Differential Forms: Unifying the Theorems of Vector Calculus

In class we have discussed the important vector calculus theorems known as Green's Theorem, Divergence Theorem, and Stokes's Theorem. Interestingly enough, all of these results, as well as the fundamental theorem for line integrals (so in particular the fundamental theorem of calculus), are merely special cases of one and the same theorem named after George Gabriel Stokes (1819-1903). This all-including theorem is stated in terms of differential forms. Without giving exact definitions, let us use the language of differential forms to unify the theorems we have learned. A striking pattern will emerge.

0**-forms.** A scalar field (i.e. a real-valued function) is also called a 0*-form.*

1**-forms.** Recall the following notation for line integrals (in 3-space, say):

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_a^b P\underbrace{x'(t)dt}_{dx} + Q\underbrace{y'(t)dt}_{dy} + R\underbrace{z'(t)dt}_{dz} = \int_C Pdx + Qdy + Rdz,$$

where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. The expression $P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$ is called a 1*-form.*

2**-forms.** In evaluating surface integrals we can introduce similar notation:

$$\begin{aligned}
&\iint_S \mathbf{F}\cdot\mathbf{n}\, dS \\
&= \iint_\Gamma \mathbf{F}\cdot\frac{\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}}{\left\|\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right\|}\left\|\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right\| dudv \\
&= \iint_\Gamma P\underbrace{\begin{vmatrix}\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\end{vmatrix} dudv}_{dy\wedge dz} - \iint_\Gamma Q\underbrace{\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\end{vmatrix} dudv}_{dx\wedge dz} + \iint_\Gamma R\underbrace{\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} dudv}_{dx\wedge dy} \\
&= \iint_S Pdy\wedge dz - Qdx\wedge dz + Rdx\wedge dy.
\end{aligned}$$

We call $P(x,y,z)dy\wedge dz - Q(x,y,z)dx\wedge dz + R(x,y,z)dx\wedge dy$ a 2*-form.*

3**-forms.** By making a change of variables (using a positive Jacobian by arranging the variables in the right order), we can write a triple integral as

$$\iiint_T f(x,y,z)\; dxdydz = \iiint_\Gamma f\underbrace{\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix} dudvdw}_{dx\wedge dy\wedge dz} = \iiint_\Gamma f dx\wedge dy\wedge dz.$$

The expression $f(x,y,z)dx\wedge dy\wedge dz$ is called a 3*-form.*

Rules. As the determinant representation suggests, differential forms follow two basic rules:

(1) Transposing two differentials in one term of a differential form, changes its sign. For example, $f\ dx \wedge dy \wedge dz = -f\ dx \wedge dz \wedge dy$.

(2) If the same differential appears twice in one term of a differential form, then that component is identically zero. For example, $f\ dx \wedge dy \wedge dx = 0$.

Exterior derivatives. We now define a derivative for differential forms.

(i) The *exterior derivative df* of a 0-form f is the 1-form whose components are those of the gradient of f:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

(ii) We obtain the exterior derivative of a p-form ($p \geq 1$) by applying the exterior derivative operator to each component, concatenating its derivative with the other differentials by the symbol "$\wedge$", and simplifying the result by means of the above rules and the usual distributive law.

To illustrate the idea, let us work out two examples of exterior derivatives.

<u>Ex 1</u> If $Pdx + Qdy + Rdz$ is a 1-form, then

$$\begin{aligned}
&d(Pdx + Qdy + Rdz) \\
&= dP \wedge dx + dQ \wedge dy + dR \wedge dz \\
&= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz\right) \wedge dy \\
&\quad + \left(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \frac{\partial R}{\partial z}dz\right) \wedge dz \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dx \wedge dz + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.
\end{aligned}$$

(The fun part here is to verify the last step by multiplying out and applying the above rules.) Notice that the result is a 2-form whose components are that of the curl of $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

Ex 2 If $Pdx \wedge dy - Qdx \wedge dz + Rdy \wedge dz$ is a 2-form, its exterior derivative comes to

$$d(Pdx \wedge dy - Qdx \wedge dz + Rdy \wedge dz) = \cdots = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dz \wedge dy \wedge dz.$$

(Check this!) Hence, we obtain the 3-form whose component is the divergence of $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$.

Summary. In 3-space, there are 0-forms, 1-forms, 2-forms, and 3-forms. Whereas in the plane there are only 0-forms, 1-forms, and 2-forms. On the real line, we have 0-forms and 1-forms. The exterior derivative of a p-form is a $(p+1)$-form. If $p = 0$, this process corresponds to taking the gradient (or simply the derivative in the case of a single variable). If $p = 1$, it means taking the curl, or in case $p = 2$, the divergence.

We know from class that the curl of a gradient is identically zero, as is the divergence of a curl. This corresponds to applying the exterior derivative operator twice. In fact, one can show that two applications of the exterior derivative operator will always render a form that is identically zero.

We are now ready to state Stokes's Theorem in its most general form:

Theorem [Stokes's general theorem]
If R is a p-dimensional region and η a $(p-1)$-form, then the integral of $d\eta$ over R equals the integral of η over the boundary of R.

This theorem holds in all dimensions, even in dimensions greater than 3. If we apply it to the plane or to 3-space, we can deduce from it the theorems we have studied in class:

(1) **The fundamental theorem for line integrals.** Let R be the curve C (so that $p = 1$). Then the boundary of R consists of the two endpoints A and B of C. As our 0-form we take the scalar field $\eta = f(\mathbf{r})$. Then $d\eta = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$. Interpreting the "integral" of f over the two points A and B as $f(B) - f(A)$, Stokes's general theorem implies

$$\int_C \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = f(B) - f(A).$$

But that is exactly the fundamental theorem for line integrals:

$$\int_C \nabla f(\mathbf{r}) \cdot \mathbf{dr} = f(\mathbf{r})|_A^B .$$

The same is true in the plane and on the real line. In the latter case, the statement reduces to the regular fundamental theorem of calculus.

(2) **Green's Theorem.** Let R be the planar region Ω (i.e. $p = 2$) and call its boundary curve C. Given a 1-form $\eta = Pdx + Qdy$, you can verify that $d\eta = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$. Therefore, above theorem reads

$$\iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy = \int_C Pdx + Qdy.$$

This is what we have been calling Green's Theorem. (To get rid of the "$\wedge$", interpret it as a change of coordinates just like in the 3-form example above.)

(3) **Stokes's Theorem.** If we let R stand for the surface S in 3-space with boundary curve C and take a 1-form $\eta = Pdx + Qdy + Rdz$, then

$$d\eta = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dx \wedge dz + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

So, if we set $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, our general theorem reads

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS = \int_C Pdx + Qdy + Rdz.$$

This is the familiar version of Stokes's Theorem.

(4) **The Divergence Theorem.** Finally, we take R to be the solid T and call its boundary surface S. If $\eta = Pdy \wedge dz - Qdx \wedge dz + Rdx \wedge dy$ is a 2 form, then its exterior derivative is given by $d\eta = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz$. Setting $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, we obtain the Divergence Theorem:

$$\iiint_T \text{div } \mathbf{F} \, dxdydz = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$