The Lebesgue Number

Definition. The *diameter* of a non-empty bounded subset S of a metric space X is defined to be the least upper bound of the set $\{d(x,y) \mid x, y \in S\}$ of real numbers. We will denote the diameter of S by $\operatorname{diam}(S)$.

Definition. A collection $\mathcal{U}$ of subsets of a topological space X is said to *cover* X, and is also called a *cover of* X, if its union $\bigcup \mathcal{U}$ equals X. A collection of open subsets of X which covers X is called an *open cover* of X.

Lemma. *For every open cover $\mathcal{U}$ of a compact metric space X there is a positive real number λ, called a* Lebesgue[1] number*, such that every subset of X of diameter less than λ is contained in some element of $\mathcal{U}$.*

Proof. Let $\mathcal{U}$ be an open cover of X and suppose, to the contrary, that there is no such λ. Then for every $n \in \mathbb{N}$, $\frac{1}{n}$ is not a Lebesgue number, that is, there is a subset $S_n \subseteq X$ such that $\operatorname{diam}(S_n) < \frac{1}{n}$ but S_n is not entirely contained in any of the elements of $\mathcal{U}$. (In particular, S_n is not empty.) For each $n \in \mathbb{N}$, we choose one point $x_n \in S_n$. Since X is compact, there is a subsequence (x'_n) of (x_n) which converges to some point $x \in X$. Since $\mathcal{U}$ covers X, there is a $U \in \mathcal{U}$ such that $x \in U$. Then there is an $\epsilon > 0$ such that $x \in N_X(x, \epsilon) \subseteq U$. Choose $N \in \mathbb{N}$ with $\frac{1}{N} < \frac{\epsilon}{2}$. Since (x'_n) converges to x, all but finitely many members of the subsequence (x'_n) must lie in $N_X(x, \frac{\epsilon}{2})$. Hence, infinitely many members of the original sequence (x_n) lie in $N_X(x, \frac{\epsilon}{2})$. So, there is an $n > N$ such that $x_n \in N_X(x, \frac{\epsilon}{2})$. This implies, however, that $S_n \subseteq N_X(x, \epsilon)$. For if $s \in S_n$, then

$$d(s,x) \leqslant d(s,x_n) + d(x_n,x) < \operatorname{diam}(S_n) + \frac{\epsilon}{2} < \frac{1}{n} + \frac{\epsilon}{2} < \frac{1}{N} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

But then $S_n \subseteq N_X(x, \epsilon) \subseteq U$, which contradicts our assumption. □

Theorem. *Suppose $f : X \to Y$ is a continuous function from a compact metric space X to a topological space Y. Let $\mathcal{U}$ be an open cover of Y. Then there is a number $\lambda > 0$ such that for every subset S of X with diam$(S) < \lambda$ there is some $U \in \mathcal{U}$ such that $f(S) \subseteq U$.*

Proof. Since every $U \in \mathcal{U}$ is an open subset of Y and since $f : X \to Y$ is continuous, each $f^{-1}(U)$, which is defined by $f^{-1}(U) = \{z \in X \mid f(z) \in U\}$, is an open subset of X. Moreover, since $\mathcal{U}$ covers Y, for every $x \in X$ there is a $U \in \mathcal{U}$ such that $f(x) \in U$, that is, $x \in f^{-1}(U)$. Consequently, $\mathcal{V} = \{f^{-1}(U) \mid U \in \mathcal{U}\}$ is an open cover of the compact metric space X. By the above lemma, we only have to let λ be a Legesgue number for $\mathcal{V}$. For if now S is any subset of X of diameter less than λ, then $S \subseteq f^{-1}(U)$ for some $U \in \mathcal{U}$, which means $f(S) \subseteq U$. □

Corollary. *Suppose $f : X \to Y$ is a continuous function from a compact metric space X to some metric space Y. Then for every $\epsilon > 0$, there is a $\delta > 0$ such that $d(f(x), f(y)) < \epsilon$ whenever $d(x,y) < \delta$.*

Proof. Consider $\mathcal{U} = \{N_Y(y, \frac{\epsilon}{2}) \mid y \in Y\}$ and take $\delta = \lambda$ from the above theorem. □

[1] Henri Léon Lebesgue, French mathematician, 1875-1941