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The stability of electricity prices: Estimation and inference of the Lyapunov exponents ☆

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Abstract

The aim of this paper is to illustrate how the stability of a stochastic dynamic system is measured using the Lyapunov exponents. Specifically, we use a feedforward neural network to estimate these exponents as well as asymptotic results for this estimator to test for unstable (chaotic) dynamics. The data set used is spot electricity prices from the Nordic power exchange market, Nord Pool, and the dynamic system that generates these prices appears to be chaotic in one case since the null hypothesis of a non-positive largest Lyapunov exponent is rejected at the 1 per cent level.

Keywords: Feedforward neural network; Lyapunov exponents; Nord pool; Spot electricity prices; Stochastic dynamic system

1. Introduction

The aim of this paper is to illustrate how the stability of a stochastic dynamic system is measured using the Lyapunov exponents. Specifically, we use a feedforward neural network to estimate these exponents as well as asymptotic results for this estimator to test for unstable (chaotic) dynamics, where a positive exponent is an operational definition of chaos. The data set used is spot electricity prices from the Nordic power exchange market, Nord Pool.

The estimation of the Lyapunov exponents using a feedforward neural network can be found in earlier studies such as Dechert and Gencay [1], Gencay and Dechert [2], McCaffrey et al. [3] and Nychka et al. [4]. The estimation of these exponents has been proved to be quite accurate when applying chaotic series with additive noise in simulations. However, the statistical properties of the Lyapunov exponent estimator were

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unknown until Shintani and Linton's 2004 paper (see Ref. [5]), and without the statistical distribution for this estimator, no statistical conclusion can be drawn on the dynamic structure of the empirical data.

This paper applies the statistical distribution derived in Shintani and Linton [5] to test the stability of spot electricity prices from Nord Pool, and the stochastic dynamic system that generates these prices appears to be chaotic in one case since the null hypothesis of a non-positive largest Lyapunov exponent is rejected at the 1% level.

The rest of this short paper is organized as follows: the Lyapunov exponents are in focus in Section 2, the empirical illustration is carried out in Section 3, and Section 4 concludes the paper with a remark.

2. The Lyapunov exponents

The aim of this section is fourfold: (i) to define the Lyapunov exponents of a stochastic dynamic system; (ii) to motivate why these exponents provide a measure of the stability of a stochastic dynamic system; (iii) to demonstrate how the Lyapunov exponents can be estimated from time series data; and (iv) to demonstrate how hypothesis tests of these exponents can be constructed.

2.1. Definition of the Lyapunov exponents

As argued in Bask and de Luna [6,7], and to be further explained in Section 2.2, the Lyapunov exponents can be used in the determination of the stability of a stochastic dynamic system. Specifically, assume that the stochastic dynamic system, $f:\mathbb{R}^n\to\mathbb{R}^n$, generating, for example, asset returns is

$$S_{t+1}=f(S_t)+\varepsilon^s_{t+1}, \tag{1}$$

where S_t and ε^s_t are the state of the system and a shock to the system, respectively, both at time $t\in[1,2,\dots,\infty]$. For an n-dimensional system as in (1), there are n Lyapunov exponents that are ranked from the largest to the smallest exponent:

$$\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n, \tag{2}$$

and it is these exponents that provide information on the stability properties of the dynamic system f in (1).

Now, how are the Lyapunov exponents in (2) defined? Temporarily, assume that there are no shocks to the dynamic system f in (1), and consider how the system amplifies a small difference between the initial states S_0 and S'_0:

$$S_j-S'_j=f^j(S_0)-f^j(S'_0)\simeq Df^j(S_0)(S_0-S'_0), \tag{3}$$

where $f^j(S_0)=f(\cdots f(f(S_0))\cdots)$ denotes j successive iterations of the dynamic system starting at state S_0, and where Df is the Jacobian of the system:

$$Df^j(S_0)=Df(S_{j-1})Df(S_{j-2})\cdots Df(S_0). \tag{4}$$

Then, associated with each Lyapunov exponent, λ_i, $i\in[1,2,\dots,n]$, there are nested subspaces $U^i\subset\mathbb{R}^n$ of dimension $n+1-i$ with the property that

$$\lambda_i\equiv\lim_{j\to\infty}\frac{\log_e\|Df^j(S_0)\|}{j}=\lim_{j\to\infty}\frac{1}{j}\sum_{k=0}^{j-1}\log_e\|Df(S_k)\|, \tag{5}$$

for all $S_0\in U^i-U^{i+1}$. Due to Oseledec's multiplicative ergodic theorem, the limits in (5) exist and are independent of S_0 almost surely with respect to the measure induced by the process $\{S_t\}_{t=1}^{\infty}$.[1] Then, allow for shocks to the dynamic system f in (1), meaning that the aforementioned measure is induced by a stochastic process.

[1]See Guckenheimer and Holmes [8] for a careful definition of the Lyapunov exponents and their properties.

2.2. Motivation of the Lyapunov exponents

The reason why the Lyapunov exponents provide a measure of the stability of a stochastic dynamic system may be seen by considering two different starting values of the system, where the difference is an exogenous shock at time $t = 0$. The largest Lyapunov exponent, λ_1, measures the slowest exponential rate of convergence of two trajectories of the dynamic system starting at these two different values at time $t = 0$, but with identical exogenous shocks at times $t > 0$. Indeed, λ_1 measures the convergence of a shock in the direction defined by the eigenvector corresponding to this exponent. If the difference between the two starting values lies in another direction of $\mathbb{R}^n$, then the convergence is faster. Thus, λ_1 measures the "worst case scenario."[2] In particular, when $\lambda_1 > 0$, the two trajectories diverge from each other, and for a bounded stochastic dynamic system, a positive exponent is an operational definition of chaotic dynamics.

2.3. Estimation of the Lyapunov exponents

Since the actual functional form of the dynamic system f in (1) is not known, it may seem like an impossible task to determine the stability of the system. However, it is possible to reconstruct the dynamics of the system using only a scalar time series, and, then, measure the stability of this reconstructed system. Therefore, associate the dynamic system f in (1) with an observer function, $g : \mathbb{R}^n \to \mathbb{R}$, that generates observed asset returns:

$$s_t = g(S_t) + \varepsilon_t^m, \tag{6}$$

where $s_t \in S_t$ and ε_t^m are the asset return and a measurement error, respectively, both at time t. Thus, (6) means that the asset return series

$$\{s_t\}_{t=1}^N, \tag{7}$$

is observed, which is used to reconstruct the dynamics of the system f in (1), where N is the number of consecutive returns in the time series.

Specifically, the observations in a scalar time series, like the asset return series in (7), contain information about unobserved state variables that can be used to define a state in present time. Therefore, let

$$T = (T_1, T_2, \ldots, T_M)', \tag{8}$$

be the reconstructed trajectory, where T_t is the reconstructed state at time t and M is the number of states on the reconstructed trajectory. Each T_t is given by

$$T_t = \{s_{t+m-1}, s_{t+m-2}, \ldots, s_t\}, \tag{9}$$

where m is the embedding dimension, and time $t \in [1, 2, \ldots, N - m + 1]$. Thus, T is an $M \times m$ matrix and the constants M, m and N are related as $M = N - m + 1$.

Takens [9] proved that the map

$$\Phi(S_t) = \{g(f^{m-1}(S_t)), g(f^{m-2}(S_t)), \ldots, g(f^0(S_t))\}, \tag{10}$$

which maps the n-dimensional state S_t onto the m-dimensional state T_t, is an embedding if $m > 2n$. This means that the map is a smooth map that performs a one-to-one coordinate transformation and has a smooth inverse. A map that is an embedding preserves topological information about the unknown dynamic system, like the Lyapunov exponents, and, in particular, the map induces a function, $h : \mathbb{R}^m \to \mathbb{R}^m$, on the reconstructed trajectory,

$$T_{t+1} = h(T_t), \tag{11}$$

which is topologically conjugate to the unknown dynamic system f in (1). That is,

$$h^j(T_t) = \Phi \circ f^j \circ \Phi^{-1}(T_t). \tag{12}$$

[2] An extensive discussion of the Lyapunov exponents as a measure of the stability of a stochastic dynamic system is provided in Bask and de Luna [6]. For example, it is argued therein that the average of the Lyapunov exponents, $\lambda \equiv (1/n)\sum_{i=1}^n \lambda_i$, is useful as a measure of an "average scenario."

Thus, h in (11) is a reconstructed dynamic system that has the same Lyapunov exponents as the unknown dynamic system f in (1).[3]

Now, to estimate the Lyapunov exponents of the dynamic system generating asset returns, one has to estimate h in (11). However, since

$$h: \begin{pmatrix} s_{t+m-1} \\ s_{t+m-2} \\ \vdots \\ s_t \end{pmatrix} \longrightarrow \begin{pmatrix} v(s_{t+m-1}, s_{t+m-2}, \ldots, s_t) \\ s_{t+m-1} \\ \vdots \\ s_{t+1} \end{pmatrix}, \tag{13}$$

the estimation of h reduces to the estimation of v:

$$s_{t+m} = v(s_{t+m-1}, s_{t+m-2}, \ldots, s_t). \tag{14}$$

Moreover, note that the Jacobian of h at the reconstructed state T_t is

$$Dh(T_t) = \begin{pmatrix} \frac{\partial v}{\partial s_{t+m-1}} & \frac{\partial v}{\partial s_{t+m-2}} & \frac{\partial v}{\partial s_{t+m-3}} & \cdots & \frac{\partial v}{\partial s_{t+1}} & \frac{\partial v}{\partial s_t} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{15}$$

We use a feedforward neural network to estimate the above derivatives and to derive the Lyapunov exponents in (5) (see Refs. [1–4]). A neural network model with q hidden units, u_{it}, and m inputs, x_{jt}, can be represented as

$$\begin{cases} s_t = \beta_0 + \sum_{i=1}^{q} \beta_i u_{it} + \varepsilon_t, \\ u_{it} = \dfrac{1}{1+\exp(-w_{it})}, \\ w_{it} = \gamma_{0t} + \sum_{j=1}^{m} \gamma_{ij} x_{jt}, \end{cases} \tag{16}$$

where ε_t is a random error, and time $t \in [1, 2, \ldots, N-m+1]$. The input variable x_{jt} in the estimation of a dynamic system are the lagged dependent variables, $s_{t-1}, s_{t-2}, \ldots, s_{t-m}$. The parameters to be estimated in the model are β_i, γ_{ij} and the variance of ε_t, and we use nonlinear least squares to estimate these parameters.

Hornik et al. [13] show that the mapping and its derivatives of any unknown functional form can be approximated by the neural network model in (16). This universal approximation property enables us to apply the estimates of the derivatives from the neural network for the estimates of the derivatives in (15), and the estimation of the Lyapunov exponents in (5) can be derived. In choosing the best model, we use the Schwarz information criterion (SIC) as in Nychka et al. [4] to determine the numbers of hidden units and inputs.

2.4. *Inference of the Lyapunov exponents*

Shintani and Linton [5] derive the asymptotic distribution of a neural network estimator of the Lyapunov exponents. Specifically, given some technical conditions (see Ref. [5] for details), they show that

$$\sqrt{M}(\widehat{\lambda}_{iM} - \lambda_i) \Longrightarrow \mathbb{N}(0, V_i), \tag{17}$$

where $\widehat{\lambda}_{iM}$ is the estimator of the ith Lyapunov exponent, based on the M reconstructed states on the trajectory, V_i is the variance of the ith Lyapunov exponent, and $i \in [1, 2, \ldots, n]$. The stability of a stochastic dynamic system can be measured by the estimates of these exponents, and if the value of the largest exponent is positive, then the system appears to be chaotic.

[3] Since the m-dimensional system h in (11) has a larger dimension than the n-dimensional system f in (1), the number of spurious Lyapunov exponents are $m-n$. This issue is discussed in Dechert and Gencay [10,11] and Gencay and Dechert [12].

To test the stability of a dynamic system, we consider the following null and alternative hypotheses,

$$\mathbf{H}_0 : \lambda_i \leqslant 0, \quad \mathbf{H}_1 : \lambda_i > 0, \tag{18}$$

and the test statistic is

$$\widehat{t}_i = \frac{\widehat{\lambda}_{iM}}{\sqrt{\frac{\widehat{V}_i}{M}}}, \tag{19}$$

where $\widehat{V}_i$ is a consistent estimator of V_i (see Ref. [14]), and $i \in [1, 2, \ldots, n]$. Thus, the null hypothesis is rejected when

$$\widehat{t}_i \geqslant z_\alpha, \tag{20}$$

where the significance level is

$$Pr[\mathbb{Z} \geqslant z_\alpha] = \alpha, \tag{21}$$

where $\mathbb{Z}$ is the standard normal random variable, and $i \in [1, 2, \ldots, n]$.

3. Illustration: stability of electricity prices

The Nordic power exchange market and the data set used are described in Section 3.1, and the empirical results are found in Section 3.2 that also includes a sensitivity analysis of the results.

3.1. Nord Pool and data set used

Nord Pool is a multi-national exchange for trade in power, joining the Nordic countries. Norway was, in 1991, the first of the Nordic countries to deregulate the power market, and Nord Pool ASA was established in 1993, then under the name Statnett Marked AS. Sweden started the deregulation process in 1991, and went step-wise to a deregulated power market. January 1, 1996, was the start-up of the joint Norwegian–Swedish power exchange market, renamed to Nord Pool ASA.

Finland started a power exchange market of its own, EL-EX, in 1996, and joined Nord Pool in 1997. In 1999, Elbas is launched as a separate market for power balance adjustments in Sweden and Finland, giving a fully integrated market between Norway, Sweden and Finland. Denmark Nord Pool Consulting is established in 1998, and western Denmark joins the market in 1999 as a Nordic power exchange price area. When eastern Denmark joins in 2000, the Nordic power exchange market becomes fully integrated.

The data set used is spot electricity prices from Nord Pool. Specifically, it is the daily average of the hourly system price for the period January 1, 1993, to December 31, 2005. The data are analyzed split in parts with the natural breakpoints when a new country is joining the common market. Since the prices are not stationary, we use the returns, which is the logarithm-difference of the prices, in the empirical analysis. See Table 1 for the specific dates in the integration process and for the results of the stationarity tests of the time series.

3.2. Empirical results

For each time series, we estimated the Lyapunov exponents making use of 4, 8 and 12 inputs, respectively, to the feedforward neural network. Moreover, the number of hidden units in the neural network in each case runs from 1 unit to 12 units.[4] In Table 2, the estimates of the Lyapunov exponents that minimizes SIC in each subperiod in the integration process in the power market is reported, including three types of standard errors. The three types of standard errors, $\sqrt{\widehat{V}_i/M}$ in (19), are the heteroskedasticity

[4]We have used NETLE 4, a computer program developed by C.-M. Kuan, T. Liu and R. Gencay, when estimating the Lyapunov exponents (see Refs. [2,15] for details).

Table 1
The Dickey–Fuller unit root test for the system price and the logarithm-difference of the system price (return) at Nord Pool

Countries	Date of entry of new country	Price	Return
Norway	1/1/93	−0.70	−10.67*
Norway and Sweden	1/1/96	−0.44	−9.59*
Norway, Sweden and Finland	12/29/97	−1.30	−8.67*
Norway, Sweden, Finland and western Denmark	7/1/99	−0.36	−6.13*
Norway, Sweden, Finland and Denmark**	10/1/00	−1.19	−15.66*

*Indicates that the t-test is significant at the 1% level.
**Indicates that it is eastern Denmark that joins the power exchange market at this date.

Table 2
Estimates of the Lyapunov exponents (LE) and three standard errors (SE). The top, middle and bottom SEs are the estimates based on the Newey-West, Parzen and Quadratic Spectral kernel, respectively

	1/1/93-12/31/95		1/1/96-12/28/97		12/29/97-6/30/99		7/1/99-9/30/00		10/1/00-12/31/05	
	LE	SE	LE	SE	LE	SE	LE	SE	LE	SE
λ_1	−0.0606	0.00452	−0.0623	0.00776	−0.0421	0.00740	−0.0664	0.00134	−0.0319	0.00568
		0.00473		0.00782		0.00753		0.00321		0.00525
		0.00444		0.00787		0.00685		0.00376		0.00504
λ_2	−0.0743	0.00442	−0.116	0.00840	−0.0588	0.00821	−0.0677	0.00172	−0.101	0.00426
		0.00447		0.00844		0.00840		0.00323		0.00420
		0.00437		0.00858		0.00731		0.00378		0.00431
λ_3	−0.130	0.00661	−0.148	0.0110	−0.0994	0.00495	−0.0988	0.00609	−0.125	0.00439
		0.00664		0.0109		0.00533		0.00539		0.00439
		0.00660		0.0111		0.00556		0.00531		0.00460
λ_4	−0.160	0.00651	−0.183	0.0109	−0.107	0.00508	−0.102	0.00682	−0.157	0.00517
		0.00652		0.0109		0.00558		0.00582		0.00512
		0.00650		0.0109		0.00585		0.00573		0.00520
λ_5	−0.169	0.00709	−0.235	0.0130	−0.124	0.00750	−0.171	NA	−0.176	0.00580
		0.00708		0.0129		0.00741		NA		0.00580
		0.00709		0.0131		0.00732		NA		0.00581
λ_6	−0.199	0.00811	−0.291	0.0151	−0.135	0.00778	−0.174	NA	−0.277	0.00707
		0.00811		0.0149		0.00807		NA		0.00707
		0.00812		0.0153		0.00816		NA		0.00706
λ_7	−0.211	0.00872	−0.423	0.0177	−0.145	0.00790	−0.281	0.00283	−0.323	0.00901
		0.00869		0.0172		0.00789		0.00393		0.00898
		0.00882		0.0178		0.00791		0.00418		0.00903
λ_8	−0.231	0.00847	−1.41	0.0265	−0.166	0.00850	−1.23	0.00480	−1.01	0.0169
		0.00837		0.0312		0.00850		0.00645		0.0166
		0.00865		0.0331		0.00849		0.00707		0.0190
λ_9	−0.253	0.00928			−0.267	0.0135				
		0.00941				0.0132				
		0.00968				0.0141				
λ_{10}	−0.286	0.0112			−0.284	0.00935				
		0.0110				0.0117				
		0.0114				0.0133				
λ_{11}	−0.367	0.0146			−0.290	0.00978				
		0.0145				0.0132				
		0.0148				0.0145				
λ_{12}	−1.07	0.0355			−0.296	0.0139				
		0.0360				0.0139				
		0.0336				0.0140				

NA or "not available" means that the estimated variance is negative. Note that the kernel estimator of a variance may be negative, meaning that the SE does not exist.

Table 3
Estimates of the Lyapunov exponents (LE) and three standard errors (SE)

	1/1/93-12/31/95		1/1/96-12/28/97		12/29/97-6/30/99		7/1/99-9/30/00		10/1/00-12/31/05	
	LE	SE	LE	SE	LE	SE	LE	SE	LE	SE
λ_1	−0.0806	0.00410	−0.0623	0.00776	−0.0215	0.00554	0.0670	0.0169	−0.0386	0.00294
		0.00408		0.00782		0.00548		0.0167		0.00286
		0.00398		0.00787		0.00551		0.0168		0.00306
λ_2	−0.0855	0.00435	−0.116	0.00840	−0.0482	0.00594	−0.0193	0.00852	−0.0775	0.00336
		0.00432		0.00844		0.00600		0.00862		0.00336
		0.00432		0.00858		0.00594		0.00843		0.00336
λ_3	−0.118	0.00545	−0.148	0.0110	−0.0734	0.00663	−0.0451	0.00762	−0.119	0.00400
		0.00544		0.0109		0.00665		0.00769		0.00395
		0.00541		0.0111		0.00665		0.00761		0.00405
λ_4	−0.134	0.00521	−0.183	0.0109	−0.0940	0.00650	−0.0757	0.00811	−0.131	0.00413
		0.00515		0.0109		0.00650		0.00806		0.00408
		0.00550		0.0109		0.00663		0.00822		0.00420
λ_5	−0.176	0.00653	−0.235	0.0130	−0.100	0.00769	−0.130	0.0113	−0.147	0.00468
		0.00648		0.0129		0.00769		0.0114		0.00460
		0.00667		0.0131		0.00769		0.0113		0.00473
λ_6	−0.201	0.00715	−0.291	0.0151	−0.124	0.00724	−0.148	0.0118	−0.170	0.00534
		0.00704		0.0149		0.00717		0.0117		0.00532
		0.00719		0.0153		0.00743		0.0119		0.00537
λ_7	−0.213	0.00789	−0.423	0.0177	−0.143	0.00875	−0.271	0.0195	−0.196	0.00621
		0.00783		0.0172		0.00875		0.0196		0.00617
		0.00793		0.0178		0.00918		0.0197		0.00625
λ_8	−0.237	0.00875	−1.41	0.0265	−0.148	0.00931	−1.12	0.0576	−0.263	0.00679
		0.00860		0.0312		0.00925		0.0589		0.00689
		0.00878		0.0331		0.00944		0.0562		0.00706
λ_9	−0.284	0.00956			−0.175	0.0102			−0.300	0.00768
		0.00943				0.0102				0.00781
		0.00961				0.0103				0.00788
λ_{10}	−0.330	0.00937			−0.206	0.0132			−0.344	0.00863
		0.00982				0.0132				0.00878
		0.0105				0.0134				0.00913
λ_{11}	−0.400	0.0115			−0.319	0.0182			−0.471	0.0108
		0.0121				0.0185				0.0113
		0.0124				0.0188				0.0115
λ_{12}	−1.86	0.0710			−0.506	0.0318			−0.593	0.0165
		0.0675				0.0329				0.0164
		0.0535				0.0278				0.0160

The top, middle and bottom SEs are the estimates based on the Newey-West, Parzen and Quadratic Spectral kernel, respectively. Outliers are eliminated in the estimations.

and autocorrelation consistent estimators based on Newey-West, Parzen and Quadratic Spectral kernels (see Ref. [14]).[5]

Clearly, there is no unstable (chaotic) dynamics in the time series since all estimates of the largest Lyapunov exponent are negative.

When inspecting the time series, it is clear that there are some extreme values, outliers. To see their impact on the result, we eliminated the outliers from the time series and performed the same analysis as above.[6] See Table 3 for the results.

[5] Detailed results of the estimations are available on request from the authors.

[6] The excluded outliers are from February 28, 1994, to March 2, 1994, December 8, 1998, January 24, 2000, February 5, 2001, from December 5, 2002, to January 14, 2003. In total, 44 outliers are excluded.

When eliminating the outliers, the dynamic system appears to be chaotic for the period July 1, 1999, to September 30, 2000, since the null hypothesis in (18) is rejected for the largest Lyapunov exponent at the 1% level. For all other time series, there is no chaotic dynamics.

4. Concluding remark

We should also mention impulse–response functions as another tool to measure the stability of a stochastic dynamic system. Specifically, Koop et al. [16] and Potter [17] extend, in an appealing way, the linear technique of impulse–response functions to the non-linear case, although they show that there is no unique definition of such a function when a non-linear dynamic system is considered. Certainly, impulse–response functions are useful graphical tools in the non-linear case, even if they are less appropriate when inference needs to be performed on a change in the stability. It is, therefore, we recommend the estimation and inference of the Lyapunov exponents to measure the stability of a stochastic dynamic system.

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