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USING THE CORRELATION EXPONENT TO DECIDE WHETHER AN ECONOMIC SERIES IS CHAOTIC

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'In Roman mythology, the god Chaos is the father of the god Time' (Robert Graves, *I Claudius*—Arthur Barker, London, 1934)

SUMMARY

We consider two ways of distinguishing deterministic time-series from stochastic white noise; the Grassberger–Procaccia correlation exponent test and the Brock, Dechert, Scheinkman (or BDS) test. Using simulated data to test the power of these tests, the correlation exponent test can distinguish white noise from chaos. It cannot distinguish white noise from chaos mixed with a small amount of white noise. With i.i.d. as the null, the BDS correctly rejects the null when the data are deterministic chaos. Although the BDS test may also reject the null even when the data are stochastic, it may be useful in distinguishing between linear and nonlinear stochastic processes.

1. INTRODUCTION

Econometricians and applied economists often take the viewpoint that unforecastable shocks and innovations continually bombard the actual economy. In other words, the economy is essentially stochastic in nature. By contrast, some models in the economic theory literature (e.g. Grandmont, 1985) suggest that an essential nonlinearity in real economic forces permits deterministic time-series to have the appearance of chaos. It is our purpose here to examine some of the tests that have been proposed to resolve the issue. The choice is whether the economy is better modelled as (1) essentially linear in structure with significant stochastic elements, or (2) having a nonlinear structure with insignificant stochastic forces or (3) having a clear nonlinear structure but with significant stochastic shocks. Much of the chaos literature only discusses the first two of these three possibilities. Our results cast doubt on the hope that stochastic shocks can be reduced to insignificance in nonlinear models when doing empirical work.

In applied economic models it is common practice for the unforecastable shocks to an economy to be equated to the residuals of a specification. Assume that the shocks are independent and identically distributed (or i.i.d., for short). Further, assume the existence of second moments. Necessary conditions for a series x_t to be i.i.d. are: (1) that the mean and variance of x_t are constant, and (2) autocovariances $\text{cov}(x_t, x_{t-k})$ are all zero for all $k \neq 0$. These are called the 'white noise conditions', and a series that has them is called white noise. Clearly an i.i.d. series is a white noise but not necessarily vice-versa, although a Gaussian white

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noise is i.i.d. It is well known that non-i.i.d. processes can have the white noise properties; see the examples in Granger (1983).

Some deterministic processes can also have white noise properties. Many find this observation interesting, and even surprising. A chaotic deterministic process is often characterized by its non-periodic trajectory. In particular, some chaos has first and second moment properties (means, variances and covariances) that are the same as a stochastic process. If these properties are the same as white noise, the process will be called here 'white chaos'. An example of such a process is the tent map, where it is generated by:

$$\begin{aligned} x_t &= a^{-1}x_{t-1}, \qquad \text{if } 0 \leqslant x_{t-1} < a \\ &= (1-a)^{-1}(1-x_{t-1}), \qquad \text{if } a \leqslant x_{t-1} \leqslant 1. \end{aligned} \tag{1}$$

Sakai and Tokumaru (1980) show that the autocorrelations for the tent map are the same as that of some first-order autoregressive process. Especially when the constant a is near to 0·5, the autocorrelations for tent map are close to that of an i.i.d. process.

Time-series data from the 'logistic map' have similar properties. The logistic map is given by:

$$x_t = 4x_{t-1}(1 - x_{t-1}) \tag{2}$$

with some suitable starting value x_0 in the range (0, 1). Table I shows the estimated autocorrelations and partial autocorrelations for a tent map and logistic map. The autocorrelations for x_t are all small and insignificantly different from zero, indicating that these series have at least the dynamic part of the white noise properties. However, x_t is clearly not i.i.d., as x_t is generated from a nonlinear deterministic process. Surveys of the relevance of chaos in economics can be found in Frank and Stengos (1988) and Brock and Sayers (1988).

Table I. Autocorrelation and partial autocorrelation function for the tent map and logistic map

	The original series				The squares of observations			
	Tent map estimated		Logistic map estimated		Tent map estimated		Logistic map estimated	
Lag	ACF	PACF	ACF	PACF	ACF	PACF	ACF	PACF
1	0·001	0·001	0·016	0·016	−0·215*	−0·215*	−0·221*	−0·221*
2	−0·006	−0·006	0·006	0·006	−0·058*	−0·110*	0·001	−0·050*
3	−0·012	−0·012	0·004	0·004	−0·024	−0·066*	0·001	−0·009
4	0·001	0·001	0·006	0·006	−0·000	−0·030*	0·005	0·004
5	0·004	0·003	−0·001	−0·001	0·005	−0·008	0·004	0·007
6	−0·008	−0·008	−0·025	−0·025	−0·008	−0·013	−0·026*	−0·025
7	−0·003	−0·003	0·003	0·004	−0·003	−0·009	0·010	−0·000
8	0·006	0·006	−0·003	−0·002	0·009	0·004	−0·003	−0·002
9	−0·006	−0·007	−0·002	−0·002	−0·008	−0·007	−0·004	−0·005
10	0·003	0·003	0·012	0·012	0·006	0·004	0·014	0·012

Note: The initial value is 0·1 and 6000 observations are generated. The first 100 observations are truncated. For the tent map, the constant a in (1) is 0·49999. Asterisks indicate significant lags.

2. CORRELATION EXPONENT TABLES

The existence of deterministic white chaos raises the question of how one can distinguish between it and a true stochastic white noise, such as an i.i.d. series. One possibility is to use a statistic known as the 'correlation exponent'. Let $\{x_t\}$ be a univariate time series. Define first the 'correlation integral' as

$$C(\varepsilon) = \lim_{N\to\infty} \frac{1}{N^2} \{\text{number of pairs } (i, j) \text{ such that } | x_i - x_j | < \varepsilon\} \tag{3}$$

Thus, all pairs of values of the series are compared and those within ε of each other are counted; they are then normalized by the number of all possible pairs N^2. The limit is taken as N grows large.

Intuitively, $C(\varepsilon)$ measures the probability that any particular pair in the time-series is close. Suppose that for small values of ε, $C(\varepsilon)$ grows exponentially at the rate v:

$$C(\varepsilon) \approx \varepsilon^{v} \tag{4}$$

The symbol v is the above-mentioned correlation exponent; it is also called the 'correlation dimension'. Grassberger and Procaccia (1983) show that the correlation exponent is bounded above by the Hausdorff dimension and information dimension.

These dimensions are measures of the local structure of fractal attractors. For some chaotic process the dimension of the attractors is fractional. Notice that the correlation exponent is used not only for distinguishing white chaos from stochastic white noise, but also for distinguishing the low-dimensional chaos from high-dimensional stochastic process.

A generalization is needed to obtain a useful set of statistics. Let $X_{t,m}$ be the vector of m consecutive terms $(x_t, x_{t+1}, \ldots, x_{t+m-1})$. Define the correlation integral as:

$$C_m(\varepsilon) = \lim_{N\to\infty} N^{-2} \{\text{number of pairs } (i, j) \text{ such that each corresponding component of } X_{i,m} \text{ and } X_{j,m} \text{ is less than } \varepsilon \text{ apart}\}. \tag{5}$$

Thus, for each $X_{i,m}$ all other lengths of m of the series are compared to it . If $X_{i,m}$ and $X_{j,m}$ are ε close to each other, then they are counted. Similarly, for small values of ε, $C_m(\varepsilon)$ grows exponentially at the rate v_m:

$$C_m(v) \approx \varepsilon^{v_m} \tag{6}$$

The length m is called the 'embedding dimension. By properly choosing m, the correlation exponent v of a deterministic white chaotic process can be numerically measured by v_m provided $m > v$. Grassberger and Procaccia (1983) give some numeral values of v_m with different m values for logistic map and Hénon map.

However, it is easily seen that, for stochastic white noise, $v_m = m$ for all m. If the correlation exponent v is very large (so that one has a high-dimensional chaotic process), then it will be very difficult to estimate v without an enormous amount of data. It is also true that it is difficult to distinguish high-dimensional chaos from stochastic white noise by just looking at estimates of v_m. The length of economic time-series is usually short by comparison with the physical sciences, and this fact diminishes the usefulness for macroeconomics of statistics based on the correlation exponent. For choosing the proper sample size, refer to the paper by Smith (1992a). The correlation exponent, v_m can be approximated by

$$\hat{v}_m = \lim_{\varepsilon\to 0} \frac{\mathrm{d}\log(C_m(\varepsilon))}{\mathrm{d}\log(\varepsilon)}. \tag{7}$$

There are several empirical ways of estimating the correlation exponent. For example, ordinary linear regression is used by Denker and Keller (1986) and by Scheinkman and Lebanon (1989); generalized least-square is used by Cutler (1991); and the random coefficient regression is used by Ramsey and Yuan (1989). In the regression method a set of $\log C_m(\varepsilon)$ and $\log \varepsilon$ are obtained from the data series by 'properly' choosing some values of ε. It is obvious that the choice of the range of ε is arbitrary and subjective. Brock and Baek (1991) also note this point.

The other type of estimation of the correlation exponent is the regression-free method. The typical examples are the point estimator presented in this paper and the binomial estimator used by Smith (1992a,b). Because of the similarity between these two estimators a comparison will be made in the following. More extensive references for these estimations of correlation exponent can be found in papers listed above.

The point estimator is defined by

$$v_{m,j} = \frac{\log(C_m(\varepsilon_j)) - \log(C_m(\varepsilon_{j+1}))}{\log(\varepsilon_j) - \log(\varepsilon_{j+1})}, \tag{8}$$

where ε_j and ε_{j+1} are constants greater than zero and less than 1. That is $\varepsilon_j = \phi^j$ with $0 < \phi < 1$, $j \geqslant 1$, and $C_m(\varepsilon_j)$ is the correlation integral defined by (5). Notice that $v_{m,j}$ is the point elasticity of $C_m(\varepsilon)$ on ε. In the following empirical work the minimum of the sample will be subtracted from each observation and then divided by the sample range. Hence, the transformed observations will take a value between zero and one. This ensures that the distance between any two points of $\{x_t\}$ is less than one. Thus the constant ε_j can also be restricted within the range of zero and one, and the possible range of ε is objectively given. As shown in (4) and (6), the correlation exponent can be observed only when ε is small enough. Let $\varepsilon_j = \phi^j$, for $0 < \phi < 1$ and $j \geqslant 1$. Then the correlation exponent is related to the value of $v_{m,j}$ for sufficiently large j.

The point estimator is also used by Brock and Baek (1991). They derived the statistical property of this estimator under the assumption that x_t is i.i.d. However, this statistical property cannot apply to low-dimensional chaos. When statistical inference for chaos is conducted, the statistic should be based on an assumption of low information dimension. Also, hypothesis testing based on the x_t being i.i.d. cannot be used for testing the difference between deterministic chaos and a stochastic process. This is because the rejection of the null is caused by dependence among the x_t. Our section on the BDS test will give details of this argument.

The assumptions on x_t can be relaxed from i.i.d. towards some degree of mixing (as in Denker and Keller, 1986 and Hiemstra, 1992). But the derived statistics are still not appropriate for statistical inference and hypothesis testing for chaos. Any statistic based on the null of stochastic x_t, instead of more general assumptions, will give the estimate $v_m = m$. If the statistic is used for low-dimensional chaos, which has $v_m = v$ and $v < m$, the statistical inference will be incorrect and the conclusion from the hypothesis testing is ambiguous. Furthermore, the correlation exponent is only approximated for ε close to zero. Any statistic based on the correlation exponent needs to consider this point.

Smith (1992b) defines his binomial estimator with this in mind. He uses the independence assumption in a different way. In his estimator for correlation exponent, independence is applied to the inter-point distance. If there are N data points for x_t, then there are $N(N-1)/2$ inter-point distances. The 'independent distance hypothesis' (IDH) implies that these inter-point distances are independent when $\varepsilon \to 0$ (Theiler, 1990). This IDH is different from an independence assumption on x_t, and it avoids the problem of $v_m = m$ if x_t is assumed to be stochastic. Let N_j be the number of inter-point distances less than ε_j, where $\varepsilon_j = \varepsilon_0\phi^j$ for $j \geqslant 0$

and $0 < \phi < 1$. Based on IDH, Smith's binomial estimator is

$$\tilde{v}_m = \frac{\log\left(\sum_{j=0}^{K} N_{j+1}\right) - \log\left(\sum_{j=0}^{K-1} N_j\right)}{\log \phi} \tag{9}$$

For sufficiently large N, equation (5) implies that

$$C(\varepsilon_j) = \frac{N_j}{[N(N-1)/2]}$$

and

$$\tilde{v}_m = \frac{\log\left(\sum_{j=0}^{K} C(\varepsilon_{j+1})\right) - \log\left(\sum_{j=0}^{K-1} C(\varepsilon_j)\right)}{\log \varepsilon_{j+1} - \log \varepsilon_j} \tag{10}$$

An alternative estimator for the correlation exponent used by Smith (1992a) is

$$\tilde{\tilde{v}}_m = \frac{\frac{1}{K}\sum_{j=1}^{K} (\log N_j - \log N_{j-1})}{\log \phi} \tag{11}$$

It is equivalent to

$$\tilde{\tilde{v}}_m = \frac{1}{K}\sum_{j=1}^{K}\left(\frac{\log C(\varepsilon_j) - (\log C(\varepsilon_{j-1})}{\log \varepsilon_j - \log \varepsilon_{j-1}}\right) \tag{12}$$

which is the average of the point estimator in (8) for some range values of ε_j. The following simulation shows the properties of the point estimator and consequentially it also provides some of the properties of the binomial estimator $\tilde{\tilde{v}}_m$.

Table II shows the point estimates of the correlation exponent, $v_{m,j}$, for six values of the embedding dimension m and 25 epsilon values. The table uses data from the logistic map with sample sizes of 500 and 5900. For most macroeconomic series, 500 is a large but plausible sample size (approximately 40 years of monthly data). A sample of 5900 observations is large compared to most economic time-series. However, financial data are often of this size (20 years of week-day daily price data). The data are chaotic and known to have a true correlation dimension of one. Thus, for $m \geqslant 2$ and small ε_j (or large j), where $\varepsilon_j = 0{\cdot}9^j$, the figures in the table should all equal one if the sample size is large enough. Using the larger sample of 5900 observations the values are indeed near one for $2 \leqslant m \leqslant 5$ and small epsilon, or $j > 20$. There does appear to be a slight downward bias, with most values under $1{\cdot}0$.

The estimate is less reliable for $m = 10$. Using a much smaller sample of 500 observations, this general pattern is the same but with higher variability. Looking at the table for $m > 1$ and small enough epsilon (or large j) gives ample visual evidence that the quantity being estimated is close to unity. It is stable as m goes from 2 to 5. This result is consistent with those of Grassberger and Procaccia (1983). In particular, they also found that v_m is underestimated when $m = 1$.

These results are thus encouraging, as tables such as these do give the correct pattern if the data are truly chaotic. The same results were found with data from the Hénon map, but these are not shown. Table III shows the same results for 'stochastic' Gaussian white noise series, of sample sizes 500 and 5900 respectively. Theory suggests that these estimates should equal

Table II. Correlation exponents for logistic map

	Sample size = 500						Sample size = 5900					
	m						m					
j	1	2	3	4	5	10	1	2	3	4	5	10
16	0·78	0·96	1·05	1·00	0·95	0·80	0·76	0·89	0·97	1·02	1·14	2·16
17	0·66	0·82	0·84	0·88	0·96	1·27	0·77	0·90	0·97	1·01	1·11	1·41
18	0·66	0·80	0·78	0·84	0·91	0·80	0·77	0·90	0·97	1·00	1·08	1·32
19	0·74	0·86	0·85	0·87	0·98	2·17	0·79	0·90	0·96	0·99	1·05	1·39
20	0·77	0·90	0·92	1·02	1·12	0·76	0·78	0·90	0·96	0·98	1·02	1·19
21	0·82	0·88	0·95	0·92	0·92	0·73	0·78	0·89	0·94	0·97	0·99	1·18
22	0·77	0·89	0·90	0·94	0·96	0·79	0·78	0·89	0·95	0·98	1·02	1·33
23	0·73	0·84	0·87	0·88	0·82	0·53	0·77	0·89	0·94	0·97	1·00	1·38
24	0·86	0·89	0·93	0·92	0·90	1·04	0·78	0·88	0·93	0·97	0·99	1·22
25	0·81	0·89	0·94	0·93	0·86	0·50	0·80	0·91	0·97	1·00	1·03	1·15
26	0·81	0·95	0·99	1·01	0·93	0·39	0·81	0·91	0·96	0·99	1·02	1·05
27	0·77	0·83	0·90	0·91	0·87	0·84	0·80	0·89	0·95	0·98	1·00	1·12
28	0·79	0·81	0·86	0·87	0·83	0·45	0·80	0·88	0·94	0·96	0·99	1·14
29	0·81	0·81	0·87	0·88	0·88	0·97	0·80	0·88	0·94	0·97	0·97	1·22
30	0·74	0·88	1·00	0·98	0·97	1·08	0·81	0·89	0·95	0·98	0·98	1·11
31	0·74	0·85	0·93	1·02	1·16	0·79	0·82	0·90	0·95	0·98	0·99	0·98
32	0·68	0·79	0·83	0·80	0·82	0·86	0·82	0·90	0·95	0·97	0·98	0·98
33	0·83	0·89	1·02	1·06	0·88	0·23	0·83	0·91	0·96	0·97	0·98	1·10
34	0·83	0·93	0·99	1·06	0·81	0·00	0·82	0·90	0·95	0·98	0·96	0·95
35	0·81	0·85	0·84	0·86	0·97	0·47	0·83	0·90	0·94	0·97	0·95	1·01
36	0·77	0·86	0·84	0·71	0·87	1·59	0·82	0·89	0·93	0·95	0·94	0·94
37	0·88	0·83	0·82	0·93	0·97	0·90	0·82	0·89	0·94	0·96	0·94	1·01
38	0·88	0·95	0·93	0·89	0·78	0·00	0·85	0·92	0·96	0·98	0·99	1·02
39	0·87	0·90	0·96	0·94	0·82	1·00	0·86	0·91	0·96	0·96	0·98	1·08
40	0·74	0·80	0·81	0·78	0·90	1·94	0·83	0·91	0·97	0·99	1·05	1·20

Note: Each column represents different embedding dimension m and each row shows different value of j such that $\varepsilon_j = 0{\cdot}9^j$. Each cell is the point estimate of the correlation exponent, $v_{m,j}$, as defined in (8).

m and thus should take the value 1, 2, 3, 4, 5 and 10 in the columns. The pattern in these tables is as predicted by the theory for small enough epsilon, say $j > 25$. Note that there is a fairly consistent downward bias in the calculated dimension. The results from the correlation exponent tables are rather similar to those from the regression approach, such as Ramsey and Yuan (1990).

Interpretation of this type of 'ocular econometrics' is not easy. One has to be selective as to which parts of the table are emphasized. Statistical inference is needed for more accurate conclusions. Brook and Baek (1991) and Smith (1991a,b) give statistical properties for the correlation exponent. The simulations as shown in Tables II and III reveal an important message on the empirical use of the correlation exponent. The choice of epsilon is important. Different ranges of epsilon may give different conclusions. Further, distinguishing stochastic white noise from white chaos based on the correlation exponent is only valid for small epsilon. Brook and Baek (1991) have similar and intensive simulations on point estimates for Gaussian white noise. It should be noticed that their statistic is for all epsilon and not only for small epsilon, as is required for the definition of the correlation exponent.

Smith (1992b) has a simulation for the binomial estimates of low-dimension chaos. It is clear from Table II that the quality of binomial estimates is related to the range of epsilon chosen.

Table III. Correlation exponents for Gaussian white noise

	Sample size = 500 *m*						Sample size = 5900 *m*					
j	1	2	3	4	5	10	1	2	3	4	5	10
16	0·78	1·56	2·34	3·13	3·91	7·49	0·66	1·32	1·97	2·63	3·30	6·61
17	0·83	1·67	2·47	3·25	4·01	9·95	0·72	1·43	2·15	2·86	3·58	7·21
18	0·88	1·80	2·76	3·74	4·73	9·79	0·76	1·52	2·28	3·04	3·81	7·61
19	0·89	1·82	2·75	3·61	4·26	5·79	0·80	1·61	2·41	3·22	4·03	8·15
20	0·88	1·75	2·62	3·46	4·39	9·70	0·84	1·68	2·52	3·35	4·20	8·43
21	0·90	1·83	2·68	3·38	3·89	10·43	0·87	1·73	2·60	3·46	4·33	8·51
22	0·95	1·86	2·87	3·87	5·17	3·85	0·89	1·78	2·67	3·55	4·43	8·46
23	0·96	1·91	2·93	4·01	4·96	—	0·91	1·83	2·75	3·67	4·60	9·43
24	0·96	1·90	2·90	3·64	4·33	—	0·93	1·86	2·79	3·73	4·69	9·24
25	0·99	1·95	2·91	3·71	4·55	—	0·94	1·88	2·81	3·76	4·69	10·61
26	0·99	2·01	2·81	4·19	5·22	—	0·95	1·90	2·85	3·79	4·75	7·14
27	0·99	2·06	3·12	3·88	4·59	—	0·96	1·92	2·87	3·79	4·67	8·84
28	0·99	1·89	2·64	3·56	2·99	—	0·97	1·94	2·91	3·85	4·86	5·88
29	0·99	2·05	3·08	5·06	8·52	—	0·97	1·95	2·93	3·93	4·94	11·89
30	0·99	1·99	3·04	4·33	5·75	—	0·98	1·97	2·95	3·92	4·98	—
31	0·99	2·13	2·88	4·17	1·73	—	0·98	1·97	2·95	3·90	4·88	—
32	0·94	1·82	2·73	4·01	4·85	—	0·99	1·98	2·97	4·00	5·23	—
33	1·00	2·07	3·14	6·09	3·85	—	0·99	1·98	3·01	4·11	5·32	—
34	0·99	1·94	2·73	3·39	6·58	—	0·99	1·99	2·92	3·88	4·95	—
35	0·96	1·83	2·46	0·00	—	—	0·99	1·97	2·93	4·00	5·53	—
36	0·96	1·82	2·69	8·04	—	—	0·99	1·98	2·98	4·00	5·62	—
37	0·96	1·97	2·08	3·85	—	—	0·99	1·98	2·93	3·87	4·81	—
38	0·97	1·84	2·67	0·00	—	—	1·00	1·97	2·94	4·00	5·37	—
39	0·95	2·26	2·99	6·58	—	—	0·99	1·98	2·91	3·85	2·86	—
40	1·00	1·71	1·94	0·00	—	—	1·00	2·02	3·03	4·15	5·38	—

Note: See footnote in Table II.

In addition, the reliability in estimating the correlation exponent varies with the sample size and the embedding dimension. Ramsey and Yuan (1989) and Ramsey, Sayers, and Rothman (1990) also recognize this point.

It follows that a chaotic series can be distinguished if it has a fairly low correlation dimension, say five or less. Random number generators on computers typically have at least this dimension. Brock (1986) reports a dimension of approximately 11 for the Gauss random number generator. It is also true that it is difficult to distinguish high-dimensional chaos from stochastic white noise just by looking at estimates of v_m. For more on choosing the proper sample sizes and embedding dimensions, refer to the papers by Smith (1992a,b), Sugihara and May (1990), and Cheng and Tong (1992).

Statistical inference on chaos is a difficult task, and it is not easy to solve all the issues at the same time. When using statistics based on the correlation exponent for chaos, one must bear in mind their limitations. The point estimate tables indicate that it may be possible to distinguish a low-dimensional chaotic process from a truly stochastic i.i.d. process. Operationally, a 'stochastic process' here is a high-dimensional chaotic process, such as the random number generators used in this experiment.

To be useful with economic data these techniques must cope with added, independent 'measurement error'. With this in mind, data were formed

$$z_t = x_t + \sigma\varepsilon_t \tag{13}$$

where x_t is white chaos generated by the logistic map as in (2), ε_t is i.i.d. Gaussian white noise and σ^2 is varied to produce four alternative 'signal to noise ratios' (S). We show the results in Table IV for various signal to noise ratios.

The point estimates for the correlation exponent are shown only for $m = 1, 3, 5$ and for a reduced set of epsilon. Note that if the data were pure white chaos, the numbers should be approximately equal (to one) for each m value. For the majority of the table the estimates increase approximately proportionally to m, suggesting that the data are stochastic. Only for the largest S values and for a narrow range of epsilon values ($10 \leqslant j \leqslant 20$, say) does the estimate seem to be approximately constant. Smith (1992b) also gives estimators of the correlation exponent and variance of noise for the chaos with additive noise. From Table IV it is found that his estimators are sensitive to the range of epsilon chosen.

In a sense the correlation technique is working too well, since the true data-generating mechanism does contain a stochastic (or high-dimensional) element, ε_t. This is what is 'seen' by the point estimates. The low-dimensional deterministic chaos component, x_t, is totally missed, even when it has a much larger variance than the noise. It may well be that when deterministic chaos is present in economic data, it can be found only if it contains very little measurement error. Further, the generating process must be of low correlation dimension for detection to take place.

A possible source of such data is stock market prices. Two series were used: daily rates of returns for IBM and the Standard and Poors 500 stock index, for the period 2 July 1962 to 31 December 1985, giving 5903 observations. The autocorrelations for both series were uniformly very small and generally insignificant, as predicted by efficient market theory. Table V shows the point estimates for the IBM returns and S&P 500 returns. The patterns of the estimates for these two returns are extremely similar. Values were small for larger epsilon. Further, for small enough epsilon, the estimates are seen to increase approximately with m, but

Table IV. Correlation exponents for logistic map with additive white noise (sample size = 5900)

	m			m			m			m		
	1	3	5	1	3	5	1	3	5	1	3	5
j	$\sigma^2 = 0.3$, $S = 0.4$			$\sigma^2 = 0.1$, $S = 1.2$			$\sigma^2 = 0.01$, $S = 12$			$\sigma^2 = 0.001$, $S = 120$		
4	0·01	0·02	0·03	0·01	0·03	0·06	0·21	0·64	1·07	0·65	2·27	3·59
8	0·13	0·40	0·66	0·21	0·64	1·08	0·67	2·02	3·46	0·69	1·40	1·83
12	0·45	1·34	2·24	0·56	1·66	2·76	0·73	1·38	1·99	0·71	1·03	1·43
16	0·72	2·15	3·57	0·79	2·30	3·80	0·80	1·49	2·10	0·76	1·01	1·18
20	0·87	2·61	4·33	0·90	2·65	4·39	0·88	2·01	3·12	0·79	1·07	1·21
24	0·94	2·83	4·74	0·96	2·85	4·72	0·94	2·50	4·03	0·84	1·39	1·78
28	0·98	2·89	4·75	0·98	2·92	4·83	0·97	2·75	4·74	0·90	1·96	2·94
32	0·99	2·98	5·03	0·99	3·00	4·90	0·98	2·85	4·72	0·95	2·45	3·95
36	0·99	3·00	5·04	1·00	2·95	4·91	1·00	2·96	5·00	0·98	2·76	4·62
40	1·00	3·11	9·16	1·00	2·92	5·31	1·00	3·02	5·46	0·99	2·82	5·39

Note: The variance of the logistic map is about 0·12 and σ^2 is the variance of the white noise. S = (variance of logistic map/variance of noise), i.e. 'signal/noise ratio'. See also footnote in Table II. Only partials of j are shown in this table.

Table V. Correlation exponents for daily IBM and S&P500 rate of returns (from 2 July 1962 to 31 December 1985 with 5903 observations)

	IBM daily returns m						S&P 500 daily returns m					
j	1	2	3	4	5	10	1	2	3	4	5	10
16	0·20	0·38	0·55	0·72	0·87	1·56	0·33	0·63	0·89	1·14	1·36	2·27
17	0·26	0·50	0·72	0·94	1·14	2·04	0·40	0·75	1·07	1·37	1·64	2·71
18	0·33	0·63	0·91	1·18	1·43	2·56	0·46	0·87	1·24	1·59	1·90	3·13
19	0·39	0·76	1·10	1·43	1·73	3·09	0·52	0·99	1·42	1·81	2·17	3·54
20	0·46	0·90	1·31	1·68	2·05	3·63	0·58	1·10	1·58	2·03	2·43	3·93
21	0·53	1·03	1·49	1·93	2·35	4·17	0·63	1·21	1·74	2·23	2·68	4·30
22	0·59	1·15	1·68	2·18	2·65	4·75	0·69	1·32	1·90	2·43	2·91	4·63
23	0·65	1·27	1·85	2·40	2·93	5·26	0·73	1·41	2·03	2·61	3·13	4·97
24	0·70	1·37	2·00	2·60	3·18	5·79	0·77	1·49	2·16	2·77	3·33	5·32
25	0·74	1·46	2·14	2·78	3·41	6·27	0·81	1·57	2·27	2·92	3·51	5·65
26	0·79	1·55	2·28	2·98	3·66	6·77	0·84	1·63	2·37	3·07	3·69	5·94
27	0·83	1·63	2·40	3·14	3·86	7·22	0·87	1·69	2·45	3·16	3·82	6·44
28	0·85	1·69	2·49	3·26	4·02	7·68	0·89	1·74	2·53	3·28	3·96	6·59
29	0·88	1·74	2·58	3·39	4·19	8·08	0·91	1·78	2·60	3·39	4·12	7·00
30	0·89	1·76	2·61	3·44	4·27	8·44	0·93	1·82	2·67	3·49	4·25	6·93
31	0·92	1·82	2·70	3·56	4·42	8·81	0·94	1·86	2·72	3·56	4·35	7·81
32	0·93	1·85	2·76	3·65	4·52	8·56	0·95	1·88	2·77	3·63	4·47	8·37
33	0·95	1·89	2·83	3·74	4·62	9·54	0·96	1·89	2·80	3·69	4·56	7·67
34	0·96	1·91	2·86	3·82	4·82	9·10	0·97	1·92	2·84	3·74	4·60	7·60
35	0·97	1·93	2·88	3·82	4·79	9·84	0·98	1·94	2·88	3·79	4·75	11·00
36	0·96	1·90	2·84	3·79	4·78	14·62	0·98	1·95	2·90	3·82	4·71	7·75
37	0·99	1·98	2·97	3·98	5·04	10·43	0·98	1·94	2·90	3·84	4·75	7·09
38	0·96	1·92	2·87	3·77	4·85	—	0·99	1·96	2·91	3·81	4·74	10·43
39	0·98	1·96	2·90	3·89	4·89	—	0·99	1·98	2·93	3·95	4·79	—
40	0·98	1·96	2·94	3·86	4·94	—	0·99	1·98	2·92	3·84	4·81	—

Note: See footnote in Table II.

again with a downward bias. The pattern is consistent either with these returns being a stochastic white noise or being chaotic with a true correlation dimension of around six. To distinguish between these alternatives higher m values would have to be used. This would require a much larger sample size. These stock price series are not low-dimensional chaos, according to this technique. Other studies involving aggregate and individual stock market time-series confirm this experience (Scheinkman and LeBaron, 1989).

3. THE BDS TEST

Looking for patterns in tables may be useful, but as different people may reach different conclusions it is preferable to have a formal test with no subjectivity. Brook and Baek (1991) describe the statistical properties of the point estimator for the correlation exponent under the i.i.d. assumption for x_t. We are interested in how well it detects the presence of chaos. Since the point estimator for the correlation exponent is equal to the point elasticity of the correlation integral, Brock and Baek's statistic is derived from a statistic using the correlation integral. Such a statistic was developed by Brock, Dechert and Scheinkman (1987) (henceforth BDS).

We examine the properties of the BDS statistic here, yielding some insight into the statistic proposed by Brock and Baek. A good discussion of a BDS application can be found in Brock and Sayers (1988).

Using the correlation integral $C_m(\varepsilon)$ defined in (5), the BDS test statistic is

$$S(m,\varepsilon) = C_m(\varepsilon) - [C_1(\varepsilon)]^m \tag{14}$$

The null hypothesis is

$$H_0: x_t \text{ is i.i.d.} \tag{15}$$

and it is shown that for large samples under the null, $S(m,\varepsilon)$ is asymptotically distributed as normal, i.e.

$$S(m,\varepsilon) \sim N(o,q) \tag{16}$$

where q is a complicated expression depending on m, ε, and sample size.

If a series is linear but has autocorrelation, then the test should reject the null. In practice the BDS test statistic is applied to the residuals of a fitted linear model. The model specification is constructed first and then tested to see if the fitted model gives i.i.d. residuals. BDS (1987) show that asymptotically, (16) still applies when residuals are used, so that there is no 'nuisance parameter' problem. However, it was pointed out by Brock *et tal.* (1991a) that the BDS test is not free of the nuisance parameter problem if heteroscedastic errors are involved. Since BDS is being used here as a test for stochastic or deterministic nonlinearity, it is necessary to remove linear components of the series before applying the test. To do this in practice, an AR(p) model is built for x_t, using some criteria such as AIC or BIC[1] to select the order p of the model. The test is then applied to the residuals of this linear fitting procedure.

Recall that the test is constructed using a null of i.i.d., and that rejection of the null does not imply chaos. The test may well have good power against other alternatives, such as stochastic nonlinear processes. Nevertheless, if the linear component has been properly removed, rejection of the null does correspond to presence of 'nonlinearity' in the data, however defined.

Lee, White and Granger (1990) have compared the BDS with several other tests of nonlinearity for a variety of univariate nonlinear processes. They find that it often has good power, but does less well than some of the other tests. However, the test used there had an embedding dimension of $m = 2$ and just a single epsilon value. Other simulations, such as Hsieh and LeBaron (1991) and Hsieh (1991), also show the size and power of BDS test for some nonlinear models. We study here how the BDS test is affected by other values of m and how sensitive it is to the choice of epsilon. Also, it is essential to look at the BDS test properties when epsilon values are small, where it is the only relevant range for the testing of chaos. Then it can also be applied to the statistical properties of the point estimator used by Brock and Baek (1991).

The following experiment was conducted. A series of length 200 is generated by some mechanism. As the BDS test is not affected by the norm used in calculating the correlation integral, each observation can be transformed within the range (0, 1) as above. The BDS test is then applied and the null rejected or not, and this procedure repeated 1000 times. The tables show the percentage of rejections with given significance levels for m = 2, 3, and 4, and for epsilon values $\varepsilon_j = 0.8^j$, with $j = 1, 2, 4, 6, 8$ and 10. Small j values correspond to large epsilons, and this is a range of no relevance for testing chaos.

[1] I.e., Akaike's Information Criterion and the Bayesian Information Criterion, respectively.

To check if the critical values used in the test (which are based on the asymptotic theorem) are unbiased, the experiment was first run in the case where the null hypothesis was correct. Machine-generated random numbers from a Gaussian distribution were used. These numbers were random shuffled to reduce any hidden non-randomness in the data. Both sets of data produced similar results, and just those for the shuffled data are shown. Table VI shows the size of the BDS test for various significance levels.

For columns with significance level $\alpha = 0{\cdot}05$, for example, if the asymptotic critical values were correct, the proportion of times the null hypothesis is rejected should be 5 per cent of the time. The approximate 95 per cent region is 0·037–0·063. The values are seen consistently biased towards too frequent rejection of H_0 with a sample size of 200.[2]

However, in most cases, with $j = 6$ and $j = 8$, the values are not badly biased. With the other values of j the critical values are so biased that they are unusable. This is not surprising. When low j values (i.e. larger epsilon values) are considered, most of the pairwise distances in (5) will be smaller than epsilon. Clearly, when epsilon is small (e.g. $j = 10$), few pairs are within an epsilon distance. In either case it is not easy to find the independence based on the relationship of $C_m(\varepsilon) = C_1(\varepsilon)^m$. It will be more likely that $C_m(\varepsilon)$ is close to $C_1(\varepsilon)$ instead of $C_1(\varepsilon)^m$. Hence $S(m, \varepsilon)$ should not have mean zero and the null hypothesis is easily rejected. The results are seen to vary little as m goes from 2 to 4. Although values are shown for all j with the other experiments, only for $j = 6$ and 8 are sensible interpretations and comments about power possible.

Further experiments are conducted for the testing i.i.d. of the fitted residuals. Table VII shows the size and power of the BDS test based on fitted residuals using 5 per cent significance level. Applying the BDS test to the residuals from a linear fitted model of autoregressive order 1 and 2, gives the size of the test. As shown in the upper part of Table VII, the size is similar to the random numbers case. The power of the test is examined by applying the test to: (1) a moving average model, (2) two white chaos series and (3) seven nonlinear stochastic processes. In the white chaos case, no linear regression is needed before applying the BDS test.

Table VI. Size of BDS test for shuffled pseudo-random numbers

	$\alpha = 1\%$ m			$\alpha = 2{\cdot}5\%$ m			$\alpha = 5\%$ m			$\alpha = 10\%$ m		
j	2	3	4	2	3	4	2	3	4	2	3	4
1	0·878	0·821	0·819	0·893	0·848	0·832	0·910	0·868	0·849	0·931	0·884	0·867
2	0·157	0·207	0·268	0·225	0·284	0·331	0·315	0·374	0·430	0·421	0·480	0·514
4	0·033	0·033	0·034	0·056	0·059	0·060	0·085	0·094	0·105	0·154	0·166	0·181
6	0·016	0·018	0·015	0·041	0·033	0·034	0·067	0·059	0·067	0·118	0·116	0·112
8	0·017	0·019	0·016	0·045	0·032	0·050	0·072	0·072	0·075	0·126	0·128	0·131
10	0·041	0·059	0·078	0·070	0·086	0·124	0·111	0·139	0·169	0·184	0·207	0·240

Note: Four significance levels (α), 0·01, 0·25, 0·05 and 0·10, are used for the BDS statistic, $S(m, \varepsilon_j)$, with different embedding dimension m and different epsilons, $\varepsilon_j = 0{\cdot}8^j$. The pseudo-random numbers are generated from Fortran subroutine, IMSL. Each observation in the replication is randomly chosen from an array of 100 dimension. The numbers in this array are randomly generated from pseudo-normal numbers and the position of the array being chosen is randomly decided by pseudo-uniform random numbers.

[2] Hsieh and LeBaron (1988) and Brock, Hsieh and LeBaron (1991b) find that the BDS test does not have good finite sample properties. The size of the test can be improved by increasing the sample size.

Table VII. Size and power of BDS test for residuals

	m			m			m			m		
	1	2	3	1	2	3	1	2	3	1	2	3
j	AR(1)			AR(2)			MA(2)			NLSIGN		
1	0·931	0·885	0·885	0·916	0·892	0·877	0·909	0·860	0·864	0·886	0·845	0·840
2	0·350	0·413	0·439	0·338	0·405	0·442	0·344	0·436	0·448	0·310	0·374	0·394
4	0·086	0·092	0·099	0·098	0·106	0·107	0·100	0·126	0·130	0·102	0·119	0·108
6	0·061	0·058	0·057	0·061	0·068	0·079	0·073	0·094	0·089	0·059	0·073	0·075
8	0·070	0·064	0·078	0·063	0·079	0·100	0·092	0·095	0·116	0·070	0·123	0·162
10	0·107	0·124	0·157	0·102	0·124	0·155	0·132	0·158	0·226	0·113	0·205	0·311
j	Logistic map			Tent map			Bilinear			BLMA		
1	1·000	0·985	0·887	0·776	0·872	0·847	0·969	0·907	0·911	0·975	0·931	0·934
2	0·955	0·961	0·978	0·999	0·802	0·592	0·546	0·584	0·579	0·393	0·391	0·400
4	1·000	1·000	1·000	1·000	1·000	0·985	0·878	0·920	0·917	0·675	0·725	0·717
6	1·000	1·000	1·000	1·000	1·000	1·000	0·988	0·996	0·995	0·971	0·988	0·990
8	1·000	1·000	1·000	1·000	1·000	1·000	0·987	0·997	0·996	0·992	0·996	0·996
10	1·000	1·000	1·000	1·000	1·000	1·000	0·981	0·993	0·991	0·986	0·997	0·996
j	NLMA1			NLAR			TAR			NLMA2		
1	0·969	0·925	0·902	0·942	0·910	0·906	0·896	0·853	0·861	0·894	0·824	0·832
2	0·405	0·480	0·518	0·381	0·486	0·516	0·325	0·408	0·453	0·375	0·426	0·442
4	0·080	0·120	0·150	0·082	0·100	0·103	0·187	0·185	0·182	0·371	0·464	0·455
6	0·081	0·165	0·194	0·126	0·171	0·209	0·145	0·134	0·132	0·328	0·435	0·436
8	0·075	0·170	0·234	0·242	0·372	0·489	0·195	0·168	0·168	0·285	0·402	0·414
10	0·065	0·182	0·245	0·417	0·711	0·914	0·464	0·443	0·391	0·273	0·400	0·420

Note: The residuals from first-order autoregressive regression for AR(1), NLSIGN, Bilinear, NLAR, and TAR models are derived for the BDS statistic $S(m, \varepsilon_j)$, $\varepsilon_j = 0{\cdot}8^j$. For AR(2), MA(2), BLMA, NLMA1, and NLMA2 models, the residuals are derived from the second-order autoregressive regression. In case of chaos, the BDS test is applied to the original series. The numbers show the percentage rejections in 1000 replications with 5 per cent significance level.

The test works very well for a fairly small sample size with true chaotic series, in that the null is rejected uniformly for smaller epsilon values. Using data from the logistic map which is chaos data, the BDS test rejected the null with a probability of 1·0 for all j values, $j \geqslant 4$. Similar results were found with chaotic data generated by the tent map.

The experiments for nonlinear stochastic process are divided into two groups. For the first group the BDS test has very good power. The BDS test rejects the null hypothesis of i.i.d. more than 90 per cent of replications. The results are shown in the centre part of Table VII along with the results on white chaos. The models in this group are bilinear (BL) and bilinear moving average (BLMA) models, which are

$$\text{(BL)} \qquad x_t = 0{\cdot}7 x_{t-1}\varepsilon_{t-2} + \varepsilon_t \tag{17}$$

$$\text{(BLMA)} \qquad x_t = 0{\cdot}4 x_{t-1} - 0{\cdot}3 x_{t-2} + 0{\cdot}5 x_{t-1}\varepsilon_{t-1} + 0{\cdot}8\varepsilon_{t-1} + \varepsilon_t \tag{18}$$

For the second group the BDS test has power smaller than 50 per cent. This means that the BDS test does not easily detect these types of nonlinearity. The models are the nonlinear sign model (NLSIGN), two nonlinear moving average models (NLMA1 and NLMA2), rational nonlinear autoregressive model (NLAR), and threshold autoregressive model (TAR), which have the following forms:

$$\text{(NLSIGN)} \quad x_t = \text{SIGN}(x_{t-1}) + \varepsilon_t, \text{SIGN}(x) = 1, 0, \text{ or } -1, \text{ if } x < 0, = 0, > 0 \tag{19}$$

$$\text{(NLMA1)} \quad x_t = \varepsilon_t - 0{\cdot}4\varepsilon_{t-1} + 0{\cdot}3\varepsilon_{t-2} + 0{\cdot}5\varepsilon_t\varepsilon_{t-2} \tag{20}$$

$$\text{(NLAR)} \quad x_t = \frac{0{\cdot}7 \mid x_{t-1} \mid}{2 + \mid x_{t-1} \mid} + \varepsilon_t \tag{21}$$

$$\text{(TAR)} \quad \begin{aligned} x_t &= 0{\cdot}9x_{t-1} + \varepsilon_t \text{ if } |x_{t-1}| \leqslant 1 \\ &= -0{\cdot}3x_{t-1} + \varepsilon_t \text{ if } |x_{t-1}| > 1 \end{aligned} \tag{22}$$

$$\text{(NLMA2)} \quad x_t = \varepsilon_t - 0{\cdot}3\varepsilon_{t-1} + 0{\cdot}2\varepsilon_{t-2} + 0{\cdot}4\varepsilon_{t-1}\varepsilon_{t-2} - 0{\cdot}25\varepsilon_{t-2}^2 \tag{23}$$

Table VII shows that the BDS test has the greatest power on the bilinear model. It rejects the null hypothesis of i.i.d. more than 90 per cent replications. But the BDS test has the least power on the nonlinear sign model. Actually, the residuals are seen to be i.i.d. by the BDS test. For the other four models the power for the NLMA1 model is slightly higher than nonlinear sign model, and the highest power is found for the NLMA2 model. The power of these four models is shown in the lower part of Table VII. The low power of NLMA1 and NLMA2 may be because of the heteroscedastic errors.

As noted before, the figures for $j = 1, 2$ and 10 are based on a biased significance level and so should be discounted. The power is seen to vary widely with the type of nonlinearity that is present, as is found to occur with other tests of nonlinearity. There is a general pattern of increasing power as j increases and as m increases, but this does not happen in all cases. It does seem that the choice of epsilon is critical in obtaining a satisfactory test, and that this is more important than the choice of the embedding dimension. Furthermore, the 'correct' range of epsilon for BDS test may or may not coincide with the range of small epsilon required for the definition of low-dimensional chaos. Other simulations, such as Hsieh and LeBaron (1988), Hsieh (1991), Brock *et al.* (1991a) and Brock, Hsieh, and LeBaron (1991b) also show the size and power of BDS test for some nonlinear models. Those results are rather similar with what has been found here. The BDS test may be useful in distinguishing linear stochastic process from nonlinear stochastic process. It cannot be used alone for distinguishing between deterministic chaos and stochastic process. In addition to the problem of choosing proper epsilon, rejection of the i.i.d. null hypothesis may be caused by dependence among x_t or stochastic nonlinearity in x_t.

4. CONCLUSIONS

Our specific results include the following:

1. Some deterministic systems behave like white noise (Table I).
2. The correlation exponent technique can be used to distinguish these systems (Table II) from stochastic white noise (Table III).
3. The correlation exponent does not work very well in uncovering even a low-dimensional deterministic process when stochastic noise is present (Table IV).
4. Real economic data fail to exhibit low-dimensional chaos (e.g. Table V).
5. A BDS test for stochastic white noise correctly rejects a null of white noise when the series deterministically generated, but rejected the null too often in cases where the data came from essentially stochastic sources (Table VI).

6. The BDS test has power to reject the stochastic nonlinearity. But its power varies as models differ (Table VII). BDS correctly rejected the i.i.d. null if the data came from bilinear processes, but had less power when series came from threshold autoregressive or nonlinear, moving-average processes. It had no power for the nonlinear sign model or the NLAR.
7. For empirical work, both the correlation exponent and the BDS test require a great deal of care in choosing epsilon, see Tables II–V and Table VII.

Our results are consistent with current practice in the economic literature. Any economy is in theory essentially nonlinear in nature with complex interactions among many variables of economic significance. However, at the current state of the art there is no good way to capture the richness of these models in testable form. At the level of applied works the models are linearized and the corresponding error terms modelled as residuals. The question remains: Do we live in an essentially linear economic world with unforecastable events exogenous to the model? Pragmatism dictates that we continue to develop better estimation methods for a world having both nonlinear interactions and unforecastable shocks.

Some general speculative remarks can be made about the difficulties of distinguishing between chaotic and stochastic processes. There are several tests, such as BDS, with stochastic white noise as the null. If the null is rejected with prewhitened data, then nonlinearity can be accepted. However, the theory is still lacking for making the choice between stochastic and deterministic. This lacuna follows from our observation that, so far as we are aware, there is no statistical test with deterministic chaos as its null hypothesis.

A common fallacy in many fields using time-series data is that: 'The data-generating process G has property P; if our data has property P it is because they are generated by process G.' Naturally, this is logically correct only if P characterizes G. That the data are consistent with G does not rule out other models of the universe. It is vital for researchers working with time-series to have a statistic that completely characterizes chaotic processes.

One can certainly argue that statistical tests are not the proper way to detect deterministic processes. In this view, evidence of 'strange attractors', say, is convincing enough. However, the sample sizes available from economic time-series data are not large enough to provide such evidence. New techniques that could cope with small sample sizes are needed here as well. We are led to the conclusion that probabilistic methods are for the time being the most appropriate technique for analysing economic time-series data. We suspect that this conclusion also applies to much data where chaos has been 'found' in the behavioural sciences, biology, health sciences and education.

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