A NORM ON THE FUNDAMENTAL GROUP OF NON-HAKEN 3-MANIFOLDS

Kerry N. Jones

Abstract. A canonical (presentation-independent) conjugacy-invariant norm is constructed on the fundamental group of any 3-manifold which is orientable, irreducible, has infinite fundamental group and contains no incompressible surface. More generally, this norm exists on any torsion-free group whose commutator quotient is finite. This norm is then computed explicitly in an example which shows that the induced metric on the group is not quasi-isometric to any word metric.

One of the most elusive areas of 3-dimensional topology is the study of non-Haken 3-manifolds, that is, of irreducible, closed 3-manifolds with infinite fundamental group which contain no incompressible surfaces. It seems that most unknown questions about 3-manifolds are reducible either to this area or to the Poincaré conjecture. One of the great difficulties involved in studying non-Haken 3-manifolds is that very few positive statements can be made about them – they seem, in general, to *lack* properties, rather than *possessing* them. The purpose of this paper is to describe a property which is possessed by some 3-manifolds, but not by others, but which *is* possessed by all non-Haken 3-manifolds with infinite fundamental group. It is not yet clear how to use this property to answer any of the numerous questions about these manifolds, but this paper is written as a possible start in a new direction of research.

The principal result of this paper is

Theorem 1. *Let M be an orientable, irreducible, closed 3-manifold with infinite fundamental group, containing no incompressible surfaces. Then, there exists a norm, $\|\cdot\|$, on $\pi_1(M)$ with the following properties:*

(1) *$\|\cdot\|$ is canonical (i.e., independent of any particular presentation of $\pi_1(M)$).*
(2) $\|aba^{-1}\| = \|b\|$
(3) $\|a^{-1}\| = \|a\|$
(4) *$\|\cdot\|$ gives rise to a left- and right-invariant metric on $\pi_1(M)$ given by $d(a,b) = \|ab^{-1}\|$*
(5) *the set of values taken on by $\|\cdot\|$ is discrete.*

We will prove this theorem as a consequence of

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Theorem 2. *Let G be a torsion-free group with $G/[G,G]$ finite. Then, there exists a norm on G possessing the five properties in the conclusion of Theorem 1.*

We will prove Theorem 1, assuming Theorem 2, and then prove Theorem 2.

Proof of Theorem 1. We need merely show that if M satisfies the hypotheses of Theorem 1, then $\pi_1(M)$ satisfies the hypotheses of Theorem 2. This follows by standard 3-manifold arguments, which we sketch here and refer the reader for details to [He]: any orientable, irreducible 3-manifold M with infinite fundamental group has contractible universal cover and so is a $K(\pi_1(M),1)$. But, if G contains an element of finite order, any $K(G,1)$ must have infinite dimension, thus, $\pi_1(M)$ must be torsion-free. If $H_1(M)=\pi_1(M)/[\pi_1(M),\pi_1(M)]$ is infinite, then $H^1(M)$ is nontrivial, and one can construct an incompressible surface in M, essentially by taking a minimal genus surface dual to a nontrivial cohomology class. Thus, if M is orientable and irreducible but not Haken, $\pi_1(M)$ is torsion-free and $H_1(M)$ is finite. $\square$

Proof of Theorem 2. First, let us define our norm: let

$$\mathbb{N}(a)=\{(n,m)\mid n\geq 1,\quad a^n=\prod_{i=1}^{m}[x_i,y_i],\quad x_i,y_i\in G\}$$

and let

$$\|a\|=\min_{(n,m)\in\mathbb{N}(a)}\ln(2n+m-1)$$

In order to show that this is a norm, we will need a bit more notation: For any $a\in G$, let $n(a)$ and $m(a)$ be the two integers realizing the norm. Note that our hypotheses imply that $n(a)$ exists and that $m(a)\geq 1$ unless $a=1$. We must verify that

(1) $\|a\|\geq 0$
(2) $\|a\|=0 \iff a=1$
(3) $\|ab\|\leq\|a\|+\|b\|$

1 and 2 are trivial, since, as observed above, $m(a)$ is positive unless $a=1$. To prove 3, we need the observation that $(ab)^{n(a)n(b)}$ may be written as the product of $m(a)n(b)+m(b)n(a)+n(a)n(b)-1$ commutators. We will do this as follows:

$$\begin{aligned}(ab)^{n(a)n(b)}&=[ab,a]a^2b^2(ab)^{n(a)n(b)-2}\\&=[ab,a][a^2b^2,a]a^3b^3(ab)^{n(a)n(b)-3}\\&=[ab,a][a^2b^2,a]\dots[a^{n(ab)-1}b^{n(ab)-1},a]a^{n(a)n(b)}b^{n(a)n(b)}\end{aligned}$$

where $a^{n(a)n(b)}$ may be written as the product of at most $m(a)n(b)$ commutators and $b^{n(a)n(b)}$ may be written as the product of at most $m(b)n(a)$ commutators.

Now, we are ready to verify that $\|\cdot\|$ is indeed a norm on G (assume for now that neither a nor b is 1):

$$\begin{aligned}
2n(ab) + m(ab) - 1 &\leq 3n(a)n(b) + m(a)n(b) + m(b)n(a) - 2 \\
&= (2n(a) + m(a) - 1)(2n(b) + m(b) - 1) \\
&\quad - (n(a) + m(a) - 1)(n(b) + m(b) - 1) + n(a) + n(b) - 2 \\
&\leq (2n(a) + m(a) - 1)(2n(b) + m(b) - 1) - n(a)n(b) \\
&\quad + n(a) + n(b) - 2 \\
&= (2n(a) + m(a) - 1)(2n(b) + m(b) - 1) - 1 \\
&\quad - (n(a) - 1)(n(b) - 1) \\
&< (2n(a) + m(a) - 1)(2n(b) + m(b) - 1)
\end{aligned}$$

Thus, if neither a nor b are the identity, $\|ab\| \leq \|a\| + \|b\|$ and if either one is the identity, then this inequality holds trivially, so $\|\cdot\|$ is indeed a norm on G. It should be noted, that, in fact, the "triangle inequality" holds *strictly* if neither a nor b are the identity.

The other parts of the conclusion of the theorem are clear with this definition of $\|\cdot\|$, since it is clearly conjugacy-invariant and inverse-invariant, and any conjugacy-invariant, inverse-invariant norm gives rise to a left- and right-invariant metric by the definition given in the theorem ($d(ac, bc) = \|acc^{-1}b^{-1}\| = \|ab^{-1}\| = d(a, b)$ and $d(ca, cb) = \|cab^{-1}c^{-1}\| = \|ab^{-1}\| = d(a, b)$). It is worth noting here that the "opposite" definition for d (namely, $d(x, y) = \|x^{-1}y\|$) gives the same metric, since xy^{-1} and $x^{-1}y$ are conjugate inverses ($xy^{-1} = x(y^{-1}x)x^{-1} = x(x^{-1}y)^{-1}x^{-1}$). □

We will give one general result about this norm before proceeding to an example.

Theorem 3. *Let G, H be torsion-free groups with finite abelianization and let $\phi : G \to H$ be any homomorphism. Then, denoting by $\|\cdot\|_G$ and $\|\cdot\|_H$ the norms on G and H respectively, we have for any $a \in G$ that $\|\phi(a)\|_H \leq \|a\|_G$.*

Proof. The proof is fairly clear: if $a^{n(a)}$ is the product of $m(a)$ commutators, say, $[x_i, y_i]$ then $\phi(a)^{n(a)}$ is the product of the $[\phi(x_i), \phi(y_i)]$ and so $\|\phi(a)\|_H \leq \|a\|_G$. □

In general, it seems rather difficult to compute the norm of a given element in the fundamental group of an arbitrary non-Haken 3-manifold (or 3-manifold with infinite π_1 and finite H_1). Nevertheless, it *can* be completely computed in some instances, and we give one example of this:

Let M be the 3-fold cyclic branched cover of S^3, branched over the figure-eight knot. M is a Seifert-fibered manifold with infinite fundamental group and $H_1(M) = \mathbb{Z}_4 \oplus \mathbb{Z}_4$. M is in fact Haken, containing an incompressible torus, but nevertheless Theorem 2 can be applied to $\pi_1(M)$ and the norm on $\pi_1(M)$ can be completely computed. This manifold is a Euclidean manifold, and $\pi_1(M)$ may thus be regarded as a discrete subgroup of $\mathrm{Isom}(\mathbb{E}^3)$ which may in turn be regarded as a subgroup of $\mathrm{GL}(4, \mathbb{R})$. The particular subgroup which we will use is:

$$\begin{pmatrix} (-1)^\beta & 0 & 0 & 0 \\ 0 & (-1)^\alpha & 0 & 0 \\ 0 & 0 & (-1)^{\alpha+\beta} & 0 \\ 2p+\alpha & 2q+\beta & 2r+\beta & 1 \end{pmatrix} \qquad \text{where } \alpha, \beta \in \mathbb{Z}_2, p, q, r \in \mathbb{Z}$$

The elements of this subgroup may be specified more succinctly as (α, β, p, q, r). Now, we need to do some computations to try to recognize elements of $[G, G]$. If we calculate the commutators of the form $[(\alpha_1, \beta_1, p_1, q_1, r_1), (\alpha_2, \beta_2, p_2, q_2, r_2)]$ for the 16 possible combinations of α_i, β_i we see that all commutators are of one of the forms

$$\begin{gathered}(0,0,2m,2n,0)\\(0,0,2m,0,2n)\\(0,0,0,2m,2n)\\(0,0,2m+1,2n+1,2k+1)\end{gathered}$$

and that all products of such commutators are of the form $(0, 0, m, n, k)$ where $m \equiv n \equiv k \pmod 2$. We further observe that all products of elements of this latter form are again of that form. Furthermore, all elements of this form are either commutators or products of two commutators:

$$\begin{aligned}(0,0,2m,2n,0) &= [(0,0,m,n,0),(1,1,0,0,0)]\\(0,0,0,2m,2n) &= [(0,0,0,m,n),(1,0,0,0,0)]\\(0,0,2m,0,2n) &= [(0,0,m,0,n),(0,1,0,0,0)]\\(0,0,2m+1,2n+1,2k+1) &= [(1,0,m,0,0),(0,1,0,-n-1,k)]\\(0,0,2m,2n,2k) &= [(0,0,m,n,0),(1,1,0,0,0)] \quad [(0,0,0,0,k),(1,0,0,0,0)]\end{aligned}$$

where the last form cannot be improved upon unless one of m, n, k is zero.

Thus, we see that $[G, G]$ consists of all elements of the form $(0, 0, m, n, k)$ where $m \equiv n \equiv k \pmod 2$. Now, let us consider an arbitrary element of G and compute its norm.

If $x = (\alpha, \beta, p, q, r)$ and either α or β is nonzero then $x^2 = (0, 0, m, n, k)$ where exactly one of m,n,k is nonzero and odd, thus x^4 is a commutator and $\|x\| = \ln 8$.

If $\alpha = \beta = 0$, but $x \notin [G, G]$ then $x^2 \in [G, G]$ and x^2 is a commutator if and only if at least one of p, q, r is zero. So, if p, q, r are all nonzero, then $\|x\| = \ln 5$ and if at least one is zero, $\|x\| = \ln 4$.

If $x \in [G, G]$ then $\|x\| = \ln 2$ if x is a commutator (either p, q, r are odd or at least one is zero) and otherwise $\|x\| = \ln 3$.

In particular, we see that in this case the norm is bounded and so the group has a finite diameter with respect to the induced metric. One corollary of this observation is that this metric is *not* in general quasi-isometric to any of the more customary word metrics coming from a particular presentation of the group. Of course, one may always define finite norms on any group, by altering some word metric (e.g., by taking $(1 + \|x\|)^{-1}$ where $\|x\|$ is a word metric) but these norms never satisfy Theorem 1(5) and generally do not satisfy Theorem 1(1,2,4) or Theorem 3. One may certainly pick particular presentations in which Theorem 3 would hold and one *might* be able to make parts 1,2 and 4 of Theorem 1 hold by picking a particularly judicious presentation and altering the word metric to make it conjugacy-invariant, but it seems unlikely that one could ever obtain Theorem 1(5) for any finite norm derived from a word metric on a torsion-free group.

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References

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The University of Texas, Austin, TX 78712