March 27, 2010

DENSITY OF REPELLING FIXED POINTS IN THE JULIA SET OF A RATIONAL OR ENTIRE SEMIGROUP, II

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ABSTRACT. In [13] there is a survey of several methods of proof that the Julia set of a rational or entire function is the closure of the repelling cycles, along with a discussion of which of those methods can and cannot be extended to the case of semigroups. In particular that paper presents an elementary proof based on the ideas of [11] that the Julia set of either a non-elementary rational or entire semigroup is the closure of the set of repelling fixed points. This paper serves as a brief follow up to [13] by showing that the ideas of [3] can also be used to provide an elementary proof for the semigroup case. It also touches upon some key differences between the dynamics of iteration and the dynamics of semigroups.

1. INTRODUCTION

As stated in the abstract, this paper can be regarded as a follow up to [13], which was the focus of a lecture given at the Dynamical Systems II conference held at Denton, TX in 2009. It also relates to the discussions that followed, and so the author would like to thank the participants for their questions and comments, and especially thank the organizers for their efforts in hosting the event.

This paper is concerned with the dynamics of semigroups, a natural generalization of the study of the dynamics of iteration of a complex analytic map. We define a *rational (respectively, entire) semigroup* to be a semigroup generated by non-constant rational (respectively, entire) maps on the Riemann sphere $\overline{\mathbb{C}}$ (respectively, complex plane $\mathbb{C}$) with the semigroup operation being the composition of maps. We denote by $\langle h_\lambda : \lambda \in \Lambda \rangle$ the semigroup generated by the family of maps $\{h_\lambda : \lambda \in \Lambda\}$. Thus $\langle h_\lambda : \lambda \in \Lambda \rangle$ denotes the family of all maps which can be created through composition of any finite number of maps h_λ.

Research on the dynamics of rational semigroups was initiated by Hinkkanen and Martin in [7], where each rational semigroup was always taken to have at least one element of degree at least two – a restriction we do not impose here. Two main motivations for their study are given in [7]. The first motivation is to see to what extent, and in what sense, the classical iteration theory of Fatou and Julia extends to this more general setting of semigroups. The second motivation is to use this theory to study the parameter space of certain one-complex parameter Kleinian groups, where portions of such parameter spaces can be characterized as stable basins of infinity for certain polynomial semigroups (see also [4, 5]). Also, Ren, Gong, and Zhou studied such rational semigroups from the perspective of *random dynamical systems* (see [27, 6]), that is, dynamics along iteratively defined *composition sequence* of maps $h_{\lambda_n} \circ \cdots \circ h_{\lambda_1}$ where each $\lambda_k \in \Lambda$ is selected at random. Later, Kriete and Sumi in [8] studied semigroups of entire maps.

2000 Mathematics Subject Classification: Primary 37F10, 37F50, 30D05. Key words and phrases. Complex dynamics, Julia sets, Random Dynamics, Dynamics of Semigroups.

Study of semigroup dynamics, random dynamics, and their intimate connections, have produced recent results exhibiting new phenomena not possible in the classical iteration theory. Many results can be found in the works of Sumi [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]; we highlight just a few here which pertain to polynomial semigroups. A polynomial semigroup may have a bounded postcritical set, yet have a disconnected Julia set. However, in such a setting there is a natural "surrounding" order on the connected components of the Julia set [14, 15, 24], and such components are often (but not always) Julia sets of corresponding compositions sequences. It often happens that such Julia sets are Jordan curves but not quasicircles, and the basin of infinity is not a John domain [16] - something which cannot happen at all in iteration theory. Considering the space of all composition sequences (corresponding to a given semigroup G) gives rise to a "probability of escape" function $T(z)$, a function which gives the probability that z will tend to ∞ under a randomly selected composition sequence. The function $T(z)$ is often a complex analogue of the devil's staircase or Lebesgue singular function in that it is continuous on $\mathbb{C}$ and varies only on the Julia set (typically a thin fractal set) of the associated semigroup [20, 21, 24]. In [17, 22, 23], it was shown that the unique zero of the pressure function for the skew product associated with an expanding finitely generated rational semigroup can be easily greater than two. These few examples illustrate the richness of results that can occur in this new setting, but which cannot occur in the usual iteration theory dynamics. See the above references for an extended exposition, details, and precise formulations of these results.

We follow [7] in saying that for a rational (respectively, entire) semigroup G the *Fatou set* $F(G)$ is the set of points in $\overline{\mathbb{C}}$ (respectively, $\mathbb{C}$) which have a neighborhood on which G is normal, and its complement in $\overline{\mathbb{C}}$ (respectively, $\mathbb{C}$) is called the *Julia set* $J(G)$. The more classical Fatou set and Julia set of the cyclic semigroup $\langle g \rangle$ generated by a single map (i.e., the collection of iterates $\{g^n : n \geq 1\}$) is denoted by $F(g)$ and $J(g)$, respectively.

Immediately from the definitions, one can show (as done in [7]) that the Fatou set $F(G)$ is *forward invariant* under each element of G, i.e., $g(F(G)) \subset F(G)$ for all $g \in G$, and thus $J(G)$ is *backward invariant* under each element of G, i.e., $g^{-1}(J(G)) \subset J(G)$ for all $g \in G$.

This paper addresses the relationship between repelling fixed points and the Julia set of a rational or entire semigroup G. Since a point $w \in \mathbb{C}$ is called a *repelling fixed point* for the map f when $f(w) = w$ and $|f'(w)| > 1$, it is elementary to show that such a point is in $J(f)$, and hence in $J(G)$ for any semigroup G containing f. The goal of this paper is to present an elementary argument that such fixed points are dense in $J(G)$ when G is a *non-elementary* rational or *non-elementary* entire semigroup (i.e., when $J(G)$ contains three or more points). More specifically, we prove the following.

Theorem 1.1. *Let G be a non-elementary rational or non-elementary entire semigroup. Then $J(G)$ is the closure of the set of repelling fixed points.*

See [13] and its references for a discussion of various methods of proof based on the proofs of the corresponding result in the classical iteration case. In particular, Theorem 1.1 is proven there using elementary methods based on [11], which uses a key result from Nevanlinna theory. This paper simplifies that approach by following [3] and thus manages to avoid the use of Nevanlinna theory.

Lastly, we note that Theorem 1.1 is a fundamental result in both the study of iteration and semigroup dynamics. In particular, in semigroup theory it provides the key step in proving the following. Given a non-elementary rational or non-elementary entire semigroup G, we have (i) $J(G) = \overline{\cup_{g\in G} J(g)}$, (ii) $F(G)$ is precisely the set of z which has a neighborhood on which every composition sequence is normal, and (iii) when G is a rational semigroup, $J(G)$ is uniformly perfect when there is a uniform bound on the Lipschitz constants (with respect to the spherical metric) of the generators (see [12, 27, 13]). Regarding further results on the uniform perfectness of $J(G)$ the interested reader will want to consult [19]. There are various other applications to Theorem 1.1, including the structure of Julia sets and surrounding order for polynomial semigroups with bounded postcritical set as well as some results related to random complex dynamics (see [14, 15, 24]).

Acknowledgment: The author would like to sincerely thank both the referee and Hiroki Sumi for valuable comments and suggestions.

2. Background and preliminary results

A preimage z of w under a meromorphic function f maps to w with valency (local degree) denoted by $v_f(z)$. Such a point z is called a *critical point* if $v_f(z) > 1$, which in the case that both z and $f(z)$ are finite, means exactly that $f'(z) = 0$. A point w in the image of f is called *completely ramified* if $v_f(z) > 1$ for every preimage z of w.

Definition 2.1. Let $\mathcal{H}$ be a family of meromorphic functions from domain D mapping into $\overline{\mathbb{C}}$. We define the following:

(1) the *forward orbit* of $z \in D$ under $\mathcal{H}$ is $\mathcal{H}(z) = \{h(z) : h \in \mathcal{H}\}$,
(2) the *backward orbit* of $z \in \overline{\mathbb{C}}$ under $\mathcal{H}$ is $\mathcal{H}^{-1}(z) = \{w \in \overline{\mathbb{C}} :$ there exists $h \in \mathcal{H}$ such that $h(w) = z\}$,
(3) the *simple backward orbit* of $z \in \overline{\mathbb{C}}$ under $\mathcal{H}$ is $S^-_{\mathcal{H}}(z) = \{w \in \overline{\mathbb{C}} :$ there exists $h \in \mathcal{H}$ such that $h(w) = z$ and $v_h(w) = 1\}$.

For a rational or entire semigroup G we define the *exceptional set* to be $E(G) = \{z \in \overline{\mathbb{C}} : \#G^{-1}(z) < 3\}$ where $\#A$ denotes the cardinality of the set A.

It is well known that if a semigroup G contains either a transcendental entire map or a rational map of degree two or more, then it is non-elementary (see [9], p. 69). The number three in the definition of non-elementary is special because of the role it plays in Montel's theorem, which can be used to give the following well known facts (see, for example, [13]).

Proposition 2.2. *Let G be a non-elementary rational (respectively, entire) semigroup G. Then*

(i) $J(G)$ is the smallest closed subset of $\overline{\mathbb{C}}$ (respectively, $\mathbb{C}$) which contains three or more points and is backward invariant.
(ii) $J(G) \subset \overline{G^{-1}(z)}$ for any $z \in \overline{\mathbb{C}} \setminus E(G)$ (respectively, $z \in \mathbb{C} \setminus E(G)$).
(iii) $J(G) = \overline{G^{-1}(z)}$ for any $z \in J(G) \setminus E(G)$.
(iv) $J(G)$ is perfect, and hence uncountable.

Lemma 2.3. *When G is a non-elementary rational or non-elementary entire semigroup, we have $\#E(G) < 3$.*

Proof. Suppose a, b, c are distinct points in $E(G)$ and consider the finite set $A = \{a, b, c\} \cup G^{-1}(a) \cup G^{-1}(b) \cup G^{-1}(c)$. Since A is backward invariant under G, we must have $J(G) \subset A$ by Proposition 2.2(i), which is a contradiction since $J(G)$ is perfect by Proposition 2.2(iv). □

In classical iteration theory the Julia set and exceptional set are disjoint, i.e., $E(\langle g\rangle) \cap J(\langle g\rangle) = \emptyset$, but this need not be the case for semigroups (see Example 2.6).

For a rational or entire semigroup G we let $A(G)$ be the set of $z \in J(G)$ such that $S_G^-(z)$ has three or more accumulation points in $\overline{\mathbb{C}}$. The importance of the defining property of $A(G)$ is given by the following lemma.

Lemma 2.4. *Given a non-constant meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ and a set S in $\overline{\mathbb{C}}$ which has three or more accumulation points in $\overline{\mathbb{C}}$, the set S must contain at least one point with a simple preimage under f.*

Remark 1. Using Nevanlinna theory, one can see that any set S containing five or more points would be enough to satisfy the conclusion of the above lemma. (See [1] and [2] for a very nice presentation of simple proofs of the key results and a discussion of the uses of both Nevanlinna theory and Ahlfors covering theory in dynamics.) However, we are trying to obtain our results with the simplest means possible and so we continue with the lemma stated above whose proof is elementary, but which we provide anyway for the sake of completeness.

Proof. Let V be the set of points in $f(\mathbb{C})$ which are not completely ramified by f, and note that V is open. The set $f(\mathbb{C}) \setminus V$ of completely ramified image points has no accumulation points in $f(\mathbb{C})$ (since if $w = f(z)$ were such an accumulation point, then any neighborhood of z mapping onto a neighborhood of w would have to contain a sequence of critical points tending to z thus contradicting the fact that f is non-constant). Since Picard's Theorem implies that $\overline{\mathbb{C}} \setminus f(\mathbb{C})$ has at most two points, we see that $\overline{\mathbb{C}} \setminus V = (\overline{\mathbb{C}} \setminus f(\mathbb{C})) \cup (f(\mathbb{C}) \setminus V)$ is a set which has at most two accumulation points in $\overline{\mathbb{C}}$. Since S has three or more accumulation points in $\overline{\mathbb{C}}$, it follows that S must meet V, which is the desired conclusion. □

As we shall see, the key to making use of the definition of $A(G)$ is that often one can easily show that $A(G)$ is dense in $J(G)$.

Lemma 2.5. *Let G be a non-elementary rational or non-elementary entire semigroup. Then $A(G)$ is dense in $J(G)$.*

Proof. **Case 1.** Suppose there exists a non-Möbius $g \in G$. Observe that the postcritical set $P(\langle g\rangle) = \{g^n(z) : v_{g^n}(z) > 1\}$ (note that we do not take the closure here as is sometimes done in the literature) is countable since each map g^n has only a countable number of critical points. Also, since it is well known that $\langle g\rangle$ is non-elementary, Lemma 2.3 implies that the set $E(\langle g\rangle)$ contains at most two points. Setting $B = P(\langle g\rangle) \cup E(\langle g\rangle)$ we see that for any $z_0 \in J(G) \setminus B$ we have $S_G^-(z_0) \supset S_{\langle g\rangle}^-(z_0) = \langle g\rangle^{-1}(z_0)$, which by Proposition 2.2 has a closure which contains the uncountable set $J(g)$. Hence $J(G) \setminus B \subset A(G)$.

We also note that the Baire Category Theorem implies $J(G) \setminus B$ is dense in $J(G)$ since B is countable (labeling $B = \{b_n : n \in \mathbb{N}\}$ we see that since $J(G)$ is perfect each $O_n = J(G) \setminus \{b_n\}$ is an open and dense set in $J(G)$, and hence $J(G) \setminus B = \cap_{n\in\mathbb{N}} O_n$ is also dense in $J(G)$). Thus, $A(G)$ is dense in $J(G)$.

Case 2. We suppose that G consists entirely of Möbius maps. Note that for any $z_0 \in J(G) \setminus E(G)$, the set $S_G^-(z_0) = G^{-1}(z_0)$ is dense in the uncountable set $J(G)$ by Proposition 2.2. Hence in this case $A(G) \supset J(G) \setminus E(G)$, which is dense in $J(G)$ since $E(G)$ is finite by Lemma 2.3 and $J(G)$ is perfect by Lemma 2.2(iv). □

Remark 2. We note that we could have defined instead $A(G)$ to be the set of $z \in J(G)$ such that $S_G^-(z)$ has an uncountable number of accumulation points, and then the corresponding version of Lemma 2.5 would still follow from the given proof. This would allow one to get by with a correspondingly weaker version of Lemma 2.4.

We also note that the proof of Lemma 2.5 would still carry over for countable semigroups G if we instead replaced the set B in the proof with the set $P(G) \cup E(G)$, where $P(G) = \{g(z) : v_g(z) > 1 \text{ for some } g \in G\}$ is the postcritical set of G. However, this altered proof would not necessarily apply for semigroups G which are uncountable. In particular, as we see in the next example, it is possible for $J(G) \setminus P(G)$ to not be dense in $J(G)$.

Example 2.6. For each $a \in \Delta(0,1)$, let f_a be a polynomial whose Julia set is the circle $\{z : |z-a| = (1-|a|)/2\}$ and such that $\overline{\mathbb{C}} \setminus \overline{\Delta(0,1)}$ is forward invariant under f_a. Also, letting $g_r(z) = z^2/r$ for each $0 < r < 1$, we see that $J(g_r) = \{z : |z| = r\}$. Letting $G = \langle f_a, g_r : a \in \Delta(0,1), 0 < r < 1\rangle$, we see that $P(G) \supset \Delta(0,1)$, since each a is a critical value for f_a. Also, since $\overline{\mathbb{C}} \setminus \overline{\Delta(0,1)}$ is forward invariant under each of the maps in G, Montel's theorem shows that $\overline{\mathbb{C}} \setminus \overline{\Delta(0,1)} \subset F(G)$. Clearly then $J(G) = \overline{\Delta(0,1)}$ since $J(G)$ contains each $J(g_r)$. Hence for this rational semigroup $J(G) \setminus P(G)$ is not dense in $J(G)$.

We also note that the semigroup $G' = \langle g_r : 0 < r < 1\rangle$ which has $J(G') = \overline{\Delta(0,1)}$ and $E(G') = \{0, \infty\}$, illustrates that the Julia set can meet the exceptional set even if the semigroup is rational.

3. Proof of the main result

The following important result known as Zalcman's Rescaling lemma provides, through an elegantly simple argument, the key perspective on the non-normality condition to be employed (see [26] for the original statement and also see [10] and [1] for the slightly modified statements which we adopt here).

Theorem 3.1. *Let $\mathcal{F}$ denote a family of meromorphic functions on domain $U \subset \mathbb{C}$. Then $\mathcal{F}$ is not normal on U if and only if there exists a sequence $f_j \in \mathcal{F}$, a point $z_0 \in U$, a sequence $z_j \to z_0$, a sequence of positive real numbers $\rho_j \to 0$ and a nonconstant meromorphic function f on $\mathbb{C}$ such that*

$$f_j(z_j + \rho_j z) \to f(z)$$

locally uniformly on $\mathbb{C}$. Moreover, f can be chosen to have $f^\#(z) \leq 1 = f^\#(0)$ for all $z \in \mathbb{C}$, where $f^\#$ denotes the spherical derivative.

With regard to Theorem 3.1, we set $r_j(z) = z_j + \rho_j z$ and note that $r_j(z) \to z_0$ uniformly on compact subsets of $\mathbb{C}$. Further, we say that f, the limit of $f_j \circ r_j$, *absorbs* the point z_0 if there exists a simple solution in $\mathbb{C}$ to the equation $f(z) = z_0$, i.e., z_0 is a point in the image of f which is not completely ramified.

Proof of Theorem 1.1. Let $z_0 \in J(G)$ and apply Theorem 3.1 to obtain maps $f_k \in G$, linear maps $r_k \to z_0$, and a nonconstant meromorphic function f on $\mathbb{C}$. We consider two cases.

Case 1. Suppose z_0 is absorbed by f. By this assumption there exists an open disk D contained in a strictly larger open disk on which f is univalent and such that $\Delta = f(D)$ is a neighborhood of z_0. Since $f_k \circ r_k \to f$, it follows then that for large k, the maps $f_k \circ r_k$ are univalent on D and the image $\Delta_k = f_k \circ r_k(D)$ is close to Δ. Note that since the maps $r_k \to z_0$, we have that $\overline{r_k(D)}$ is contained in the interior of Δ_k when k is sufficiently large. Since f_k maps $r_k(D)$ conformally onto $\Delta_k \supset \overline{r_k(D)}$ we see that f_k must have a repelling fixed point a_k in $r_k(D)$ (since the inverse of the map $f_k : r_k(D) \to \Delta_k$ is a strict contraction of the Poincarè metric on Δ_k which must then have an attracting fixed point). Since $a_k \in r_k(D)$, we see that $a_k \to z_0$ and thus z_0 is a limit of repelling fixed points.

Case 2. Suppose $z_0 \in A(G)$, i.e., $S_G^-(z_0)$ has three or more accumulation points in $\overline{\mathbb{C}}$. By Lemma 2.4 there exists a point $w_0 \in S_G^-(z_0)$ which has a simple preimage under f. Since $w_0 \in S_G^-(z_0)$ there exists $g \in G$ such that $g(w_0) = z_0$ and $v_g(w_0) = 1$. We note then that the maps $g \circ f_k \circ r_k$ converge to $g \circ f$, where this limit map absorbs z_0. Thus we see from the proof in Case 1 that z_0 is a limit of repelling fixed points of $g \circ f_k \in G$.

Since each point in $A(G)$ is a limit of repelling fixed points and, by Lemma 2.5, $A(G)$ is dense in $J(G)$ the result of the theorem holds. □

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