamma) = \int_0^1 \rho(x + iy) dy \leq \left[\int_0^1 \rho^2(x + iy) dy \right]^{\frac{1}{2}}$ where γ is the vertical path $\gamma(y) = x + iy$ for $0 \leq y \leq 1$. Then $a = \int_0^a 1^2 dx \leq \int_0^a \int_0^1 \rho^2(x + iy) dy dx = \iint_{R(a)} \rho^2(x + iy) dy dx = A_\rho(R(a))$. So, taking infimum, $a \leq M_1(R(a))$. But if $\rho \equiv 1$, which can easily be seen to be admissible, then $M_1(R(a)) \leq \text{area}(R(a)) = a$. Thus $M_1(R(a)) = a = M(R(a))$ for the rectangle $R(a)$.

Since every quadrilateral Q is mapped conformally onto some $R(a)$, the general case follows from conformal invariance of both M and M_1 . \square

THEOREM 6.10. [*Rengel's Inequality*] *Consider the Jordan arcs $(z_1 z_2)$ and $(z_3 z_4)$ for a quadrilateral $Q(z_1, z_2, z_3, z_4)$ such that $\overline{Q} \subset \mathbb{C}$. Let s be the Euclidean distance between $(z_1 z_2)$ and $(z_3 z_4)$, i.e., $s = \inf |w - z|$ where $w \in (z_1 z_2)$ and $z \in (z_3 z_4)$, and let t be the Euclidean distance between $(z_2 z_3)$ and $(z_4 z_1)$. Then*

$$\frac{t^2}{\text{area}(Q)} \leq M(Q(z_1, z_2, z_3, z_4)) \leq \frac{\text{area}(Q)}{s^2}.$$

Also, equality holds if and only if Q is a rectangle.

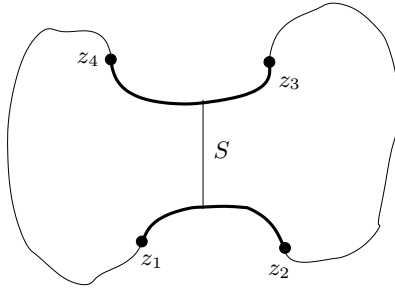


FIGURE 4

PROOF. Notice that $\ell(\gamma) \geq s > 0$ for every $\gamma \in \Gamma(Q)$. Now take $\rho(z) \equiv 1/s$ and note that $\ell_\rho(\gamma) = \frac{\ell(\gamma)}{s} \geq 1$ for every $\gamma \in \Gamma(Q)$. Thus ρ is admissible and so $M_1\left(Q\left(\vec{z}\right)\right) \leq \iint_Q \rho^2(z) dx dy = \frac{1}{s^2} \iint_Q dx dy = \frac{1}{s^2}(\text{area}Q)$.

By reordering the vertices and setting $\tilde{Q} = Q(z_2, z_3, z_4, z_1)$ we see then that $\frac{1}{M(Q)} = M(\tilde{Q}) \leq \frac{\text{area}\tilde{Q}}{t^2} = \frac{\text{area}Q}{t^2}$, which gives the lower bound on $M(Q)$.

Furthermore, if Q is a rectangle, then equality holds by the Remark 6.5. For the converse, when equality holds, we refer the reader to ([3], pp. 23) for the proof that Q must be a rectangle. \square

We are now able to state the **geometric** definition of quasiconformal maps, which we will take to be our main definition.

DEFINITION 6.11. *Let $f : \Omega_1 \rightarrow f(\Omega_1)$ be a sense-preserving homeomorphism of a plane domain and let $K \geq 1$. We say f is K -quasiconformal on Ω_1 (or $f \in K - QC$) if*

$$K_f := \sup_{\bar{Q} \subseteq \Omega_1} \frac{M(f(Q))}{M(Q)} \leq K,$$

where \bar{Q} denotes the closure of quadrilateral Q in the plane $\bar{\mathbb{C}}$. We call K_f the maximal dilatation of f . A map is called quasiconformal if it is K -quasiconformal for some K .

LEMMA 6.12. *Let $f : \Omega_1 \rightarrow f(\Omega_1)$ be K -quasiconformal. Suppose that quadrilateral $Q(z_1, z_2, z_3, z_4) \subseteq \bar{Q} \subseteq \Omega_1$. Let $\tilde{Q} = Q(z_2, z_3, z_4, z_1)$ and $f(\tilde{Q}) = f(Q)(f(z_2), f(z_3), f(z_4), f(z_1))$ denote a new cyclic order on Q and $f(Q)$ respectively. Then $M(f(\tilde{Q})) \geq \frac{1}{K}M(\tilde{Q})$.*

PROOF. By Lemma 6.6 part (a), we see that $M(f(\tilde{Q})) = \frac{1}{M(f(Q))}$ since this is merely a new cyclic order on the same quadrilateral. Also, $M(\tilde{Q}) = \frac{1}{M(Q)}$, and so $M(\tilde{Q}) = \frac{1}{M(Q)} \leq K \frac{1}{M(f(Q))}$, which follows from f being K -quasiconformal. Therefore $\frac{1}{K}M(\tilde{Q}) \leq M(f(\tilde{Q}))$. \square

THEOREM 6.13. *Let $f : \Omega_1 \rightarrow \Omega_2$ be K -quasiconformal and let quadrilateral $Q(z_1, z_2, z_3, z_4) \subseteq \Omega_1$ with $\bar{Q} \subseteq \Omega_1$. Then,*

- a) $\frac{1}{K}M(Q) \leq M(f(Q)) \leq K \cdot M(Q)$.
- b) $f^{-1} : \Omega_2 \rightarrow \Omega_1$ is K -quasiconformal.
- c) If $g : \Omega_2 \rightarrow \Omega_3$ is K_0 -quasiconformal, then $g \circ f : \Omega_1 \rightarrow \Omega_3$ is at least $K_0 \cdot K$ -quasiconformal, that is, $K_{g \circ f} \leq K_0 K$.

REMARK 6.14. *By Theorem 6.13 (a) we see that a 1-QC map f on Ω satisfies $M(f(Q)) = M(Q)$ for all quadrilaterals Q with $\bar{Q} \subseteq \Omega$.*

PROOF. a) By the geometric definition of quasiconformal maps we get that $\frac{M(f(Q))}{M(Q)} \leq K$. Therefore $M(f(Q)) \leq K \cdot M(Q)$. The remaining inequality follows directly from Lemma 6.12.

b) Let quadrilateral $Q_2 \subseteq \bar{Q}_2 \subseteq \Omega_2$ and set $Q_1 = f^{-1}(Q_2) \subseteq \Omega_1$. Then $\frac{M(f^{-1}(Q_2))}{M(Q_2)} = \frac{M(Q_1)}{M(f(Q_1))} \leq K$, by the first inequality in part (a). Thus f^{-1} is $K - QC$.

c) Since $g \in K_0 - QC$, we get that for any quadrilateral Q with $\bar{Q} \subseteq \Omega_1$ we have $M(g(f(Q))) \leq K_0 M(f(Q)) \leq K_0 K \cdot M(Q)$, since $f \in K - QC$. \square

THEOREM 6.15. *Let $f : \Omega_1 \rightarrow \Omega_2$ be a function between plane domains. Then f is 1-quasiconformal if and only if f is conformal.*

PROOF. “ \Leftarrow ” Since we have shown that the conformal module is conformally invariant we see that $\frac{M(f(Q))}{M(Q)} = 1$ for all quadrilaterals $Q \subseteq \Omega_1$ with $\overline{Q} \subseteq \Omega_1$.

“ \Rightarrow ” Since f is 1-quasiconformal, it is therefore bijective. So we need only show that $f : \Omega_1 \rightarrow \Omega_2$ is analytic. Let $z_0 \in \Omega_1$ and let quadrilateral $Q \subseteq \overline{Q} \subseteq \Omega_1$ be such that $z_0 \in Q$. Let $f_1 : Q \rightarrow R(a)$ and $f_2 : f(Q) \rightarrow R(a')$ be conformal maps as in Theorem 6.3. Let $g = f_2 \circ f \circ f_1^{-1}$. By Theorem 6.13(c) and the fact that $f_1 : Q \rightarrow R(a)$ and $f_2 : f(Q) \rightarrow R(a')$ are both conformal (and hence 1- QC), we see that g is also 1-quasiconformal.

We will show that the function g must be the identity function. This proves that f is conformal on Q since $f = f_2^{-1} \circ g \circ f_1$ is a composition of conformal maps. Since quadrilateral Q was arbitrary in Ω_1 , we get that f is analytic on all of Ω_1 .

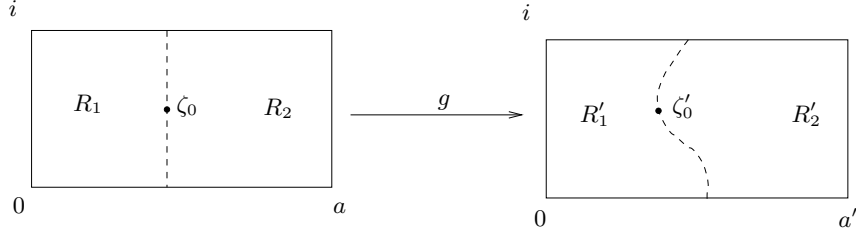


FIGURE 5

Since $f \in 1 - QC$, we see by Remark 6.14 that $a = M(Q) = M(f(Q)) = a'$. Take $\zeta_0 \in R(a)$. Let $R_1 = R(a) \cap \{z | \operatorname{Re} z < \operatorname{Re} \zeta_0\}$ and $R_2 = R(a) \cap \{z | \operatorname{Re} z > \operatorname{Re} \zeta_0\}$. Let $\zeta'_0 = g(\zeta_0)$ and $R'_n = g(R_n)$ for $n = 1, 2$. Then, since g is 1- QC , we may use continuity of the module ([3], p. 26 and 30) to obtain $M(R_n) = M(R'_n)$ for $n = 1, 2$ even though the rectangles R_n are not compactly inside the domain of g . Now Rengel's Inequality, with $s = 1$, tells us that $M(R'_n) \leq \operatorname{area}(R'_n)$. So, $a = a' = \operatorname{area}(R(a')) \geq \operatorname{area}(R'_1) + \operatorname{area}(R'_2) \geq M(R'_1) + M(R'_2) = M(R_1) + M(R_2) = \operatorname{Re} \zeta_0 + (a - \operatorname{Re} \zeta_0) = a$. Therefore, equality must hold at each step and so $M(R'_n) = \operatorname{area}(R'_n)$ for $n = 1, 2$. Then the equality of Rengel's Inequality tells us that R'_n are rectangles for $n = 1, 2$. But since $M(R_n) = M(R'_n)$ for $n = 1, 2$, we must have that $R_n = R'_n$ for $n = 1, 2$. In particular, $\operatorname{Re} \zeta_0 = \operatorname{Re} \zeta'_0$.

Similarly, we obtain that $\operatorname{Im} \zeta_0 = \operatorname{Im} \zeta'_0$. Thus g is the identity map, which completes the proof. \square

We now state and prove several chain rules that will be used to prove two important properties of quasiconformal maps.

LEMMA 6.16. *[Chain rule for ∂ and $\bar{\partial}$] Let f be real-differentiable at z_0 and let g be real-differentiable at $w_0 = f(z_0)$. Then \bar{f} and $g \circ f$ are also real-differentiable at z_0 . Further, the following hold at z_0 ,*

- a) $\partial \bar{f} = \bar{\partial} f$ and $\bar{\partial} \bar{f} = \bar{\partial} \bar{f}$;
- b) $\bar{\partial}(g \circ f) = [(\partial g) \circ f] \bar{\partial} f + [(\bar{\partial} g) \circ f] \bar{\partial} \bar{f}$;
- c) $\partial(g \circ f) = [(\partial g) \circ f] \partial f + [(\bar{\partial} g) \circ f] \partial \bar{f}$.

PROOF. (a) We consider $f(z) = \overline{f(z_0) + A(z - z_0) + B(\bar{z} - \bar{z}_0) + o(z - z_0)}$ as in equation (4). Then, noting that $\overline{o(z - z_0)} = o(\bar{z} - \bar{z}_0)$, we see that $\overline{f(z)} = \overline{f(z_0) + A(z - z_0) + B(\bar{z} - \bar{z}_0) + o(z - z_0)}$, which shows that \overline{f} is \mathbb{R} -differentiable. We examine the z and \bar{z} coefficients to then conclude that $\partial \overline{f}(z) = \overline{B} = \overline{\partial f}$ and $\bar{\partial} \overline{f}(z) = \overline{A} = \overline{\partial f}$.

To prove (b) and (c) we first note that since f and g are real-differentiable functions they can be written in the form

$$(6.1) \quad f(z) = f(z_0) + A_1(z - z_0) + B_1(\bar{z} - \bar{z}_0) + \alpha(z)$$

$$(6.2) \quad g(w) = g(w_0) + A_2(w - w_0) + B_2(\bar{w} - \bar{w}_0) + \beta(w)$$

where $\alpha(z) = o(z - z_0)$, $\beta(w) = o(w - w_0)$ and $w_0 = f(z_0)$.

Now set $L_1(z) = A_1(z - z_0) + B_1(\bar{z} - \bar{z}_0)$ and $\tilde{L}_1(z) = df_{z_0}(z) = A_1z + B_1\bar{z} = L_1(z + z_0)$. Similarly set $L_2(w) = A_2(w - w_0) + B_2(\bar{w} - \bar{w}_0)$ and $\tilde{L}_2(w) = dg_{w_0}(w) = A_2w + B_2\bar{w} = L_2(w + w_0)$.

Note that $\tilde{L}_1(z_1 + z_2) = \tilde{L}_1(z_1) + \tilde{L}_1(z_2)$ and $\tilde{L}_2(w_1 + w_2) = \tilde{L}_2(w_1) + \tilde{L}_2(w_2)$. Thus

$$\begin{aligned} g(f(z)) &= g(w_0) + L_2(f(z)) + \beta(f(z)) \\ &= g(w_0) + L_2(f(z_0) + L_1(z) + \alpha(z)) + \beta(f(z)) \\ &= g(w_0) + \tilde{L}_2(L_1(z) + \alpha(z)) + \beta(f(z)) \\ &= g(w_0) + \tilde{L}_2(L_1(z)) + \tilde{L}_2(\alpha(z)) + \beta(f(z)) \\ &= g(w_0) + \tilde{L}_2(\tilde{L}_1(z - z_0)) + \tilde{L}_2(\alpha(z)) + \beta(f(z)). \end{aligned}$$

But

$$\begin{aligned} \tilde{L}_2(\tilde{L}_1(z - z_0)) &= A_2[A_1(z - z_0) + B_1(\bar{z} - \bar{z}_0)] + B_2[\overline{A_1}(\bar{z} - \bar{z}_0) + \overline{B_1}(z - z_0)] \\ &= (A_2A_1 + B_2\overline{B_1})(z - z_0) + (A_2B_1 + B_2\overline{A_1})(\bar{z} - \bar{z}_0). \end{aligned}$$

We then examine the z and \bar{z} coefficients to get that

$$(6.3) \quad \bar{\partial}(g \circ f)(z_0) = A_2B_1 + B_2\overline{A_1}$$

and

$$(6.4) \quad \partial(g \circ f)(z_0) = A_2A_1 + B_2\overline{B_1},$$

provided that both $\tilde{L}_2(\alpha(z)) = o(z - z_0)$ and $\beta(f(z)) = o(z - z_0)$, which we will now show.

As $z \rightarrow z_0$ we see that $\frac{\tilde{L}_2(\alpha(z))}{z - z_0} = \frac{A_2 \cdot \alpha(z)}{z - z_0} + \frac{B_2 \overline{\alpha(\bar{z})}}{z - z_0} \rightarrow 0$ since $\alpha = o(z - z_0)$.

Thus $\tilde{L}_2(\alpha(z)) = o(z - z_0)$.

We next consider $\frac{\beta(f(z))}{z - z_0}$ as $z \rightarrow z_0$. When $f(z) = w_0$, we have that $\beta(f(z)) = \beta(w_0) = 0$ and so we only need to consider those z tending to z_0 such that $f(z) \neq w_0$. For such z we have

$$(6.5) \quad \frac{\beta(f(z))}{z - z_0} = \frac{\beta(f(z))}{f(z) - w_0} \cdot \frac{f(z) - w_0}{z - z_0}.$$

However

$$\frac{f(z) - w_0}{z - z_0} = \frac{f(z_0) + A_1(z - z_0) + B_1(\bar{z} - \bar{z}_0) + \alpha(z) - w_0}{z - z_0} = A_1 + B_1 \frac{\bar{z} - \bar{z}_0}{z - z_0} + \frac{\alpha(z)}{z - z_0}$$

is bounded by $|A_1| + |B_1|$ as $z \rightarrow z_0$. Also $\beta = o(w - w_0)$, and so using $w = f(z) \rightarrow w_0$ as $z \rightarrow z_0$ we see that $\frac{\beta(f(z))}{f(z) - w_0} \rightarrow 0$. Therefore, by examining Equation (6.5) we may conclude that $\beta(f(z)) = o(z - z_0)$.

Note equations (6.1) and (6.2) where we see that $A_2 = \partial g(w_0) = (\partial g \circ f)(z_0)$, $A_1 = \partial f(z)$, $B_2 = \bar{\partial} g(w_0) = (\bar{\partial} g \circ f)(z_0)$, and $\bar{B}_1 = \overline{\partial f(z_0)} = \bar{\partial} \bar{f}(z_0)$ by part (a). Parts (b) and (c) now follow from (6.3) and (6.4). \square

REMARK 6.17 (Chain rule for differentials). *The proof of (b) and (c) above shows that the linear approximation of a composition is the composition of the respective linear approximations, as one would expect. In terms of differentials we may express this as $d(g \circ f)_{z_0} = dg_{f(z_0)} \circ df_{z_0}$.*

LEMMA 6.18 (Chain rule for complex dilatations). *We have the following:*

a) *Let μ_f and μ_h be complex dilatations for function $f : \Omega_1 \rightarrow \Omega_2$ and $h : \Omega_2 \rightarrow \Omega_3$ which are real-differentiable at z and $f(z)$, respectively, such that $\partial f(z)$, $\partial h(f(z))$, and $\partial(h \circ f)(z)$ are all nonzero. Then*

$$(i) \quad \mu_{h \circ f}(z) = \frac{\mu_f(z) + \mu_h(f(z)) \cdot \frac{\overline{\partial f(z)}}{\partial f(z)}}{1 + \mu_h(f(z)) \cdot \mu_f(z) \cdot \frac{\overline{\partial f(z)}}{\partial f(z)}} \quad \text{when } \bar{\partial} f(z) \neq 0, \text{ and}$$

$$(ii) \quad \mu_{h \circ f}(z) = \mu_h \circ f(z) \cdot \frac{\overline{\partial f(z)}}{\partial f(z)} \quad \text{when } \bar{\partial} f(z) = 0.$$

b) *Let μ_f be the complex dilatation for homeomorphism $f : \Omega_1 \rightarrow \Omega_2$ that is real-differentiable at z such that $\partial f(z)$ is nonzero and $|\partial f(z)| \neq |\bar{\partial} f(z)|$ (i.e. $J_f(z) \neq 0$). Then, assuming f^{-1} is real-differentiable at $f(z)$, we have that*

$$\mu_{f^{-1}}(f(z)) = -\mu_f(z) \cdot \frac{\partial f(z)}{\bar{\partial} f(z)}.$$

c) *Let μ_f and μ_h be complex dilatations for homeomorphisms $f : \Omega_1 \rightarrow \Omega_2$ and $h : \Omega_2 \rightarrow \Omega_3$ that are real-differentiable at z such that $\partial f(z)$, $\partial h(z)$, and $\partial(h \circ f^{-1})(f(z))$ are all nonzero and $|\partial f(z)| \neq |\bar{\partial} f(z)|$. Then, if f^{-1} is real-differentiable at $f(z)$, we have*

$$\mu_{h \circ f^{-1}}(f(z)) = \frac{\mu_h(z) - \mu_f(z)}{1 - \mu_h(z) \cdot \overline{\mu_f(z)}} \cdot \frac{\partial f(z)}{\bar{\partial} f(z)}.$$

REMARK 6.19. *When a real-differentiable homeomorphism f satisfies $J_f(z_0) \neq 0$, then it can be shown that f^{-1} must be real-differentiable at $f(z_0)$. So the hypotheses in (b) and (c) on the differentiability of f^{-1} is not necessary. However, for our applications to quasiconformal maps, differentiability of f^{-1} will follow from Lemma 8.3, and so the above weaker result will suffice for our purposes.*

PROOF. (a) By applying Lemma 6.16 in the case when $\bar{\partial}f \neq 0$, we have

$$\begin{aligned} \mu_{h \circ f} &= \frac{\bar{\partial}(h \circ f)}{\partial(h \circ f)} = \frac{[(\partial h) \circ f] \bar{\partial}f + [(\bar{\partial}h) \circ f] \bar{\partial}f}{[(\partial h) \circ f] \bar{\partial}f + [(\bar{\partial}h) \circ f] \bar{\partial}f} \\ &= \frac{[(\partial h) \circ f] \bar{\partial}f + [(\bar{\partial}h) \circ f] \bar{\partial}f}{[(\partial h) \circ f] \bar{\partial}f + [(\bar{\partial}h) \circ f] \bar{\partial}f} \cdot \frac{\frac{1}{\partial f[(\partial h) \circ f]}}{\frac{1}{\partial f[(\partial h) \circ f]}} = \frac{\mu_f + \mu_h \circ f \cdot \frac{\bar{\partial}f}{\partial f}}{1 + \mu_h \circ f \cdot \frac{\bar{\partial}f}{\partial f}} \\ &= \frac{\mu_f + \mu_h \circ f \cdot \frac{\bar{\partial}f}{\partial f}}{1 + \mu_h \circ f \cdot \frac{\bar{\partial}f}{\partial f} \cdot \frac{\bar{\partial}f}{\partial f}} = \frac{\mu_f + \mu_h \circ f \cdot \frac{\bar{\partial}f}{\partial f}}{1 + \mu_h \circ f \cdot \mu_f \cdot \frac{\bar{\partial}f}{\partial f}}. \end{aligned}$$

When $\bar{\partial}f = 0$ we obtain the second formula in part (a) also from the above work.

(b) Consider the analytic function $f \circ f^{-1}(z) = z$. By Lemma 6.16 (b) and (c) we obtain (at $f(z)$)

$$\begin{aligned} 1 &= \partial(f \circ f^{-1}) = (\partial f \circ f^{-1})(\partial f^{-1}) + (\bar{\partial}f \circ f^{-1})(\bar{\partial}f^{-1}) \\ 0 &= \bar{\partial}(f \circ f^{-1}) = (\partial f \circ f^{-1})(\bar{\partial}f^{-1}) + (\bar{\partial}f \circ f^{-1})(\bar{\partial}f^{-1}). \end{aligned}$$

We will let $A = \partial f \circ f^{-1}$, $B = \partial f^{-1}$, $C = \bar{\partial}f \circ f^{-1}$, and $D = \bar{\partial}f^{-1}$. Then, by Lemma 6.16 (a) we have that $1 = AB + CD$ and $0 = A\bar{D} + C\bar{B}$. Using the conjugate of the second equation we solve for B and D in terms of A and C obtaining $B = \frac{\bar{A}}{|A|^2 - |C|^2}$ and $D = \frac{-\bar{C}}{|A|^2 - |C|^2}$. Note that here we use the hypothesis that $|A| = |\partial f(z)| \neq |\bar{\partial}f(z)| = |C|$. Therefore, again using Lemma 6.16 (a), we get that $\mu_{f^{-1}}(f(z)) = \frac{\bar{\partial}f^{-1}}{\partial f^{-1}} = \frac{\bar{D}}{B} = -\frac{C}{A} = -\frac{CA}{AA} = -\mu_f(z) \frac{\partial f}{\bar{\partial}f}$, since $\mu_f = \frac{\bar{\partial}f}{\partial f} = \frac{C}{A}$.

(c) Now we use (a) and (b) to compute the following when $C = \bar{\partial}f(z) \neq 0$ (and hence $D \neq 0$).

$$\begin{aligned} \mu_{h \circ f^{-1}}(f(z)) &= \frac{\mu_{f^{-1}}(f(z)) + \mu_h \circ f^{-1}(f(z)) \cdot \frac{\bar{\partial}f^{-1}}{\partial f^{-1}}}{1 + \mu_h \circ f^{-1}(f(z)) \cdot \mu_{f^{-1}}(f(z)) \cdot \frac{\bar{\partial}f^{-1}}{\partial f^{-1}}} \\ &= \frac{-\mu_f(z) \frac{\partial f}{\bar{\partial}f} + \mu_h(z) \frac{\bar{\partial}f^{-1}}{\partial f^{-1}}}{1 - \mu_h(z) \mu_f(z) \frac{\partial f}{\bar{\partial}f} \frac{\bar{\partial}f^{-1}}{\partial f^{-1}}} = \frac{-\mu_f(z) \frac{A}{A} + \mu_h(z) \frac{\bar{B}}{B}}{1 - \mu_h(z) \mu_f(z) \frac{A}{A} \frac{D}{D}} \\ &= \frac{(-\mu_f(z) + \mu_h(z)) \frac{A}{A}}{1 - \mu_h(z) \mu_f(z) \frac{A}{A} \frac{(-\bar{C})}{(-C)}} = \frac{(-\mu_f(z) + \mu_h(z)) \frac{A}{A}}{1 - \mu_h(z) \bar{\mu}_f(z)} \\ &= \frac{\mu_h(z) - \mu_f(z)}{1 - \mu_h(z) \cdot \bar{\mu}_f(z)} \cdot \frac{\partial f}{\bar{\partial}f}. \end{aligned}$$

When $C = \bar{\partial}f(z) = 0$ (and hence $D = 0$) we see that $\mu_f(z) = 0$ and so our desired formula in (c) readily follows from formula (ii) in (a). \square

COROLLARY 6.20. *Let $f : \Omega_1 \rightarrow \Omega_2$ and $h : \Omega_2 \rightarrow \Omega_3$ be real-differentiable functions with complex dilatations μ_f and μ_h , respectively. Then we have*

- a) *If h is conformal, then $\mu_{h \circ f} = \mu_f$.*
- b) *If f is conformal, then $\mu_{h \circ f} = (\mu_h \circ f) \frac{\bar{f}'}{f'}$.*

PROOF. Both (a) and (b) follow directly from Lemma 6.18 (a) noting that $\mu_h \equiv 0$ when h is conformal and noting that $\bar{\partial}f(z) = 0$ and $\partial f(z) = f'(z)$ when f is conformal. \square

REMARK 6.21. *By Corollary 6.20 we see that the modulus of the complex dilatation is preserved by pre and post compositions with conformal maps. Hence, \tilde{K}_f is similarly preserved.*

We will now investigate how μ_f is related to \tilde{K}_f for certain maps. We begin with an obvious lemma.

LEMMA 6.22. *Let $f : \Omega_1 \rightarrow \Omega_2$ be a sense-preserving function between plane domains that is differentiable in the real sense at $z_0 \in \Omega_1$ and let $K \geq 1$. Then the statements*

$$(i) \quad |\mu_f(z_0)| \leq \frac{K-1}{K+1} \quad \text{and} \quad (ii) \quad \tilde{K}_f(z_0) = \frac{|\partial f(z_0)| + |\bar{\partial}f(z_0)|}{|\partial f(z_0)| - |\bar{\partial}f(z_0)|} \leq K$$

are equivalent.

PROOF. Note that if in either statement $K = 1$ holds, then $\mu(z_0) = 0$ and thus $\bar{\partial}f(z_0) = 0$. So we only need to consider the case where $K > 1$. Assuming the first inequality, we find

$$\begin{aligned} \left| \frac{\bar{\partial}f(z_0)}{\partial f(z_0)} \right| = |\mu_f(z_0)| &\leq \frac{K-1}{K+1} \Rightarrow \frac{K+1}{K-1} \left| \frac{\bar{\partial}f(z_0)}{\partial f(z_0)} \right| \leq 1 \\ \Rightarrow \frac{K|\bar{\partial}f(z_0)| + |\bar{\partial}f(z_0)|}{K|\partial f(z_0)| - |\partial f(z_0)|} &\leq 1 \Rightarrow K|\bar{\partial}f(z_0)| + |\bar{\partial}f(z_0)| \leq K|\partial f(z_0)| - |\partial f(z_0)| \\ \Rightarrow |\bar{\partial}f(z_0)| + |\partial f(z_0)| &\leq K(|\partial f(z_0)| - |\bar{\partial}f(z_0)|) \Rightarrow \frac{|\partial f(z_0)| + |\bar{\partial}f(z_0)|}{|\partial f(z_0)| - |\bar{\partial}f(z_0)|} \leq K, \end{aligned}$$

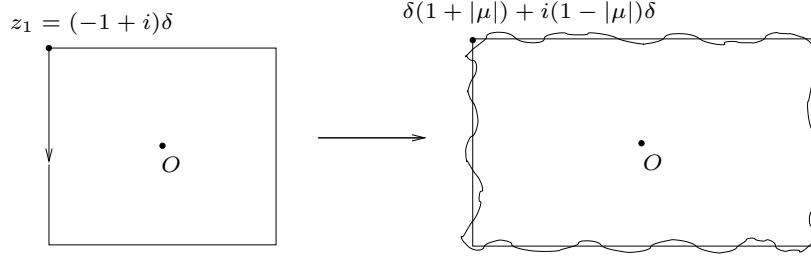
noting that the denominator in the last inequality is nonzero since $|\mu_f(z_0)| < 1$. We then unravel the implications in reverse order to see the reverse implication. \square

THEOREM 6.23. *Let $f : \Omega_1 \rightarrow \Omega_2$ be a K -QC function between plane domains. Suppose that f is differentiable in the real sense at $z_0 \in \Omega_1$ and $df_{z_0}(z) \neq 0$. Then*

$$\tilde{K}_f(z_0) = \frac{|\partial f(z_0)| + |\bar{\partial}f(z_0)|}{|\partial f(z_0)| - |\bar{\partial}f(z_0)|} \leq K, \quad \text{or equivalently,} \quad |\mu_f(z_0)| \leq \frac{K-1}{K+1}.$$

PROOF. Under the assumptions of the theorem we want to show that $|\mu_f(z_0)| \leq \frac{K-1}{K+1}$. Since translations are conformal and pre and post composition with conformal maps preserves the modulus of complex dilatation (Corollary 6.20), we can, without loss of generality, assume that $f(z_0) = 0 = z_0$. Since f is real-differentiable at z_0 we can then write $f(z) = Az + B\bar{z} + o(z)$. Therefore $|\mu_f(0)| = \frac{|B|}{|A|} = |\mu_L(0)|$ where $L(z) = Az + B\bar{z}$. Recall that equation 3.10 allows us to write $L = g \circ L_{|\mu|} \circ h$ where $h(z) = e^{-i\frac{\theta}{2}} \cdot z$, $g(z) = |A| e^{i(\alpha + \frac{\theta}{2})} \cdot z$ and $L_{|\mu|}(z) = z + |\mu|\bar{z}$, where $\mu = \mu_f(0)$.

Consider $H = g^{-1} \circ f \circ h^{-1}$. Since h and g are just rotations we have that $dh^{-1} = h^{-1}$, $dg^{-1} = g^{-1}$ and, recalling Remark 6.17, we have that $dH = dg^{-1} \circ df \circ dh^{-1} = g^{-1} \circ L \circ h^{-1} = L_{|\mu|}$. This shows that $H(z) = z + |\mu|\bar{z} + o(z)$. This last expression of H together with the fact that $H = g^{-1} \circ f \circ h^{-1} \in K$ -QC (since g^{-1} and h^{-1} are conformal) will allow us below to show that $|\mu_f(0)| = |\mu_H(0)| \leq \frac{K-1}{K+1}$ which will finish the proof of the theorem.

FIGURE 6. Illustration of mapping $H(z)$

For $\delta > 0$ consider the oriented square $Q((-1+i)\delta, (-1-i)\delta, (1-i)\delta, (1+i)\delta)$. Then $H((1+i)\delta) = \delta(1+|\mu|) + i(1-|\mu|)\delta + o(\delta)$ (see Figure 6). By Rengel's inequality we have that $M(H(Q)) \leq \frac{\text{area}(H(Q))}{s^2} = \frac{[4\delta^2(1-|\mu|^2) + o(\delta^2)]}{[2\delta(1+|\mu|) + o(\delta)]^2} = \frac{[4(1-|\mu|^2) + o(1)]\delta^2}{[2(1+|\mu|) + o(1)]^2\delta^2} = \frac{4(1-|\mu|^2) + o(1)}{[2(1+|\mu|)]^2 + o(1)} = \frac{1-|\mu|}{1+|\mu|} + o(1)$ as we let $\delta \rightarrow 0$ (using Lemma 2.2 (d) in the last step). Also, by Theorem 6.13 (a), noting that the conformal module of the square Q is 1, we have $\frac{1}{K} = \frac{M(Q)}{K} \leq M(H(Q)) \leq \frac{1-|\mu|}{1+|\mu|} + o(1)$. Letting $\delta \rightarrow 0$ we see that $\frac{1+|\mu|}{1-|\mu|} \leq K$. Therefore $\frac{K-1}{K+1} \geq |\mu_H(z_0)|$, as desired. \square

THEOREM 6.24. *Let $f : \Omega_1 \rightarrow \Omega_2$ be a C^1 sense-preserving bijection between plane domains with $J_f > 0$ on Ω_1 . If $\sup_{z \in \Omega_1} |\mu_f(z)| \leq \frac{K-1}{K+1}$ (equivalently $\tilde{K}_f \leq K$) for some K , then f is K -quasiconformal.*

REMARK 6.25. *Theorem 6.24 shows that a C^1 function that is K -QC under the tentative definition in Definition 4.3 is quasiconformal under our main geometric definition in Definition 6.11. However, not all quasiconformal maps are C^1 (or even everywhere real differentiable) and so these definitions are not equivalent.*

REMARK 6.26. *The complex dilatation μ_f for a real-differentiable function f measures the deviation from conformality. So for a function to be quasiconformal, there must be a bound on μ_f . Likewise, the closer μ_f is to zero, the closer f is to being conformal.*

PROOF. (Theorem 6.24) Let quadrilateral $Q \subseteq \bar{Q} \subseteq \Omega_1$. Let $f_1 : Q \rightarrow R(a)$ and $f_2 : f(Q) \rightarrow R(a')$ be as in Theorem 6.3. Quasiconformality of f is shown if we can show that $a' \leq Ka$. Set $g = f_2 \circ f \circ f_1^{-1} : R(a) \rightarrow R(a')$.

Since f is a C^1 bijection between plane domains with $J_f > 0$ and f_1, f_2 are both conformal, g is also a C^1 bijection between plane domains with $J_g > 0$ (since J_g is the product of $J(f_2), J(f)$ and $J(f_1^{-1})$ evaluated at corresponding points according to the chain rule). Therefore the partial derivative $\frac{\partial g}{\partial x}$ exists and is continuous in $R(a)$. By considering Figure 7 we see that, fixing $0 < y_0 < 1$, we have

$$a' \leq |g(a + iy_0) - g(iy_0)| \leq \int_0^a \left| \frac{\partial g}{\partial x}(x + iy_0) \right| dx \leq \left\{ \int_0^a \left[\frac{\partial g}{\partial x}(x + iy_0) \right]^2 dx \right\}^{\frac{1}{2}} \cdot a^{\frac{1}{2}},$$

where the last inequality follows from the Cauchy-Schwarz inequality (see Appendix I). Since $\sup_{z \in \Omega_1} |\mu_f(z)| \leq \frac{K-1}{K+1}$ and f_1^{-1}, f_2 are both conformal, by Corollary 6.20 (a)

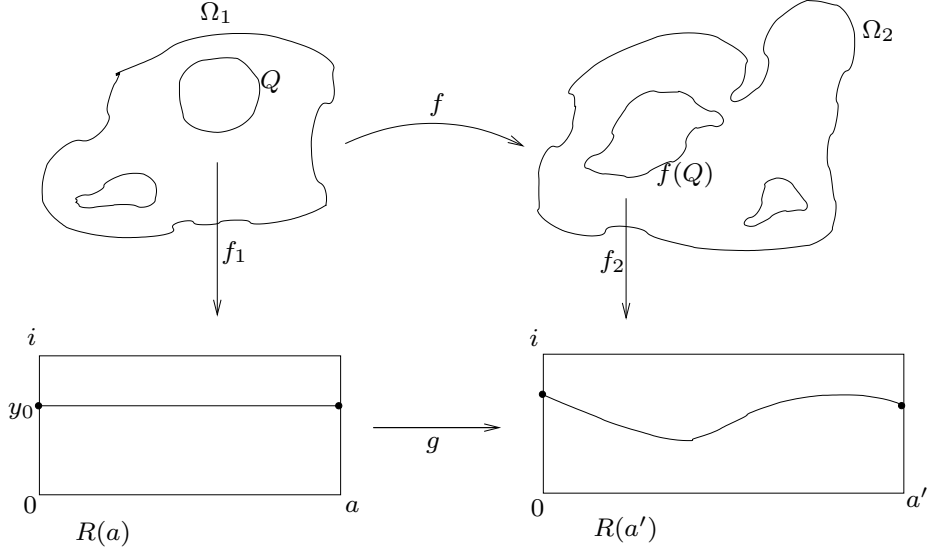


FIGURE 7

and (b), we see that $\sup_{z \in \Omega_1} |\mu_g(z)| \leq \frac{K-1}{K+1}$. But Lemma 6.22 lets us alternatively assume for all $z \in R(a)$ that $\frac{|\partial g(z)| + |\bar{\partial} g(z)|}{|\partial g(z)| - |\bar{\partial} g(z)|} \leq K$ which implies that $\frac{(|\partial g(z)| + |\bar{\partial} g(z)|)^2}{|\partial g(z)|^2 - |\bar{\partial} g(z)|^2} \leq K$. But $J_g(z) = |\partial g(z)|^2 - |\bar{\partial} g(z)|^2$ and $\left| \frac{\partial g}{\partial x}(z) \right| = |\partial g(z) + \bar{\partial} g(z)| \leq |\partial g(z)| + |\bar{\partial} g(z)|$ yields $\left| \frac{\partial g}{\partial x} \right|^2 \leq J_g \cdot K$. We use this fact in the following list of inequalities. We have

$$(a')^2 = \int_0^1 (a')^2 dy \leq a \int_0^1 \int_0^a \left| \frac{\partial g}{\partial x} \right|^2 (x + iy) dx dy = a \iint_{R(a)} \left| \frac{\partial g}{\partial x} \right|^2 (x + iy) dx dy$$

$$\leq a \iint_{R(a)} K \cdot J_g(x + iy) dx dy = aK \iint_{R(a')} 1 dx dy = aKa'. \quad \square$$

Thus $a' \leq Ka$ as desired. \square

We now discuss two basic examples of quasiconformal maps.

EXAMPLE 6.27. a) Let g map the plane onto itself by $g(z) = Az + B\bar{z}$ for some complex numbers A and B such that $|A| > |B| \geq 0$. Then, by Theorem 6.24, g is K -quasiconformal with $K = \frac{|A| + |B|}{|A| - |B|}$.

b) Let $f(z) = kx + iy$ where $k \geq 1$ and $z = x + iy$. Then $f(z) = k\left(\frac{z + \bar{z}}{2}\right) + i\left(\frac{z - \bar{z}}{2i}\right) = \left(\frac{k}{2} + \frac{1}{2}\right)z + \left(\frac{k}{2} - \frac{1}{2}\right)\bar{z}$. So $\frac{|\partial f(z)| + |\bar{\partial} f(z)|}{|\partial f(z)| - |\bar{\partial} f(z)|} = \frac{\frac{k+1}{2} + \frac{k-1}{2}}{\frac{k+1}{2} - \frac{k-1}{2}} = k$. Therefore f is k -quasiconformal by Theorem 6.24. If $0 < k < 1$, then f is $\frac{1}{k}$ -QC by a similar calculation.

Essentially, for a function f to be quasiconformal there must be a bound, locally, on how much f can stretch in any direction compared to any other direction. However this is not a global bound. For example, there are conformal maps $f : \{z : |z| < 1\} \rightarrow \{z : \operatorname{Re} z > 0\}$. So here there is an unbounded stretch in the global sense even though conformality implies no stretch in one direction more than any other locally.

7. Metric Definition of Quasiconformal Maps

Recall from the introduction that there are several equivalent definitions of quasiconformal maps. One of these is the metric definition. We previously mentioned the complex dilatation of f at z_0 . But for the metric definition we will now introduce the notion of *circular* dilatation at z_0 .

DEFINITION 7.1. *Let $f : \Omega_1 \rightarrow \Omega_2$ be a function between plane domains. For $z_0 \in \Omega_1$, we define $H_f(z_0) = \limsup_{r \rightarrow 0} \frac{\sup_{\theta} |f(z_0 + re^{i\theta}) - f(z_0)|}{\inf_{\theta} |f(z_0 + re^{i\theta}) - f(z_0)|}$ to be the circular dilatation of f at z_0 .*

REMARK 7.2. *Intuitively, H_f measures the local ratio of the maximum stretch to the minimum stretch of f . Thus, we expect $H_f(z_0)$ to equal $\tilde{K}_f(z_0) = \frac{\max_{\theta} |\partial_{\theta} f|}{\min_{\theta} |\partial_{\theta} f|}$, as calculated in Section 4, when f is real-differentiable at z_0 . However, we also note that $H_f(z_0)$ exists whether or not f is differentiable at z_0 .*

LEMMA 7.3. *If $f : \Omega_1 \rightarrow \Omega_2$ between plane domains is differentiable in the real sense at z_0 , and $J_f(z_0) > 0$, then $H_f(z_0) = \frac{|\partial f(z_0)| + |\bar{\partial} f(z_0)|}{|\partial f(z_0)| - |\bar{\partial} f(z_0)|} = \frac{\max_{\theta} |\partial_{\theta} f|}{\min_{\theta} |\partial_{\theta} f|} = \frac{1 + |\mu_f(z_0)|}{1 - |\mu_f(z_0)|} = \tilde{K}_f(z_0)$.*

The proof follows the same line of calculation as the proof of Lemma 3.9 and so will be omitted.

We will now state two theorems without proof. The reader may refer to [3], p. 177 for the first theorem and [3], p. 178 for the second. We will simply quote these theorems as needed.

THEOREM 7.4. *Suppose $f : \Omega_1 \rightarrow \Omega_2$ is K -quasiconformal. Then,*
a) for every $z_0 \in \Omega_1$, we have $H_f(z_0) \leq \lambda(K)$ for some positive number $\lambda(K)$; and
b) we have $\|H_f\|_{\infty} \leq K$.

THEOREM 7.5. *Suppose $f : \Omega_1 \rightarrow \Omega_2$ is a sense-preserving homeomorphism such that $H_f(z) < +\infty$ at each point of Ω_1 and $H_f(z) \leq K$ almost everywhere in Ω_1 . Then f is K -quasiconformal.*

REMARK 7.6. *It is actually sufficient, in Theorem 7.5, to replace the assumption that that $H_f(z) < +\infty$ everywhere on Ω_1 by the assumption that $H_f(z) < +\infty$ **off** a set of σ -finite linear measure ([3], p. 178). However, once f is shown to be \bar{K} -quasiconformal, we see, by Theorem 7.4, that $H_f(z)$ must be bounded everywhere.*

COROLLARY 7.7. *If $f : \Omega_1 \rightarrow \Omega_2$ is K -quasiconformal and has circular dilatation of 1 almost everywhere, then f is conformal.*

PROOF. By Theorem 7.4 we have that $H_f(z_0) \leq \lambda(K)$ for all $z_0 \in \Omega_1$. Also $\|H_f(z)\|_{\infty} = 1$. So by Theorem 7.5 f is 1-quasiconformal. Hence by Theorem 6.15 we see that f is conformal. \square

Caution – If a homeomorphism f has circular dilatation equal to 1 almost everywhere, but is not known to be quasiconformal, it is not necessarily true that f is conformal. A counterexample for this is the function $f(z) = z + ig(x)$ defined on the unit square where $g(x)$ is the Cantor function and $x = \text{Re}z$.

8. Analytic Characterization

In this section we address the connections between differentiability conditions and quasiconformality. Since the details are not provided, the reader may wish to refer the discussion beginning on page 162 in [3] for further examination. We begin with a definition.

DEFINITION 8.1. *We say that a complex valued continuous map $f(x, y)$ on a domain $\Omega \subset \overline{\mathbb{C}}$ is absolutely continuous on lines if in every rectangle $R = \{x + iy : a < x < b, c < y < d\}$, with $\overline{R} \subset \Omega \setminus \{\infty, f^{-1}(\infty)\}$, the map $f(x, \cdot)$ is absolutely continuous for almost all $a < x < b$ and the map $f(\cdot, y)$ is absolutely continuous for almost all $c < y < d$.*

Clearly any map satisfying Definition 8.1 has both partial derivatives f_x and f_y existing a.e. in Ω . But it is well known that a function with partial derivatives defined at a point z_0 is not necessarily real differentiable at z_0 . However, we do have the following somewhat surprising result whose proof we omit.

THEOREM 8.2 ([3], p. 128 and 130). *A continuous open map f on a domain Ω in $\overline{\mathbb{C}}$ having finite partial derivatives a.e. in Ω is real differentiable a.e. in Ω .*

By showing that a quasiconformal map f is indeed absolutely continuous on lines we can conclude a.e. real differentiability and, indeed, in the manner of the proof of Theorem 6.23 obtain a bound on μ_f , which in turn implies conditions on J_f and df . The details can be found in [3] starting in p. 162. We summarize these results as follows.

LEMMA 8.3. *Let $f : \Omega_1 \rightarrow \Omega_2$ be K -quasiconformal between plane domains. Then we have all of the following*

- a) *f is absolutely continuous on lines;*
- b) *f is differentiable in the real sense almost everywhere;*
- c) *$|\partial f(z)| \neq 0$ a.e. and $|\mu_f(z)| \leq \frac{K-1}{K+1}$ almost everywhere.*
- d) *$J_f(z) > 0$ a.e.*
- e) *$df_{z_0} \neq 0$ a.e.*

It turns out that conditions (a) and (c) are enough to ensure that a map f is quasiconformal. This leads us to the so-called analytic definition of quasiconformality which we now state as a lemma.

LEMMA 8.4. ([3], p. 168) *Suppose a homeomorphism $f : \Omega_1 \rightarrow \Omega_2$ between domains in $\overline{\mathbb{C}}$ is absolutely continuous on lines and $|\mu_f(z)| \leq \frac{K-1}{K+1}$ almost everywhere. Then f is K -quasiconformal.*

THEOREM 8.5. *Suppose $f : \Omega_1 \rightarrow \Omega_2$ is quasiconformal with $\mu_f(z) = 0$ almost everywhere in Ω_1 . Then f is conformal.*

PROOF. Since f is quasiconformal, we see by Lemma 8.3 that f is absolutely continuous on lines. Since $\mu_f(z) = 0$ almost everywhere, Lemma 8.4 applies with $K = 1$. Thus f is 1-quasiconformal and hence is conformal. \square

Caution – If a homeomorphism f has complex dilatation μ_f equal to 0 almost everywhere, but is not known to be quasiconformal, it is not necessarily true that f is conformal. A counterexample for this is the function $f(z) = z + ig(x)$ defined on the unit square where $g(x)$ is the Cantor function and $x = \operatorname{Re} z$.

9. Beltrami Equations

Lemma 8.3 shows us that a quasiconformal map f satisfies $\|\mu_f\|_\infty < 1$ on its domain. We will now ask, and answer (see Theorem 9.8) the question: Given a complex valued function μ with $\|\mu\|_\infty < 1$ on an open set Ω , can we find a quasiconformal map f such that $\mu_f = \mu$.

We begin with a definition.

DEFINITION 9.1. *We call a complex valued function μ on a domain Ω a Beltrami coefficient if μ is Lebesgue measurable and $\|\mu\|_\infty < 1$.*

REMARK 9.2. *Given Beltrami coefficients μ and ν on an open set Ω_1 , we will often write $\mu = \nu$ to mean $\mu = \nu$ almost everywhere since in what follows it is only important that equality holds almost everywhere.*

REMARK 9.3. *By Lemma 8.3, a K -quasiconformal map f has μ_f defined and satisfying $|\mu_f| \leq \frac{K-1}{K+1}$ almost everywhere. Thus μ_f is a Beltrami coefficient.*

LEMMA 9.4. *If μ_f and μ_h are Beltrami coefficients for quasiconformal maps f and h , then,*

$$\begin{aligned} \text{a) } \mu_{h \circ f}(z) &= \frac{\mu_f(z) + \mu_h \circ f(z) \cdot \frac{\overline{\partial f(z)}}{\partial f(z)}}{1 + \mu_h \circ f(z) \cdot \mu_f(z) \cdot \frac{\overline{\partial f(z)}}{\partial f(z)}} \text{ where } f : \Omega_1 \rightarrow \Omega_2 \text{ and } h : \Omega_2 \rightarrow \Omega_3; \\ \text{b) } \mu_{h \circ f^{-1}}(f(z)) &= \frac{\mu_h(z) - \mu_f(z)}{1 - \mu_h(z) \cdot \overline{\mu_f(z)}} \cdot \frac{\partial f(z)}{\overline{\partial f(z)}} \text{ where } f : \Omega_1 \rightarrow \Omega_2, h : \Omega_2 \rightarrow \Omega_3. \end{aligned}$$

REMARK 9.5. *Note that when $\overline{\partial f(z)} = 0$ we have $\mu_f(z) = 0$ and so we regard the formula in (a) as collapsing to the formula in Lemma 6.18 (a)(ii) despite the fact that the term $\frac{\overline{\partial f(z)}}{\partial f(z)}$ is not formally defined.*

PROOF. Noting that since all maps involved are quasiconformal, Lemma 8.3 shows that Lemma 6.18 may be applied, from which (a) and (b) readily follow. However, it should be noted that we inherently used the fact (see [3], p. 165) that the image of a null set under a quasiconformal map is a null set, and so one can safely assume in (a), for example, that the set of points z_0 where both f is \mathbb{R} -differentiable at z_0 and g is \mathbb{R} -differentiable at $f(z_0)$ is of full measure in Ω_1 . Indeed, this set is $\Omega_1 \setminus (\{z : f \text{ is not differentiable}\} \cup \{f^{-1}(w) : g \text{ is not differentiable at } w\})$. \square

COROLLARY 9.6. *Let $f : \Omega_1 \rightarrow \Omega_2$, $g : \Omega_1 \rightarrow \Omega_3$, and $h : \Omega_2 \rightarrow \Omega_4$ be quasiconformal maps. Then,*

- a) *We have that $\mu_f = \mu_g$ almost everywhere if and only if $f \circ g^{-1}$ is conformal;*
- b) *If h is conformal, then $\mu_{h \circ f} = \mu_f$ almost everywhere;*
- c) *If f is conformal, then $\mu_{h \circ f} = (\mu_h \circ f) \frac{\overline{f'}}{f'}$ almost everywhere.*

PROOF. a) “ \Rightarrow ” As a composition of quasiconformal maps it follows that $f \circ g^{-1}$ is K -quasiconformal where $K = K_f \cdot K_g$. Now by the previous lemma, $\mu_{f \circ g^{-1}} = \frac{\mu_f - \mu_g}{1 - \mu_f \mu_g} \cdot \frac{\partial g}{\partial f}$. But $\mu_f = \mu_g$ almost everywhere implies, then, $\mu_{f \circ g^{-1}} = \frac{\mu_f - \mu_g}{1 - \mu_f \mu_g} \cdot \frac{\partial g}{\partial f} = 0$ almost everywhere. Hence, by Theorem 8.5 we get that $f \circ g^{-1}$ is conformal.

“ \Leftarrow ” Assuming $f \circ g^{-1}$ is conformal, we have that $\mu_{f \circ g^{-1}} = 0$ which by Lemma 9.4 (b) is true only when $\mu_f - \mu_g = 0$ (since $\partial f \neq 0$ a.e. by Lemma 8.3(c)). Therefore $\mu_f = \mu_g$.

Parts (b) and (c) are direct consequences of Lemma 9.4 (a). \square

THEOREM 9.7. *Let μ be a Beltrami coefficient on the plane $\overline{\mathbb{C}}$. Let $z_1, z_2,$ and z_3 be distinct points in the plane and let $w_1, w_2,$ and w_3 also be distinct points in the plane. Then there exists a unique quasiconformal map f from the plane to itself such that $\bar{\partial}f = \mu \cdot \partial f$ almost everywhere and $f(z_n) = w_n$ for $n = 1, 2, 3$.*

PROOF. – We will only show uniqueness. The reader may refer to ([3], p. 194) for existence. Suppose that both f_1, f_2 satisfy the statement of the theorem. Then $T = f_2^{-1} \circ f_1$ fixes $z_1, z_2,$ and z_3 . But since $\mu_{f_1} = \mu = \mu_{f_2}$, we know that T is conformal. Thus T is a Möbius map that fixes 3 distinct points and therefore is the identity map. Hence $f_1 = f_2$. \square

We will state the following theorem without proof. The reader may refer to ([3], p. 194) for the proof of this important theorem.

THEOREM 9.8 (Existence Theorem). *Let μ be a Beltrami coefficient on an open subset U of the plane $\overline{\mathbb{C}}$. Then there exists a quasiconformal map f on U with $\mu_f = \mu$ almost everywhere in U .*

10. Riemann Surface Structures

In order to understand and prove the Fundamental Theorem of Quasiconformal Surgery it will be useful to understand Riemann Surfaces. In particular, we will focus on domains in $\overline{\mathbb{C}}$ that have a “conformal structure” induced by Beltrami coefficients. Much like the way the topology on a set determines which functions are continuous, we will see that the conformal structure on a set determines which functions are analytic. We now develop the necessary ideas.

DEFINITION 10.1. *A manifold of dimension n is a connected Hausdorff space X for which every point has a neighborhood U that is homeomorphic to an open subset V of \mathbb{R}^n . Such a homeomorphism $f : U \rightarrow V$ is called a chart.*

DEFINITION 10.2. *A two-dimensional manifold is called a surface.*

DEFINITION 10.3. *Two charts $f_\alpha : U_\alpha \rightarrow \mathbb{C}$ and $f_\beta : U_\beta \rightarrow \mathbb{C}$, whose domains intersect, are called compatible if the maps $f_\beta \circ f_\alpha^{-1} : f_\alpha(U_\alpha \cap U_\beta) \rightarrow f_\beta(U_\alpha \cap U_\beta)$ and $f_\alpha \circ f_\beta^{-1} : f_\beta(U_\alpha \cap U_\beta) \rightarrow f_\alpha(U_\alpha \cap U_\beta)$ are holomorphic (i.e., analytic). If A is a collection of compatible charts and if any x in X is in the domain of some f in A , then we say that A is an atlas. When we endow X with an atlas A , we say that (X, A) is a Riemann surface. See Figure below.*

DEFINITION 10.4. *Let U and V be Riemann Surfaces and $\Phi(U)$ and $\Phi(V)$ be their respective atlases. We say that $f : U \rightarrow V$ is analytic between Riemann surfaces if for any $\varphi \in \Phi(U)$ and $\psi \in \Phi(V)$ such that $f(U)$ meets the domain of ψ we have $\psi \circ f \circ \varphi^{-1}$ is analytic (in the usual sense) on its domain. See Figure 9.*

We will be using the following notation when talking about Riemann surfaces whose underlying set is an open subset of the plane $\overline{\mathbb{C}}$. Let U be an open subset of the plane, and let μ be a Beltrami coefficient on U . Then $U[\mu]$, sometimes called U with the conformal structure μ , denotes the Riemann Surface U with atlas $A(\mu) = \{\varphi : U \rightarrow \overline{\mathbb{C}} \mid \varphi \text{ is quasiconformal with } \mu_\varphi = \mu \text{ almost everywhere}\}$, where $\overline{\mathbb{C}}$ denotes the plane. The set $A(\mu)$, also called the set of all μ -conformal maps on

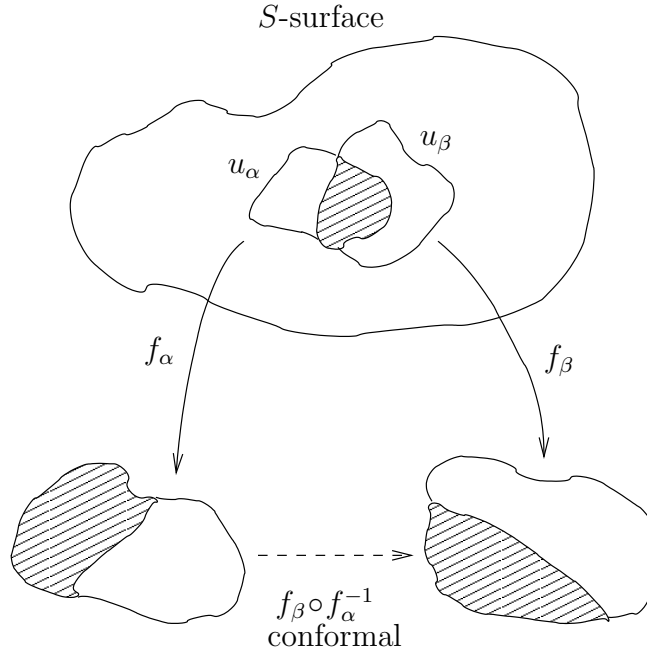


FIGURE 8

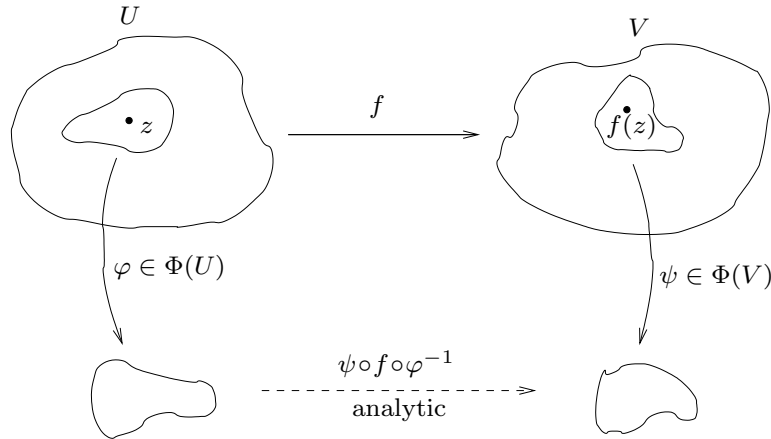


FIGURE 9

U , is nonempty by the Existence Theorem 9.8. Also, for any two such maps φ and ψ , we see that $\varphi \circ \psi^{-1}$ is conformal by Corollary 9.6 (a) and so $A(\mu)$ is indeed an atlas. We denote the Riemann surface with the atlas $A(0)$ of all conformal maps on U by $U[0]$, which we will often simply write as U .

REMARK 10.5. When checking to see if $f : U[\eta] \rightarrow V[\nu]$ is analytic between Riemann surfaces, we note that it suffices to use only a single element from each atlas, instead of checking all charts. Indeed, if $\varphi_1 \in \Phi(U)$ and $\psi_1 \in \Phi(V)$ are such that $\psi_1 \circ f \circ \varphi_1^{-1}$ is analytic, then for any $\varphi_2 \in \Phi(U)$ and $\psi_2 \in \Phi(V)$ we

must have that $\psi_2 \circ f \circ \varphi_2^{-1}$ is also analytic. This follows since $\psi_2 \circ f \circ \varphi_2^{-1} = (\psi_2 \circ \psi_1^{-1}) \circ (\psi_1 \circ f \circ \varphi_1^{-1}) \circ (\varphi_1 \circ \varphi_2^{-1})$ is a composition of analytic (in the usual sense) maps.

LEMMA 10.6. *Let U and V be domains in the plane $\overline{\mathbb{C}}$. Let ν be a Beltrami coefficient on V . If $\alpha : U \rightarrow V$ is quasiconformal with dilatation μ_α , then we can find a unique Beltrami coefficient η on U such that the map $\alpha : U[\eta] \rightarrow V[\nu]$ is analytic between Riemann surfaces. Here we call η the “pull back” of ν by α and we denote this pull back by $\eta = \alpha^*\nu$.*

PROOF. Let $\varphi \in A(\nu)$. Since both α, φ are quasiconformal, their composition $\varphi \circ \alpha$ is quasiconformal. We set $\eta = \mu_{\varphi \circ \alpha}$ which is a Beltrami coefficient by Remark 9.3. Now to show analyticity between Riemann surfaces we must choose $\psi \in A(\eta)$. Since $\psi, \varphi \circ \alpha \in A(\eta)$, we see that $\varphi \circ \alpha \circ \psi^{-1}$ is conformal. Hence $\alpha : U[\eta] \rightarrow V[\nu]$ between Riemann surfaces is analytic.

The uniqueness follows from Corollary 9.6 (a). \square

REMARK 10.7. *We note that when we pull back a Beltrami coefficient by a quasiconformal map we still get a Beltrami coefficient.*

We will use this “pull back” method often throughout the rest of this paper. This method lets us put a structure on a set so that we have the correct dilatation to insure analyticity of a given map. We will next show that we can “pull back” conformal structures by not only quasiconformal maps, but also by analytic (in the usual sense) and quasiregular maps as well. In order to show this we need the following result.

THEOREM 10.8. *Let $f : U[\mu] \rightarrow V[\nu]$ and $g : V[\nu] \rightarrow W[\eta]$ each be analytic between Riemann surfaces. Then $g \circ f : U[\mu] \rightarrow W[\eta]$ is analytic.*

PROOF. Choose $\varphi \in A(\mu)$, $\psi \in A(\nu)$, and $\beta \in A(\eta)$ arbitrarily. Since f is analytic between Riemann surfaces, we have that $\psi \circ f \circ \varphi^{-1}$ is analytic. Similarly $\beta \circ g \circ \psi^{-1}$ is analytic. Therefore $\beta \circ g \circ f \circ \varphi^{-1} = (\beta \circ g \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1})$ is analytic. Hence $g \circ f : U[\mu] \rightarrow W[\eta]$ is analytic between Riemann surfaces. \square

LEMMA 10.9. ([1], p. 182) *Let η and ν be Beltrami coefficients on domains of the plane U and V respectively. Let $g : U \rightarrow V$ be analytic and nonconstant. Then the following are equivalent:*

a) $g : U[\eta] \rightarrow V[\nu]$ is analytic.

b) $\nu(g(z)) = \frac{g'(z)}{g'(z)}\eta(z)$ almost everywhere in U .

PROOF. First, let ψ be ν -conformal on V , i.e., $\mu_\psi = \nu$, and note that $\mu_{\psi \circ g} = \mu_1$ where $\mu_1(z) = \frac{g'(z)}{g'(z)}\nu(g(z))$ almost everywhere by Corollary 9.6 (c), since the analytic map g is locally conformal away from the discrete (in U) set E of points where the multiplicity of g is strictly greater than one.

“(a) \Rightarrow (b)” Let $\varphi \in A(\eta)$. By (a), the map $(\psi \circ g) \circ \varphi^{-1}$ is analytic and therefore conformal away from a discrete set in $\varphi(U)$. Thus, by applying Corollary 9.6 (a) locally to $(\psi \circ g) \circ \varphi^{-1}$, we have $\eta(z) = \mu_\varphi(z) = \mu_{\psi \circ g}(z) = \frac{g'(z)}{g'(z)}\nu(g(z))$ almost everywhere in U . Therefore (b) holds.

“(b) \Rightarrow (a)” Let $\varphi \in A(\eta)$. Now (b) implies that $\psi \circ g$ is η -conformal in a neighborhood of each z in $U \setminus E$. Thus we see that $(\psi \circ g) \circ (\varphi^{-1}) : \varphi(U \setminus E) \rightarrow \overline{\mathbb{C}}$ is

locally conformal. Since $(\psi \circ g) \circ (\varphi^{-1})$ is continuous on all of $\varphi(U)$ and E is a set of isolated points (in U), Riemann's Theorem on removable singularities implies that $(\psi \circ g) \circ (\varphi^{-1})$ from $\varphi(U)$ into $\psi(V)$ is analytic. Thus $g : U[\eta] \rightarrow V[\nu]$ is analytic between Riemann Surfaces. \square

COROLLARY 10.10. *Let U and V be domains in \mathbb{C} and let $g : U \rightarrow V$ be surjective and analytic. Let ν be a Beltrami coefficient on V . Then there exists a unique Beltrami coefficient η on U such that $g : U[\eta] \rightarrow V[\nu]$ is analytic. We denote such η by $\eta = g^*\nu$ and call it the pull back of ν by g .*

PROOF. Set $\eta(z) = \frac{\overline{g'(z)}}{g'(z)}\nu(g(z))$ where $g'(z) \neq 0$, which is almost everywhere. But by Lemma 10.9, this is equivalent to saying that $g : U[\eta] \rightarrow V[\nu]$ is analytic provided that η is Beltrami, which holds since $\|\eta(z)\|_\infty = \|\nu(g(z))\|_\infty < 1$, using the fact that ν is a Beltrami coefficient. Again, the uniqueness follows from applying Corollary 9.6 (a) locally on the set $U \setminus \{z : g'(z) = 0\}$. \square

REMARK 10.11. *We see here that when we pull back a Beltrami coefficient by an analytic map the infinity norm of the Beltrami coefficient does not increase.*

LEMMA 10.12. *Let R be a rational map from the plane $\overline{\mathbb{C}}$ to itself, and let φ , mapping the plane to itself, be μ -conformal for some Beltrami coefficient μ . Then $\varphi R \varphi^{-1}$ is rational if and only if $\mu(R(z)) = \frac{R'(z)}{R'(z)} \cdot \mu(z)$ almost everywhere in the plane. Furthermore, when this is true, we have $\deg R = \deg \varphi R \varphi^{-1}$.*

PROOF. “ \Rightarrow ” Suppose $\varphi R \varphi^{-1}$ from the plane to itself is rational. Then $\varphi R \varphi^{-1}$ must be analytic in the usual sense because it is a rational map from the plane to itself. Therefore, by definition of a function being analytic between Riemann surfaces, noting that φ is μ -conformal, we see $R : \overline{\mathbb{C}}[\mu] \rightarrow \overline{\mathbb{C}}[\mu]$ is analytic. So by Lemma 10.9, we see that $\mu(R(z)) = \frac{R'(z)}{R'(z)} \cdot \mu(z)$ almost everywhere in the plane $\overline{\mathbb{C}}$.

“ \Leftarrow ” Now suppose that $\mu(R(z)) = \frac{R'(z)}{R'(z)} \cdot \mu(z)$ almost everywhere in the plane. By Lemma 10.9 we have that $R : \overline{\mathbb{C}}[\mu] \rightarrow \overline{\mathbb{C}}[\mu]$ is analytic. But analyticity between Riemann surfaces implies that $\varphi R \varphi^{-1}$ is analytic between the plane and itself since $\varphi \in A(\mu)$. However, the only analytic maps from the plane to itself are rational maps (see [5], p. 358).

As to the $\deg \varphi R \varphi^{-1}$, we know that φ and φ^{-1} are injective since they are quasiconformal. Therefore $\deg R = \deg \varphi R \varphi^{-1}$. \square

DEFINITION 10.13. *A quasi-regular map is a composite map $f \circ \varphi$ where φ is quasiconformal on the plane and f is rational.*

LEMMA 10.14. *Let g be a quasi-regular map. If $g : U \rightarrow V$ is surjective, and ν is a Beltrami coefficient on V , then there exists a unique Beltrami coefficient η on U such that $g : U[\eta] \rightarrow V[\nu]$ is analytic. We denote $\eta = g^*\nu$ and call η the “pull back” of ν by g .*

PROOF. Let $g = f \circ \varphi$ where φ is quasiconformal on the plane and f is rational. Using Lemma 10.6 and Corollary 10.10, we set $\eta = \varphi^*(f^*\nu)$. Thus we have $U[\varphi^*(f^*\nu)] \xrightarrow{\varphi} \varphi(U)[f^*\nu] \xrightarrow{f} V[\nu]$ and note that $g = f \circ \varphi : U[\eta] \rightarrow V[\nu]$ is analytic by Theorem 10.8.

Uniqueness follows as in the proofs of Lemma 10.6 and Corollary 10.10. \square

REMARK 10.15. We summarize Lemma 10.6, Corollary 10.10 and Lemma 10.14 by stating that for Beltrami coefficients η and ν on domains U and V , respectively, a function $f : U[\eta] \rightarrow V[\nu]$ is analytic if and only if $\eta = \mu_{\psi \circ g}$ for any $\psi \in A(\nu)$ if and only if $\eta = f^*\nu$ (almost everywhere), whether $f : U \rightarrow V$ is quasiconformal, analytic, or quasi-regular. We will use this fact in the following arguments without mention.

LEMMA 10.16. Suppose surjective $g : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is continuous and locally quasiconformal away from a finite set of points K (but not necessarily a global homeomorphism). If $\|\mu_g\|_\infty < 1$, then g is quasi-regular.

PROOF. Let $\varphi : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ be quasiconformal with $\mu_\varphi = \mu_g$, which is guaranteed by the Existence Theorem. Set $A = g \circ \varphi^{-1}$. Then away from $\varphi(K)$ we see that A is locally quasiconformal. Now since $\mu_\varphi = \mu_g$, we have that A is locally conformal away from $\varphi(K)$. Also A is continuous at each point in $\varphi(K)$ since both φ^{-1} and g are continuous. Thus by Riemann's Theorem on removable singularities, A is analytic on the entire plane, and hence A is rational. Therefore the map $g = A \circ \varphi$ is quasi-regular. \square

REMARK 10.17. Lemma 10.16 shows that quasi-regular maps are those which are globally continuous, but locally quasiconformal away from a finite set of points – just like rational maps are those which are globally continuous, but locally conformal away from a finite set of points.

11. Fundamental Theorem of Quasiconformal Surgery

Shishikura developed the idea of quasiconformal surgery “to create from given rational functions a new one preserving their dynamics (in some sense)” ([7], p. 7). The Identity Theorem (Principle of Analytic Continuation) ([5], p. 307) often restricts one from “gluing” analytic functions together to form a globally analytic map. Considering a conjugation by quasiconformal maps, however, the “gluing” process is a possibility ([7], p. 7). In this section we present this result (the Fundamental Theorem of Quasiconformal Surgery) along with an important application.

THEOREM 11.1 (Fundamental Theorem of Quasiconformal Surgery). *Let the map $g : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ be quasi-regular. Suppose, for $i = 1, \dots, m$, there are disjoint open subsets E_i of the plane $\overline{\mathbb{C}}$, quasiconformal maps $\Phi_i : E_i \rightarrow E'_i$ and an integer $N \geq 0$ satisfying the following conditions:*

i) $g(E) \subseteq E$ where $E = E_1 \cup \dots \cup E_m$;

ii) $\Phi \circ g \circ \Phi_i^{-1} : E'_i \rightarrow \cup_{i=1}^m E'_i$ is analytic where $\Phi : E \rightarrow \overline{\mathbb{C}}$ is defined by $\Phi|_{E_i} = \Phi_i$;

iii) $\bar{\partial}g = 0$ almost everywhere on $\overline{\mathbb{C}} \setminus g^{-N}(E)$ where $g^{-N}(E)$ is inverse image of E under g^N , the N^{th} iterate of g .

Then there exists a quasiconformal mapping φ of the plane $\overline{\mathbb{C}}$ such that $\varphi \circ g \circ \varphi^{-1}$ is rational. Moreover, $\varphi \circ \Phi_i^{-1}$ is conformal in E'_i and $\bar{\partial}\varphi(z) = 0$ almost everywhere on $\overline{\mathbb{C}} \setminus \cup_{n \geq 0} g^{-n}(E)$.

PROOF. Our goal is to define a Beltrami coefficient σ on the plane for which $g^*\sigma = \sigma$. Then the Existence Theorem guarantees that there is quasiconformal map φ which is a solution to the Beltrami equation $\bar{\partial}\varphi = \sigma \cdot \partial\varphi$. As we shall see, this φ will be the quasiconformal map we seek to prove the theorem.

We begin by defining, on E_i , $\sigma = \Phi_i^* \sigma_0$ where we define $\sigma_0 = 0$ on E'_i , i.e., $\sigma = \mu_{\Phi_i}$ on E_i . This says that Φ_i is a chart on $E_i[\sigma]$. So by definition of the ‘‘pull back’’ we see that $\Phi_i : E_i[\sigma] \rightarrow E'_i[\sigma_0]$ is analytic. In this way σ is defined on all of $E = \cup_{i=1}^m E_i$.

Next we will show that $g^*\sigma = \sigma$ on E , i.e., $g : E[\sigma] \rightarrow E[\sigma]$ is analytic. Without loss of generality, we will assume that E_i and E_j are components of E and that $g(E_i) \subseteq E_j$. Recall that it is enough to check analyticity with respect to one chart from each atlas. Since Φ_i is a chart on E_i and Φ_j is a chart on E_j , we have, by assumption (ii) that $\Phi_j \circ g \circ \Phi_i^{-1} : E'_i \rightarrow E'_j$ is analytic. Therefore we have shown that $g : E_i[\sigma] \rightarrow E_j[\sigma]$ is analytic and, in general, $g : E[\sigma] \rightarrow E[\sigma]$ is analytic.

We will now define σ on $\cup_{n \geq 1} g^{-n}(E)$ by successively pulling back by g . Note that the set $\cup_{n \geq 1} g^{-n}(E)$ is the set of points that are mapped into E under some iterate of g . Consider a component W_1 of E . We endow $g^{-1}(W_1)$ with the conformal structure $g^*\sigma$. We note that for any $z_0 \in g^{-1}(W_1) \cap E$ we have $\sigma(z_0) = g^*\sigma(z_0)$ from above. Thus we may extend σ to $g^{-1}(W_1) \cup E$ by setting $\sigma = g^*\sigma$ on $g^{-1}(W_1) \setminus E$. Performing this procedure on each component of E , we extend σ to all of $g^{-1}(E)$ by $g^*\sigma$. With σ now defined on all of $g^{-1}(E)$, we can similarly extend σ to all of $g^{-2}(E)$ by defining $\sigma|_{g^{-2}(E)} = g^*(\sigma|_{g^{-1}(E)})$. In this manner, we can inductively define σ such that $g^*\sigma = \sigma$ on $\cup_{n \geq 0} g^{-n}(E)$ and so $g : \cup_{n \geq 0} g^{-n}(E)[\sigma] \rightarrow \cup_{n \geq 0} g^{-n}(E)[\sigma]$ is analytic.

Now we set $\sigma = 0$ on the rest of $\bar{\mathbb{C}}$.

We now have σ defined everywhere, but we must now show that σ is a Beltrami coefficient. Note that σ is clearly Lebesgue measurable. So we must show that $\|\sigma\|_\infty < 1$. We know that on E , σ is a Beltrami coefficient. On $g^{-1}(E)$, since we pulled back by a quasi-regular map, σ is again Beltrami. We continue this up to $g^{-N}(E)$ and since we pulled back a finite number of times, we still have σ Beltrami here. However, we claim that for $n > N$ we have $\|\sigma|_{g^{-n}(E)}\|_\infty = \|\sigma|_{g^{-N}(E)}\|_\infty$. Indeed, for any $z \in g^{-(N+1)}(E) \setminus g^{-N}(E)$ condition (iii) gives $\bar{\partial}g(z) = 0$. Since $\sigma(z) = \mu_{\psi \circ g}(z)$ for any $\psi \in A(\sigma|_{g^{-N}(E)})$, we then see by Lemma 6.18(a)(ii) that $|\sigma(z)| = |\sigma(g(z))|$ and thus the pullback by g of $\sigma|_{g^{-N}(E)}$ to $\sigma|_{g^{-(N+1)}(E)}$ does not increase in norm. The claim for general $n > N$ now follows by induction. Hence, noting that $\sigma = 0$ off of $\cup_{n \geq 0} g^{-n}(E)$, we see that σ is a Beltrami coefficient with $\|\sigma\|_\infty = \max_{k=0, \dots, N} \|\sigma|_{g^{-k}(E)}\|_\infty = \|\sigma|_{g^{-N}(E)}\|_\infty < 1$.

Now the Existence Theorem provides a quasiconformal map φ on the plane such that φ is a solution to $\bar{\partial}\varphi = \sigma \cdot \partial\varphi$ on the plane, i.e., $\sigma = \varphi^* \sigma_0$ on the plane where $\sigma_0 \equiv 0$.

We will now show that $\varphi \circ g \circ \varphi^{-1} : \bar{\mathbb{C}}[0] \rightarrow \bar{\mathbb{C}}[0]$ is analytic and hence rational. As in Remark 10.15, it suffices to show that $\mu_{\varphi \circ g} = \mu_\varphi = \sigma$ a.e. in $\bar{\mathbb{C}}$. Indeed, on $\cup_{n \geq 0} g^{-n}(E)$, this equality holds since $g^*\sigma = \sigma$ there. For $z \notin \cup_{n \geq 0} g^{-n}(E)$ we see that $g(z) \notin \cup_{n \geq 0} g^{-n}(E)$ and also $z \notin g^{-N}(E)$, which by (iii), implies $\bar{\partial}g(z) = 0$ for almost all such z . By Lemma 6.18 we then have $\mu_{\varphi \circ g}(z) = \mu_\varphi(g(z)) \frac{\bar{\partial}g(z)}{\partial g(z)} =$

$\sigma(g(z)) \frac{\partial \overline{g(z)}}{\partial g(z)} = 0 = \sigma(z)$ for almost all $z \notin \cup_{n \geq 0} g^{-n}(E)$. Hence we have shown that $\varphi \circ g \circ \varphi^{-1}$ is rational.

Now since φ and Φ_i are both σ -conformal on E_i , we have that $\varphi \circ \Phi_i^{-1}$ is conformal on E'_i . Lastly we note that $\bar{\partial}\varphi = 0$ on $\bar{\mathbb{C}} \setminus \cup_{n \geq 0} g^{-n}(E)$ as $\sigma = 0$ there. The proof of the theorem is now complete. \square

DEFINITION 11.2. *A function $f : U \rightarrow V$ between domains in $\bar{\mathbb{C}}$ is called proper if for any compact $K \subseteq V$ we have that $f^{-1}(K)$ is compact in U .*

The following characterization of proper maps will be useful later. The proof is left to the reader.

LEMMA 11.3. *Let $f : U \rightarrow V$ be a map between domains in $\bar{\mathbb{C}}$. Then f is proper if and only if $f(z_n) \rightarrow \partial V$ as $z_n \rightarrow \partial U$.*

Note that $z_n \rightarrow \partial U$ means that given any compact $K \subset U$ we have that $z_n \notin K$ for all sufficiently large n .

DEFINITION 11.4. *Let U_1 and U_2 be simply connected bounded domains of the finite plane \mathbb{C} such that $\partial U_1 = \gamma_1$ and $\partial U_2 = \gamma_2$ are analytic (see Appendix II Definition 13.1 for definition) simple closed curves and $\bar{U}_1 \subseteq U_2$. We call $f : U_1 \rightarrow U_2$ polynomial-like of degree d if*

i) f is proper,

ii) f has degree d , i.e., every point in U_2 is taken d times counting multiplicity by points in U_1 under f ,

iii) f is analytic.

REMARK 11.5. *By Lemma 13.12 in Appendix II, a polynomial-like map f extends analytically to domain $U'_1 \supseteq \bar{U}_1$ and this extension maps γ_1 onto γ_2 precisely d times.*

In the proof of the next theorem we will require the following fact.

Fact: Let d be a positive integer. For any C^∞ -smooth curve $\Gamma(t)$ for $0 \leq t \leq 1$ with $\Gamma(t) \in C(0, R^d)$, there exists a C^∞ -smooth curve $\gamma(t)$ with $\gamma(t) \in C(0, R)$ such that $[\gamma(t)]^d = \Gamma(t)$. Such can easily be accomplished by noting that locally $\gamma(t)$ will be given by $h(\Gamma(t))$ where h is a suitably chosen branch of $z \mapsto z^{1/d}$. Note that if $\Gamma(t)$ traverses $C(0, R^d)$ exactly d times, then $\gamma(t)$ will traverse $C(0, R)$ exactly once.

THEOREM 11.6. *If $f : U_1 \rightarrow U_2$ is polynomial-like of degree d , then there exists a quasiconformal map of the plane φ such that $f = \varphi \circ P \circ \varphi^{-1}$ on U_1 where P is a polynomial of degree d .*

PROOF. Fix $r > 1$. Let Φ_1 map $\bar{\mathbb{C}} \setminus \bar{U}_2$ conformally onto $\bar{\mathbb{C}} \setminus \overline{\Delta(0, r^d)}$ such that $\Phi_1(\infty) = \infty$. The existence of this map is guaranteed by the Riemann Mapping Theorem. Now by the Caratheodory-Osgood Theorem we extend Φ_1 to ∂U_2 so that $\Phi_1(\partial U_2) = \partial(\Delta(0, r^d))$, and by Corollary 13.6 in Appendix II, Φ_1 extends analytically to a domain containing $\bar{\mathbb{C}} \setminus U_2$.

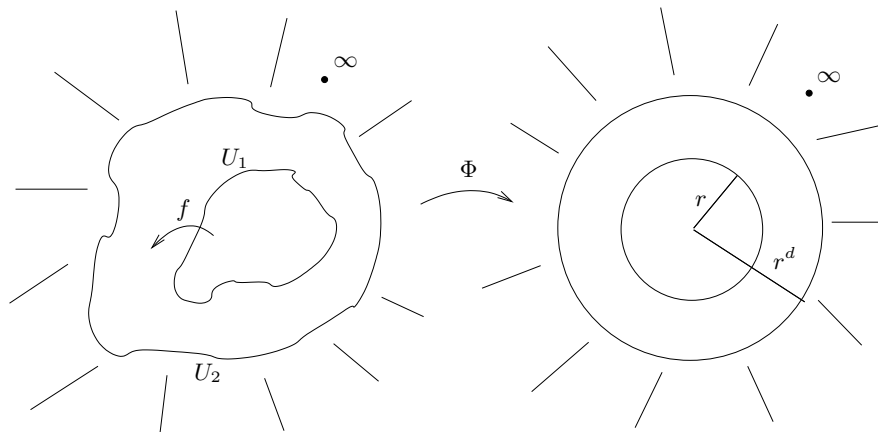


FIGURE 10

Next we define a map Φ_2 on $\partial U_1 = \gamma_1$ as follows. We note that $f(z)$ will d times traverse γ_2 as z traverses γ_1 once. Setting $\Gamma(t) = \Phi_1(f(\gamma_1(t)))$, which is C^∞ smooth and d times traverses the circle $C(0, r^d)$, we see by the Fact given just before the statement of this theorem, that there exists a C^∞ smooth parametrization $\gamma(t)$ of $C(0, r)$ such that $[\gamma(t)]^d = \Gamma(t)$. We then define Φ_2 on γ_1 by $\Phi_2(\gamma_1(t)) = \gamma(t)$ and note that Φ_2 is C^∞ smooth here. Thus $[\Phi_2(z)]^d = \Phi_1(f(z))$ on $\gamma_1 = \partial U_1$. Setting Φ_2 equal to Φ_1 on γ_2 we may then extend Φ_2 , in a C^∞ smooth way, such that $\Phi_2 : \overline{U_2} \setminus \overline{U_1} \rightarrow \overline{\Delta(0, r^d)} \setminus \overline{\Delta(0, r)}$ as in Theorem 13.9 from Appendix II. We note that on $U_2 \setminus \overline{U_1}$ the map Φ_2 is K -QC for some K by Remark 13.10 from Appendix II. We now sew Φ_1 and Φ_2 together to obtain a map Φ on $\overline{\mathbb{C}} \setminus U_1$ defined by $\Phi = \Phi_1$ on $\overline{\mathbb{C}} \setminus U_2$ and $\Phi = \Phi_2$ on $\overline{U_2} \setminus \overline{U_1}$. By Lemma 13.13, the map Φ is K -QC on $\overline{\mathbb{C}} \setminus \overline{U_1}$.

We now define a map $g(z)$ on $\overline{\mathbb{C}}$ and show that it is quasiregular. Set

$$g(z) = \begin{cases} f(z) & \text{for } z \in U_1 \\ \Phi^{-1}[(\Phi(z))^d] & \text{for } z \in \overline{\mathbb{C}} \setminus U_1. \end{cases}$$

On U_1 , g is analytic and so conformal away from the finite set $\{z : f'(z) = 0\}$. Therefore $\|\mu_g\|_\infty = 0$ on U_1 . On $\overline{\mathbb{C}} \setminus \overline{U_1}$ we see that g is a composition of K -QC Φ , followed by analytic $z \mapsto z^d$ (which is locally conformal), followed by conformal Φ^{-1} (on $\overline{\mathbb{C}} \setminus \Delta(0, r^d)$). Hence $\|\mu_g\|_\infty \leq \|\mu_\Phi\|_\infty < 1$ on $\overline{\mathbb{C}} \setminus \overline{U_1}$. (Upon closer inspection we can actually see that $\mu_g = 0$ on $\overline{\mathbb{C}} \setminus \overline{U_2}$.) We now show that g is locally quasiconformal away from the set $\{z : f'(z) = 0\} \cup \{\infty\}$. On U_1 and $\overline{\mathbb{C}} \setminus \overline{U_1}$, this is clear. For $z_0 \in \gamma_1$, we see g defined on $\Delta(z_0, \epsilon)$ for a sufficiently small ϵ is K -QC since g is the map created when the conformal map f and the K -QC map $\Phi^{-1}[(\Phi(z))^d]$ are sewn together (see Lemma 13.13 in Appendix II). We may now apply Lemma 10.16 to see that g is quasi-regular.

One can see that the Fundamental Theorem of Quasiconformal Surgery 11.1 can be applied, with $E = \overline{\mathbb{C}} \setminus \overline{U_1}$ and $N = 1$, to see that there exists a quasiconformal φ such that $P = \varphi \circ g \circ \varphi^{-1}$ is rational. But since only ∞ maps to ∞ with multiplicity d we see that P is actually a polynomial of degree d . \square

REMARK 11.7. We have, in a sense, glued $g|_{\overline{\mathbb{C}} \setminus E} = f|_{U_1}$ to $\Phi^{-1} \circ z^d \circ \Phi|_{\overline{\mathbb{C}} \setminus U_2}$ which is $z \mapsto z^d$, at least up to conjugation by a conformal map. Now pasting them together did not make a polynomial. However, when we conjugated this “pasting” by a quasiconformal map, we then have a true polynomial.

12. Appendix I

Here we present the Cauchy-Schwarz inequality.

If f and g are real-valued functions that are Lebesgue Measurable over the interval $[c, d]$, then $\int_c^d |f(y)g(y)| dy \leq \left\{ \int_c^d [f(y)]^2 dy \right\}^{\frac{1}{2}} \left\{ \int_c^d [g(y)]^2 dy \right\}^{\frac{1}{2}}$.

Furthermore, equality holds if and only if there exist constants α and β (not both zero) such that $\alpha |f|^2 = \beta |g|^2$ almost everywhere.

13. Appendix II

DEFINITION 13.1 ([4], p. 186). We say that a curve $\gamma(t) = x(t) + iy(t)$ from the interval $[0, 1]$ into \mathbb{C} is **analytic** if for each $t_0 \in (0, 1)$ we have that both the real part $x(t)$ and the imaginary part $y(t)$ can be represented by a power series centered at t_0 ; we further assume that $x'(t)$ and $y'(t)$ do not vanish at the same time.

REMARK 13.2. We note that analytic curves are C^∞ -smooth.

REMARK 13.3. We note that an open analytic curve γ defined on the open interval $(0, 1)$ in \mathbb{R} is the restriction of an analytic function f , and thus the set of image points of this curve (which, by an abuse of notation, we also call γ when no confusion will arise) is the image of an open interval under a complex analytic map. To see this we first fix a point t_0 of $(0, 1)$. We may express $x(t) = \sum_{n=0}^{\infty} a_n(t-t_0)^n$ and $y(t) = \sum_{n=0}^{\infty} b_n(t-t_0)^n$ by convergent power series in some real neighborhood of t_0 . Thus the power series $f(z) = \sum_{n=0}^{\infty} (a_n + ib_n)(z-t_0)^n$ is then convergent in some complex neighborhood of t_0 and therefore defines an analytic function there. Also note that $f'(t_0) = x'(t_0) + iy'(t_0) \neq 0$. Doing this at each point in $(0, 1)$ clearly defines an analytic map f in a complex neighborhood of $(0, 1)$.

The Schwarz reflection principle then allows one to extend analytic maps across analytic curves as well. The details can be seen in the cited reference of the following.

LEMMA 13.4. [4], p. 184] Let U and V be domains in \mathbb{C} and let $g : U \rightarrow V$ be an analytic function. Suppose ∂U includes an open analytic arc α , ∂V includes an open analytic arc β , and that g extends continuously to $U \cup \alpha$ where $g(z) \in \beta$ for all $z \in \alpha$. Then g analytically extends across α , i.e., g can be extended analytically in a neighborhood U' which contains α .

REMARK 13.5. By considering the details of the proof one can see that in Lemma 13.4 we have that $g' \neq 0$ on α .

COROLLARY 13.6. If $f : D_1 \rightarrow D_2$ is a conformal map between simply or multiply connected domains in \mathbb{C} each with boundaries that consist of a finite number of analytic curves, then f analytically extends to a domain containing $\overline{D_1}$. Furthermore, f^{-1} analytically extends to a domain containing $\overline{D_2}$.

PROOF. Suppose that D_1 has analytic boundary arcs $\gamma_1, \dots, \gamma_n$ and D_2 has analytic boundary arcs $\Gamma_1, \dots, \Gamma_m$. Then by applying the techniques used in the proof of Theorem 4.7 in [5] (which is used in the proof of the Carathéodory-Osgood Theorem), we see that f continuously extends to each γ_j , mapping it into some Γ_k . We then apply Lemma 13.4 to show the existence of an analytic extension across each γ_j and thus to a domain containing $\overline{D_1}$.

We note that we may apply the same argument to f^{-1} to conclude that f^{-1} analytically extends to a domain containing $\overline{D_2}$. \square

We note that we get the following similar result if our boundary curves are merely C^∞ -smooth instead of analytic.

THEOREM 13.7 ([2], p. 23 and 24). *If $f : D_1 \rightarrow D_2$ is a conformal map between simply or multiply connected domains in \mathbb{C} each with C^∞ -smooth boundaries, then f and all of its derivatives extend continuously to $\overline{D_1}$. Furthermore, f^{-1} and all of its derivatives have continuous extensions to $\overline{D_2}$.*

By employing the same techniques as the proof of Lemma 13.4 and using the harmonic reflection principle stated on p. 234 in [5], we see that harmonic functions can also be reflected over analytic curves. Specifically we have the following:

LEMMA 13.8. *Let U be a domain in \mathbb{C} and let $g : U \rightarrow \mathbb{R}$ be a harmonic function. Suppose ∂U includes an open analytic arc α such that g extends continuously to $U \cup \alpha$ where $g(z) = 0$ for all $z \in \alpha$. Then g harmonically extends across α , i.e., g can be extended harmonically in a neighborhood U' which contains α .*

THEOREM 13.9. *Let U_1 and U_2 be simply connected domains in \mathbb{C} bounded by closed analytic Jordan curves γ_1 and γ_2 , respectively. Suppose further that $\overline{U_1} \subset U_2$. Given C^∞ -smooth homeomorphisms $g_j : \gamma_j \rightarrow C(0, r_j)$, for $j = 1, 2$ and $r_1 < r_2$, each which preserves orientation, there exists a C^∞ -smooth homeomorphism G from $\overline{U_2} \setminus \overline{U_1}$ onto the closed true annulus $\overline{\text{Ann}(0; r_1, r_2)}$ which extends both g_1 and g_2 . Furthermore, G and all of its derivatives extend continuously to $\overline{U_2} \setminus \overline{U_1}$. We also note that there exists $\eta > 0$ (which depends on r_1, r_2, g_1, g_2, U_1 and U_2) such that the Jacobian $J_G(z) \geq \eta$ for all $z \in \overline{U_2} \setminus \overline{U_1}$.*

REMARK 13.10. *We note that the map G in Theorem 13.9 satisfies $\|\mu_G\|_\infty < 1$ on $\overline{U_2} \setminus \overline{U_1}$ and is therefore quasiconformal on $\overline{U_2} \setminus \overline{U_1}$. Since $|\partial G(z)|^2 - |\bar{\partial} G(z)|^2 = J_G(z) \geq \eta > 0$, we have that $|\partial G(z)| \geq \sqrt{\eta} > 0$. Using these two facts then allows us to note that $|\mu_G(z)| = |\bar{\partial} G(z)/\partial G(z)|$ is bounded above by 1 and continuous on the compact set $\overline{U_2} \setminus \overline{U_1}$. Thus it follows that $\|\mu_G\|_\infty < 1$.*

LEMMA 13.11 ([4], p. 333). *Every doubly connected domain A in $\overline{\mathbb{C}}$ whose boundary components each contain more than one point can be mapped conformally onto a true annulus $\text{Ann}(0; 1, R)$ for some $0 < R < \infty$. We call $\log R$ the modulus of A .*

PROOF OF THEOREM 13.9. We begin by first proving Theorem 13.9 in the special case that $\gamma_1 = C(0, r_1)$ and $\gamma_2 = C(0, r_2)$. Under the given assumptions on g_1 and g_2 we may now express $g_1(r_1 e^{i\theta}) = r_1 e^{if_1(\theta)}$ and $g_2(r_2 e^{i\theta}) = r_2 e^{if_2(\theta)}$ where for $j = 1, 2$ we have

(i) f_j maps \mathbb{R} into \mathbb{R} and is a C^∞ -smooth increasing function of θ (since g_j is C^∞ -smooth and orientation preserving), and

(ii) $f_j(\theta + 2\pi) = f_j(\theta) + 2\pi$ (since g_j is a homeomorphism of a circle, f_j must map any interval of length 2π onto an interval of length 2π).

We now define $f : [r_1, r_2] \times \mathbb{R} \rightarrow \mathbb{R}$ to be an interpolation of f_1 and f_2 given by

$$f(r, \theta) = \frac{r - r_2}{r_1 - r_2} f_1(\theta) + \left(1 - \frac{r - r_2}{r_1 - r_2}\right) f_2(\theta).$$

Note that for fixed r , the map $f(r, \theta)$ is a C^∞ -smooth function which is increasing in θ (by (i)) and $f(r, \theta + 2\pi) = f(r, \theta) + 2\pi$ (by (ii)). We now see that $G(re^{i\theta}) = re^{if(r, \theta)}$ will satisfy the conclusion of our theorem.

Now we return to considering the general case without the assumptions on γ_1 and γ_2 given above. By Lemma 13.11 there exists a conformal map h from $U_2 \setminus \overline{U_1}$ onto an annulus of the form $Ann(0; 1, R)$, which by Corollary 13.6 extends C^∞ -smoothly to all of $\overline{U_2 \setminus U_1}$. Note also that Corollary 13.6 allows one to conclude that h^{-1} is also a C^∞ -smooth homeomorphism on $\overline{Ann(0; 1, R)}$.

Now consider the C^∞ -smooth homeomorphic mapping k from $\overline{Ann(0; r_1, r_2)}$ onto $\overline{Ann(0; 1, R)}$ given by $re^{i\theta} \mapsto [\frac{R-1}{r_2-r_1}(r-r_1) + 1]e^{i\theta}$, which merely stretches radially. Hence we see that the maps $g_1 \circ h^{-1} \circ k$ and $g_2 \circ h^{-1} \circ k$ are C^∞ -smooth homeomorphisms from $C(0, r_1)$ and $C(0, r_2)$ into themselves, respectively. This case was already handled above, and so we let \tilde{G} be the C^∞ -smooth homeomorphism from $\overline{Ann(0; r_1, r_2)}$ to itself, which extends the maps $g_1 \circ h^{-1} \circ k$ and $g_2 \circ h^{-1} \circ k$. We then see that $G = \tilde{G} \circ k^{-1} \circ h$ has the required properties.

Finally, we conclude the proof by noting that since each map \tilde{G}, k^{-1} and h has Jacobian bounded below by some positive constant (which can be shown through direct calculation), so does G since $J_G(z_0) = J_{\tilde{G}}(k^{-1}(h(z_0)))J_{k^{-1}}(h(z_0))J_h(z_0)$ for any $z_0 \in \overline{U_2 \setminus U_1}$. \square

LEMMA 13.12. *Let $f : U \rightarrow V$ be a proper analytic map of degree d where U and V are simply connected domains in \mathbb{C} bounded by closed analytic Jordan curves γ_1 and γ_2 , respectively. Then f can be extended analytically to a map $f : U' \rightarrow \mathbb{C}$ such that domain $U' \supset \overline{U}$ and $f(U')$ contains \overline{V} . We also note that this extension of f then maps γ_1 onto γ_2 precisely d times.*

PROOF. We first consider the case where $U = V = \Delta(0, 1)$. Fix a point $z_0 \in \partial\Delta(0, 1)$. Since f is proper, there exists a small disk $\Delta = \Delta(z_0, \delta)$ such that $f(z) \neq 0$ on $\Delta \cap \Delta(0, 1)$. Consider the harmonic function $h(z) = \log |f(z)| = \text{Re } \text{Log} f(z)$ defined on $\Delta \cap \Delta(0, 1)$, where $\text{Log}(z)$ denotes the principal logarithm. Since $|f(z)| \rightarrow 1$ as $|z| \rightarrow 1$ (f is proper), we see that $h(z) \rightarrow 0$ as $|z| \rightarrow 1$. Thus we may reflect $h(z)$ across $|z| = 1$ to extend $h(z)$ to be harmonic on all of Δ (Theorem 3.4 in [5] states this reflection principle for real line segments, but it also holds for arcs of circles by a simple application of a Möbius transformation). Letting $v(z)$ be a harmonic conjugate of $h(z)$ in Δ , which on $\Delta \cap \Delta(0, 1)$ must equal $\text{Im } \text{Log} f(z) + c$ for some real constant c , we see that the analytic function $e^{h(z)+i(v(z)-c)}$ defined on all of Δ must equal $f(z)$ on $\Delta \cap \Delta(0, 1)$. Thus extending f in this way in a disk about each point in $\partial\Delta(0, 1)$ we see that f can be extended to a domain U'

containing $\overline{\Delta(0,1)}$. Clearly, since f is an open continuous map, we must then have that $f(U')$ contains the compact set $f(\overline{\Delta}) = \overline{\Delta(0,1)}$.

Note that by using the Argument Principle and the hypothesis that f maps U onto V in a d -to-1 fashion, we see that as the curve $\gamma(t)$ traverses $C(0,1)$ once, $f(\gamma(t))$ traverses $C(0,1)$ exactly d times.

Now we address the general case. Let $\phi : U \rightarrow \Delta(0,1)$ and $\psi : V \rightarrow \Delta(0,1)$ be Riemann maps. Since ∂U and ∂V are analytic curves we can see by Corollary 13.6 that ϕ, ψ, ϕ^{-1} , and ψ^{-1} all analytically extend across the boundary of their respective domains. Note that we may assume that these extensions are also conformal, even on these extended domains. From the special case above, we see that $g = \psi \circ f \circ \phi^{-1} : \Delta(0,1) \rightarrow \Delta(0,1)$ extends across the boundary, and so one can easily show that $\psi^{-1} \circ g \circ \phi$ (using the extensions of each of these maps) extends f to a domain containing \overline{U} . \square

The following lemma expresses the fact that we may sometimes “sew” quasiconformal maps together along a common boundary arc (where the maps agree) and obtain a resulting map which is also quasiconformal.

LEMMA 13.13. *Let Δ be an open disk which is met by an analytic arc γ such that there are two components D_1 and D_2 of $\Delta \setminus \gamma$. Suppose f_1 and f_2 are quasiconformal in D_1 and D_2 , respectively, such that $f_1(D_1)$ and $f_2(D_2)$ are disjoint and such that each map extends continuously and injectively to $\Delta \cap \gamma$ on which $f_1 = f_2$. Then the map $F(z)$ defined by f_1 on D_1 and f_2 on D_2 and $f_1 = f_2$ on $\Delta \cap \gamma$ is quasiconformal on all of Δ .*

PROOF. Since f_1 and f_2 are each absolutely continuous on lines, one can easily show, by continuity and the fact that $f_1 = f_2$ on $\Delta \cap \gamma$, that so is F . Since F is a continuous injective map of an open set Δ , it is a homeomorphism (see [3], p. 6). Since γ has measure zero, it is easy to see that by using Lemma 8.3 we have that $|\mu_F|$ is bounded almost everywhere by $\frac{K-1}{K+1}$ where $K = \max\{K_{f_1}, K_{f_2}\}$. Hence by Lemma 8.4 the map F is K -quasiconformal. \square

REMARK 13.14. *One could also apply Theorem I.8.3 in [3] to obtain the above result.*

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