

THE STRUCTURE OF CLOSED NONPOSITIVELY CURVED EUCLIDEAN CONE 3-MANIFOLDS

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ABSTRACT. A structure theorem is proven for closed Euclidean 3-dimensional cone manifolds with all cone angles greater than 2π and cone locus a link (no vertices) which allows one to deduce precisely when such a manifold is homotopically atoroidal, and to construct its characteristic submanifold (torus decomposition) when it is not. A by-product of this structure theorem is the result that any Seifert-fibered submanifold of such a manifold admits a fibration with fibers parallel to the cone locus. This structure theorem is applied to several examples arising as branched covers over universal links.

Section 0 - Introduction.

Much of the recent progress in 3-manifold topology has to do with the link between topology and geometry in 3-manifolds. There has been a great deal of work in the last decade on homogeneous Riemannian metrics on 3-manifolds, spurred on by the tantalizing prospect of the Thurston Geometrization Conjecture. At the same time, there has been a renewed interest in branched covers, as a result of the notion of a *universal link*, a link in S^3 which has the property that all closed, orientable 3-manifolds are obtained as branched covers over S^3 , branched over this fixed link (see, for example, [HLM]). It had, of course, long been known that all such 3-manifolds were representable as branched covers over the 3-sphere, but in the older construction, it was a very simple kind of branched cover (namely a 3-fold cover) over a possibly very complicated link in the 3-sphere. One advantage of the newer branched cover construction is that many geometric structures on the fixed link in S^3 lift to the branched covers and thus, to all 3-manifolds. So, it seems likely that by moving the complication from the link to the branched covering map itself we may gain some real insight into the geometry of 3-manifolds.

One particular kind of geometric structure which has this lifting property is that of a *cone manifold* structure (see, for example, [A-R], [Ho] and [Jo1]). The purpose of this paper is to give a structure theorem for 3-manifolds possessing a certain type of cone manifold structure, namely, a Euclidean cone manifold structure without vertices and with cone angles greater than 2π . These are the “nonpositively curved” cone manifolds referred to in the title. It will become clear subsequently why we refer to these as nonpositively curved. This kind of cone manifold structure is possessed, for example, by all branched covers over the figure-eight knot with branching indices greater than 2 and all branched covers over

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the Borromean rings with branching indices greater than 1 (both the figure-eight knot and the Borromean rings are universal).

More specifically, we will prove

Theorem 2.1. *Let M be a closed, orientable 3-dimensional Euclidean cone manifold with no vertices and all cone angles $> 2\pi$. Then there is a canonical compact 2-complex C in M such that*

- (1) *the components of the complement of C (denoted by M_1, \dots, M_n) are each the interior of a compact Seifert-fibered manifold (possibly with boundary)*
- (2) *each M_i may be given a convex Euclidean cone metric*
- (3) *M is atoroidal if and only if each M_i is an open solid torus*

Note that here (and consistently throughout this paper) *atoroidal* means homotopically atoroidal, i.e., admitting no nonperipheral π_1 -injectively *immersed* tori.

We will also deduce some corollaries of this structure theorem, including results related to the Jaco-Shalen/Johannson torus decomposition of these manifolds, restricting the kinds of geometric structures that can be present in these manifolds. We will also be able to reproduce (only for manifolds of this type) Casson and Gabai's recent result (see [Ga]) that manifolds with π_1 -injectively immersed tori but no incompressible tori must be Seifert-fibered.

We will then apply this theorem to several illustrative examples. The manifolds to which this theorem applies are known to be irreducible and in fact to have universal cover \mathbb{R}^3 , so finding the tori in these manifolds is the key to understanding how they fit into the Thurston Geometrization Program.

Section 1 - Cone Manifolds.

We will begin by making a few brief definitions and state some preliminary results. More details may be found in [Jo2].

Definition. *A Euclidean cone manifold is a metric space obtained as the quotient space of a disjoint union of a collection of geodesic n -simplices in \mathbb{E}^n by an isometric pairing of codimension-one faces in such a combinatorial fashion that the underlying topological space is a manifold.*

Such a space possesses a flat Riemannian metric on the union of the top-dimensional cells and the codimension-1 cells. On each codimension-2 cell, the structure is completely described by an angle, which is the sum of the dihedral angles around all of the codimension-2 simplicial faces which are identified to give the cell. The *cone locus* of a cone manifold is the closure of all the codimension-2 cells for which this angle is not 2π (the Riemannian metric may be extended smoothly over all cells whose angle is 2π). For the purposes of this paper, we are interested in the 3-dimensional case in which the singular locus is a link (which must have constant cone angle on each component) and we make this blanket assumption throughout the remainder of the paper.

One particularly useful feature of the cone manifold structure is its close relationship with the notion of a branched cover. Recall that a branched covering map is a continuous map of pairs $\rho : (\hat{M}, \hat{L}) \rightarrow (M, L)$ where \hat{M}, M are n -manifolds, and \hat{L}, L are $(n-2)$ -complexes, which restricts to a covering map both on \hat{L} and on the complement of \hat{L} (we

will make the stipulation that \hat{L} be saturated with respect to ρ for technical convenience). The important result is that if M is a cone manifold with the cone locus contained in L , then \hat{M} is a cone manifold with the cone locus contained in \hat{L} . In particular, cone metrics may be lifted to true covers as well as branched covers (a covering map is clearly a branched covering map with any downstairs branch set whatever). Branched covering maps of degree d , branched over a fixed branch set L are in one-to-one correspondence with conjugacy classes of transitive representations of $\pi_1(M-L)$ into S_d (that is, representations whose image acts transitively on the set $\{0, 1, \dots, d-1\}$). We also note that the cone angles in the lifted cone manifold structure are the downstairs cone angles multiplied by the branching indices of the branched covering (we will need this in our examples).

Geodesics in a Euclidean cone manifold are of three different types: straight lines joining points on the cone locus which join in such a way as to have an angle of at least π measured in either direction, straight lines disjoint from the cone locus, and straight lines contained in the cone locus. One consequence of the nature of geodesics in Euclidean cone manifolds is that when a geodesic encounters a point of cone angle less than 2π , that geodesic may not be extended beyond that point, since no possible direction of an extension will have the required angle measure. Conversely, however, when a geodesic encounters a cone point with angle greater than 2π there are an infinite number of distinct ways to continue.

As mentioned earlier, there is a very strong analogy between cone angle and curvature, as one might expect by considering, for example, the Gauss-Bonnet theorem. More specifically, cone angles greater than 2π act like negative curvature and cone angles less than 2π act like positive curvature. To be precise, we have the following

Proposition 1.1. *Let M be a Euclidean cone 3-manifold with cone locus a link. If all the cone angles of M are less than 2π , M admits a smooth Riemannian metric of nonnegative sectional curvature. If all the cone angles of M are greater than 2π , M admits a smooth Riemannian metric of nonpositive sectional curvature.*

Proof. One constructs a metric of bounded sectional curvature which is flat outside of a tubular neighborhood of the cone locus. See [Jo1], Theorems 2.1 and 2.2. Similar techniques are used in [G-Th] with hyperbolic cone manifolds. \square

One of the most useful aspects of this smoothing technique is that it gives us immediately that the universal cover of a Euclidean cone manifold with singular locus a link and all cone angles greater than 2π is \mathbb{R}^3 (apply the Cartan-Hadamard theorem to the smooth metric). In particular, such a manifold is irreducible.

By being a bit more careful with the smoothing, we can also deduce the following theorem, which is an analogue (and consequence) of a minimal surface result in Riemannian geometry due to Schoen and Yau [S-Y].

Proposition 1.2. *Let M be a compact Euclidean cone 3-manifold with cone locus a link and all cone angles greater than 2π . Then, any π_1 -injective map of a torus into M is homotopic to a totally geodesic torus (in the cone metric) which contains some component of the cone locus.*

Proof. See [Jo2], Lemma 3.1 for the details. Essentially, one shows that one can take a sufficiently tight smoothing to which one applies the Schoen and Yau minimality result and obtains a totally geodesic torus in the smooth metric which is homotopic to a totally

geodesic torus in the cone metric. This torus can be translated in a normal direction and remains totally geodesic until it hits some component of the cone locus, which it must in fact contain. \square

This result will be the key to the proof of part (3) of Theorem 2.1.

Section 2 - Structure Theorem.

Theorem 2.1. *Let M be a closed, orientable 3-dimensional Euclidean cone manifold with no vertices and all cone angles $> 2\pi$. Then there is a canonical compact 2-complex C in M such that*

- (1) *the components of the complement of C (denoted by M_1, \dots, M_n) are each the interior of a compact Seifert-fibered manifold (possibly with boundary)*
- (2) *each M_i may be given a convex Euclidean cone metric*
- (3) *M is atoroidal if and only if each M_i is an open solid torus*

Proof. We will construct this decomposition by working in \hat{M} , the universal cover of M . We will mimic, in some sense, the usual Dirichlet domain construction of differential geometry.

Begin with disjoint metrically regular tubular neighborhoods of the cone locus in \hat{M} . Expand the radius of these tubular neighborhoods equivariantly. When two of the neighborhoods touch, continue expanding in such a way as to maintain the product structure of each neighborhood. That is, after the first point at which two of these bump into each other, each neighborhood will be a round tubular neighborhood with a flat side cut off by a plane parallel to the core geodesics of both of the intersecting neighborhoods (see Fig. 2.1). These boundary “ribbons” intersect (nontransversely) in parallelograms (generically - they *coincide* if the core geodesics of the intersecting neighborhoods are parallel) and, as the neighborhoods continue to expand, the ribbons widen until they bump into another ribbon (or possibly the round part of another neighborhood if a tangency of the round parts occurs exactly at a “corner” of the cross section). Note that at all times the cross section of each neighborhood is convex. Note also that this expansion cannot continue indefinitely (all cross sections must eventually be compact polygons) since a regular neighborhood of the cross section is imbedded under the projection to M , which has finite volume.

When the expansion of these convex product neighborhoods has been carried as far as it will go, the union of all the boundaries form an invariant (under the actions of the deck transformations on \hat{M}) 2-complex C_1 whose complement is a collection of open parallelepipeds with convex base (and a singular core geodesic) and a collection of open Euclidean solid polyhedra. We note that each of these Euclidean polyhedra (the components that do *not* contain a cone geodesic) has compact faces, since each face is the portion of a ribbon *between* two of the nontransverse intersections with other ribbons. We need to eliminate these Euclidean polyhedra. First, however, we will note the following lemma, which will be useful subsequently.

Lemma 2.2. *Let α be a cone geodesic in a Euclidean cone manifold M satisfying the hypotheses of Theorem 2.1. Let $\hat{\alpha}$ be a component of the preimage of α in \hat{M} and let Γ_α be the deck transformation on \hat{M} with minimum translation distance which leaves $\hat{\alpha}$ invariant (i.e., the deck transformation that “rolls up” $\hat{\alpha}$ into α). Then, Γ_α rotates a tubular neighborhood of $\hat{\alpha}$ by an angle rationally related to the cone angle at α*

Figure 2.1

Proof of Lemma. Since the deck transformations act by isometries and the preceding construction is geometrically canonical, any deck transformation that leaves a cone geodesic invariant must leave the component of the complement of C_1 containing that cone geodesic invariant also. In particular, the isometry must take polygonal cross sections to polygonal cross sections and so must act locally as a translation composed with a rotation rationally related to the cone angle at the center point (other symmetries of the polygon are ruled out by orientability). \square

Now, we will eliminate the Euclidean polyhedra in the complement of C_1 (at the cost of convexity of the complement) by cutting each of these Euclidean regions up by considering the shortest path from an interior point to the boundary. The set of points that admit shortest paths to two or more faces (including those whose unique shortest path is to the intersection of two faces) is an invariant 2-complex which decomposes the polyhedron into contractible bounded polyhedra. We now alter C_1 by removing the faces which are part of the boundary of one of these Euclidean polyhedra and adding in the 2-complex which subdivides each polyhedron to yield a 2-complex C_2 . The complement of C_2 consists entirely of polyhedra which retract to a cone geodesic. They are convex parallelepipeds with non-convex “warts” attached to them along the faces which were between the intersections with the other ribbons. C_2 is still invariant under the action of the deck transformations on \hat{M} and, since each component of the complement has exactly one cone geodesic in it, has the property that the components of the complement are left invariant only by a deck transformation that has an invariant cone geodesic. In particular, using Lemma 2.2, we see that the complementary regions project to open solid tori in M which may be canonically Seifert-fibered by the projections of lines parallel to the singular core geodesic (actually the Seifert-fibration is canonical only on the complement of C_1 , but it may be extended to the complement of C_2 in an obvious, but noncanonical, fashion – this will cause us no difficulties, as we will only need the fibration to be canonical near faces which are in both C_1 and C_2).

Next, we will define a new invariant 2-complex C_3 by removing all the interiors of

all the noncompact faces from C_2 . These are all infinite strips which bisect an infinite strip cobounded by two parallel cone geodesics. We note that this can be done without disturbing the Seifert-fibration on the complement, since the Seifert-fibrations on the two sides of all of the removed faces agree. If this face removal leaves any isolated geodesics in C_3 , remove them also. Note that these may be additional singular fibers for the complement of C_3 – it is no longer true that all singular fibers of the fibration are cone geodesics. Singular fibers of order 2 can also be introduced which bisect a type-II face (see definition below) if that face is glued to an image of itself under a deck transformation.

Now, let C be the projection of C_3 to M . We claim that C has the desired properties.

Let us now proceed to verify the conclusions of the Theorem: (1) is clear from the construction. (2) follows from the following construction: let $\hat{\alpha}$ be a cone geodesic in \hat{M} . Let $N(\hat{\alpha})$ be the convex parallelepiped obtained by expanding a tubular neighborhood of $\hat{\alpha}$ until it hits either another cone geodesic *or* the perpendicular bisector of the strip cobounded by $\hat{\alpha}$ and some parallel cone geodesic $\hat{\beta}$. We will refer to the former faces as “type-I faces” and the latter as “type-II faces.” $N(\hat{\alpha})$ has compact cross section since a regular neighborhood of a polygon similar to the cross section, but shrunk by a factor of two is imbedded under projection to M . Now, consider the collection of $N(\hat{\beta})$ for all $\hat{\beta}$ parallel to $\hat{\alpha}$ (here parallel means “cobounding a totally geodesic flat strip”). These may be glued along the type-II faces to give a new parallelepiped $P(\hat{\alpha})$ which is still convex since any type-I face which is adjacent to a type-II face corresponds to a cone geodesic $\hat{\gamma}_1$ which is not parallel to the core geodesic and thus causes a type-I face adjacent to the corresponding type-II face in the adjacent parallelepiped (adjacent across the type-II face) making an angle of π with the first type-I face (see Fig. 2.2) unless another geodesic $\hat{\gamma}_2$ cuts it off exactly at the vertex, causing an angle less than π .

Figure 2.2

Now, it need not be the case that $N(\hat{\alpha})$ projects to an open solid torus in M , or that $P(\hat{\alpha})$ projects to a Seifert-fibered subset of M , but it is true that $P(\hat{\alpha})$ is homeomorphic to a component $M_{\hat{\alpha}}$ of the complement of C_3 whose stabilizer Γ leaves $P(\hat{\alpha})$ invariant (it is

generated by deck transformations that either “roll up” or permute the cone geodesics of $M_{\hat{\alpha}}$ which are also the cone geodesics of $P(\hat{\alpha})$ and thus, $M_{\hat{\alpha}}$ projects to a Seifert-fibered subspace of M that is homeomorphic to $P(\hat{\alpha})/\Gamma$ which is the interior of a compact convex Euclidean cone manifold.

(3) is somewhat more difficult to verify: we will define an associated convex cone 2-manifold (similar to the technique used in [Jo2]) which has the property that M is atoroidal if and only if the 2-manifold has no closed geodesics. (3) will follow from this. First, we will define the *associated 2-orbifold* for M and subsequently define the *associated 2-manifold* for M .

For each cone geodesic α in M , take a copy of the cross section of $N(\hat{\alpha})$, then take a quotient of this cross section under the rotation guaranteed by Lemma 2.2 and denote this quotient by $O(\alpha)$. $O(\alpha)$ is a convex “cone orbifold” – an orbifold in which the cone angles at singular points are not necessarily $2\pi/n$ where n is the order of the isotropy group. Thus, in a cone orbifold, one needs to record the cone angle at a singularity separately from the order of the local isotropy group. Now, some boundary edges of the collection of cone orbifolds will correspond to type-II faces of the $N(\hat{\alpha})$ and some will correspond to type-I faces (note that the rotation of which $O(\alpha)$ is the quotient preserves face type). Take the collection of $O(\alpha)$ for all cone geodesics α in M and glue corresponding type-II faces together - this will perhaps introduce new orbifold singularities at vertices of the $O(\alpha)$ and perhaps at the midpoints of edges (these must have isotropy order 2). Note that we must orient the cone locus to fix a normal direction for the $O(\alpha)$ in order to insure that the gluing is well-defined. The components of this new cone orbifold (which we will denote by $O(M)$) are the base orbifolds for the Seifert fibrations on the various M_i .

Now, we are ready to define an *associated 2-manifold* for M , which we will denote by $\hat{O}(M)$ (note that this is slightly different from the definition in [Jo2] – the 2-manifold in [Jo2] is the union of the cross sections of the $P(\hat{\alpha})$ which is the universal cover of the 2-manifold we will define here). We use the fact that all orbifolds (with two families of exceptions) have a finite cover which is a manifold and take $\hat{O}(M)$ to be the union of the minimal-degree manifold covers for each component of $O(M)$. This is perhaps not uniquely defined, but we really only need some compact manifold cover, so our definition will be sufficient for our purposes here. We need only show that none of the components of $O(M)$ are “bad” orbifolds (in Thurston’s terminology, see [Sc],[Th]). The bad orbifolds, however, all have underlying space S^2 and a simple Gauss-Bonnet argument shows that S^2 can admit a Euclidean cone metric only when there are at least 3 cone points with cone angles less than 2π . But the only cone points on $O(M)$ that have cone angle less than 2π are points that have nontrivial isotropy groups, and thus the orbifold structure must have at least 3 singularities. But all of the bad orbifolds have fewer than 3 singularities.

To see that $\hat{O}(M)$ has the property claimed, we use Proposition 1.2 to see that any injectively immersed torus is homotopic to a totally geodesic torus containing some cone geodesic α and thus corresponds to a closed geodesic in any component of the 2-manifold which contains a cross section of $N(\hat{\alpha})$. To see this, lift the torus to a totally geodesic plane in \hat{M} which contains a geodesic $\hat{\alpha}$ and observe that this plane stays entirely in $P(\hat{\alpha})$ and thus meets any cross section of $P(\hat{\alpha})$ in a geodesic which projects to a closed geodesic in $\hat{O}(M)$. Furthermore, any closed geodesic in $\hat{O}(M)$ corresponds to a totally geodesic (and hence π_1 -injective) immersed torus in M . Thus, M is atoroidal if and only if there

are no closed geodesics in the associated 2-manifold. It should be noted that, in general, a torus corresponds to several distinct geodesics in $\hat{O}(M)$ which form an equivariant family with respect to the orbifold covering projection to $O(M)$.

It only remains to show that the associated 2-manifold of M contains no closed geodesics if and only if each component of the complement of C is a solid torus. Since each component of $\hat{O}(M)$ is a Euclidean cone manifold with all cone angles greater than 2π , there will be closed geodesics in each free homotopy class of loops in $\hat{O}(M)$. Thus, M is atoroidal if and only if each component of $\hat{O}(M)$ is simply connected. Since the 2-sphere does not admit a Euclidean cone metric with all cone angles greater than 2π , no component of $\hat{O}(M)$ can be a 2-sphere. Thus, the only obstruction to the existence of tori in M is the possibility that each component of $\hat{O}(M)$ is a disk. But, the only orbifolds that are covered by a manifold disk are disks with a single orbifold singularity and all of the Seifert-fibered spaces corresponding to these bases are solid tori (again, see [Sc]). \square

Actually, somewhat more can be said than the preceding theorem. For each one of the M_i which is not an open solid torus, we observe that we can find a collection of disjoint 2-sided embedded tori (one for each end of M_i) which are parallel to C and saturated with respect to the Seifert fibration on M_i (since each end of the interior of an orientable Seifert-fibered manifold with boundary is a product of a torus with an open interval). Each of these tori must in fact be incompressible, since this torus fibers over a boundary curve of the associated 2-manifold to M_i . This boundary curve is homotopically nontrivial and hence homotopic to a geodesic in the 2-manifold which is covered by a totally geodesic torus (hence π_1 -injective) in M .

Thus, if there is more than one M_i , the manifold must be Haken unless all M_i are solid tori, in which case the manifold is atoroidal. In particular, if M admits an injectively immersed torus, there must be some M_i that is not a solid torus, and if M admits no incompressible tori, there must be only one M_i . Thus, we recover the result (only for manifolds of this form) that a manifold that admits an injectively immersed torus but not an incompressible torus must be Seifert-fibered (see [Ga]).

Furthermore, these tori form a collection T containing the canonical collection of tori in the Jaco-Shalen/Johannson torus decomposition (see [J-S] and [Jh]). To see this, we observe that each torus in T cuts off a "collar" from its associated M_i . The components of the complement of T thus fall into one of three categories:

- (1) a manifold homeomorphic to a non-solid torus component of the complement of C
- (2) a manifold consisting of a union of solid torus components of the complement of C , together with one or more collars and components of C .
- (3) a manifold consisting of collars and components of C .

We observe that each of these components must be Seifert-fibered or atoroidal: a component in the first category is clearly Seifert-fibered. For a component, N , in the second or third category, we observe that each collar may be extended metrically (away from the component in question) until the torus boundary is totally geodesic in the cone metric. This cannot necessarily be accomplished in M , since the geodesic homotopic to the boundary curve in the associated 2-manifold need not be simple (also, the surface covering the geodesic might be a one-sided Klein bottle instead of a torus), but it can certainly be done metrically by working (for example) in the cover of M corresponding to the fundamental

group of the particular torus in question. This metric extension is homeomorphic to N . Repeat this procedure for all collars of N . We now have a Euclidean cone manifold with totally geodesic boundary (note that it may have cone locus on the boundary) which we may double to obtain a closed Euclidean cone manifold (call it N') which either has no cone locus (possible only if N was in the third category) and is hence a Euclidean manifold and thus Seifert-fibered or has nonempty cone locus and satisfies the hypotheses of Theorem 2.1. Note now that in N' , all π_1 -injective tori may be homotoped to the doubling tori and, thus, all tori are peripheral in each half (using standard free product with amalgamation results). Note that we are *not* asserting that the atoroidal pieces obtained in this way are not Seifert-fibered also – there are some spaces that are both atoroidal and Seifert-fibered (the I -bundles over the torus and Klein bottle).

Finally, we observe that there is a restriction on the kinds of geometries that the Seifert-fibered pieces can possess - the base orbifold *must* be negatively curved (since there *are* cone points on the associated 2-manifold it must have negative Euler characteristic). So, a maximal proper Seifert-fibered submanifold of a manifold of this type must have $\mathbb{H}^2 \times \mathbb{R}$ or \mathbb{E}^3 geometry (for the components that consist of collars only and have empty cone locus) and, if the whole manifold is Seifert-fibered, it must have $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{SL}_2\mathbb{R}$ geometry (again, see [Sc] for the relevant definitions – for a different proof of a slightly weaker result, see [Jo1], Chapter 5).

We collect these results in the following

Corollary 2.2. *If M is a Euclidean cone manifold satisfying the hypotheses of Theorem 2.1, then*

- (1) *if M admits a π_1 -injective torus but no incompressible torus, M must be Seifert-fibered*
- (2) *the collection of boundary-parallel tori in each non-solid torus component of M_i forms a collection of tori containing the Jaco-Shalen/ Johannson characteristic tori*
- (3) *if M is Seifert-fibered, it must have $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{SL}_2\mathbb{R}$ geometry*
- (4) *a maximal proper Seifert-fibered submanifold of M must have \mathbb{E}^3 or $\mathbb{H}^2 \times \mathbb{R}$ geometry*

Section 3 - Examples.

The easiest way to get examples of cone manifolds of this type is to consider *sufficiently branched* covers over Euclidean orbifolds, that is, branched covers over a topological space which admits a Euclidean orbifold structure in which the downstairs branching locus is equal to the singular locus of the orbifold and the branching indices over each component are greater than or equal to the order of the isotropy group of that component in the orbifold fundamental group of the base. Two particularly accessible orbifolds to use in this context are the figure-eight knot and the 6_2^2 link (see the link tables in [Ro]) since both of these have had their lattice of branched covers calculated up to degree 10 ([He], [Jo3]). These links are of interest since they are both non-torus rational links and hence universal [HLM].

First, we note that much of the actual calculation of the 2-complex C is unnecessary if all we are interested in is, say, the homeomorphism types of the various components of the complement of C . In this case, we really need only calculate the associated 2-manifolds

of M corresponding to the various parallel classes of cone geodesics and look at how the parallelepipeds over them fit together. This can be done quite conveniently in the case of sufficiently branched covers over orbifolds by simply examining the monodromy of the branched cover.

First, the figure-eight knot (a more detailed development of whose geometry may be found in [Jo2]): S^3 admits a Euclidean orbifold structure with cone angle $2\pi/3$ along the figure-eight knot. Therefore, any branched cover over S^3 , branched over the figure-eight knot with all branching indices greater than 2 admits a Euclidean cone manifold structure satisfying the hypotheses of Theorem 2.1. Let us fix some notation by letting K denote the figure-eight knot and $\varphi : \pi_1(S^3 - K) \rightarrow S_d$ be a homomorphism with transitive image in S_d (that is, whose image acts transitively on $\{0, 1, \dots, d-1\}$). Then, φ is the monodromy of a degree d cover of $S^3 - K$ and thus a degree d branched cover of S^3 , branched over K . We will use the presentation

$$\langle a, b, c, d : d^{-1}b^{-1}c, b^{-1}aba^{-1}c, a^{-1}d^{-1}c \rangle$$

for $\pi_1(S^3 - K)$ and note that the group is generated by a and c so that we need only specify φ on these generators. Then, a component of the cone locus corresponds to a cycle in $\varphi(a)$ of length 4 or greater. For each such cycle of length q , we have a parallelepiped with base a $2q$ -gon which is the universal cover of a product neighborhood of the component of the cone locus. It is possible that two or more of these cycles represent the same component of the cone locus if φ of the longitude of the knot ($ba^{-1}c^{-1}ad$) takes one cycle to another. Let us label the vertices of each polygon in the order of each cycle of $\varphi(a)$ by the labels $0, 1, \dots, d-1$ alternating with $0', 1', \dots, (d-1)'$. We may ascertain which vertices of the polygonal cross-section correspond to type-I faces and which correspond to type-II faces by the following calculation: writing permutation actions on the right, and denoting the set of fixed points of a permutation σ by $fix(\sigma)$ we define

$$F = \begin{array}{l} fix(\varphi(d^3))\varphi(a^{-1}cab^{-1}) \cap fix(\varphi(d^3))\varphi(b^{-1}) \\ \cap fix(\varphi(a^3))\varphi(b^{-1}) \cap fix(\varphi(a^3))\varphi(cab^{-1}) \end{array}$$

Then, we set

$$G = \{j \mid orbit(\langle \varphi(ba^{-1}c^{-1}ad) \rangle, j) \subset F\}$$

Then, a vertex with a label i is a type-II face if and only if $i \in G$ and it is glued to the vertex with label $(i\varphi(bd^{-1}))'$. From this information, we can compute the associated 2-manifold.

For example, if we set

$$\varphi(a) = (021)(347)(5698)$$

and

$$\varphi(c) = (052497)(68)$$

(which is branched cover number 43 in [He]) We find that there is one component of cone locus (cone angle = $8\pi/3$) whose associated 2-orbifold is a disk with 2 orbifold singularities, of orders 2 and 3. The 2-fold singularity comes from the fact that the monodromy of the longitude in this cover rotates the disk normal to the cone locus through an angle of $4\pi/3$,

yielding a quotient orbifold with 4 vertices in the boundary, each having angle $2\pi/3$. The 3-fold singularity comes from the fact that two adjacent faces of this orbifold correspond to type-II faces which are glued to each other, yielding the orbifold asserted above. Thus, the torus decomposition of this space consists of an atoroidal Euclidean piece (which is in fact a twisted I -bundle over the Klein bottle) and the Seifert-fibered space which fibers over the disk with two exceptional fibers, of orders 2 and 3 (the trefoil knot complement).

Using another of Hempel's examples (number 37), we set

$$\varphi(a) = (0\ 2\ 1)(3\ 7\ 5\ 8\ 4\ 9\ 6)$$

and

$$\varphi(c) = (0\ 3\ 5\ 6)(2\ 4\ 8\ 7)$$

and calculate that here there is also one component of cone locus (this time with cone angle $14\pi/3$) whose associated 2-manifold is a disk (there are no type-II faces) with only one cone point and thus we have an atoroidal manifold (which is in fact computed to be hyperbolic by Jeff Weeks' computer program *snapped*).

At this point, a remark is in order about how the definitions for F and G were obtained: this computation is done in detail in [Jo2] and consists of examining the flat planes extending out from the cone locus in the direction of a potentially parallel component of cone locus and checking which components of the branching locus are intersected transversely along the way – for the two components to be truly parallel (and thus separated by a type-II face) it must be the case that all components of the branching locus encountered must *not* be in the cone locus (in this case, they must be in the 3-fold branching locus). The definitions for F and G are merely codifications of these intersection conditions in terms of the monodromy of the branched cover.

The 6_2^2 link is somewhat more complicated than the figure-eight knot because it is a 2-component link. In fact, the Euclidean orbifold structure has different cone angles on the two link components even though there is an involution of S^3 that takes one component to the other. The Euclidean orbifold structure has cone angle $2\pi/3$ on one component and π on the other.

We will use the presentation

$$\langle a, b, c, d, e : ab^{-1}d^{-1}ba^{-1}b^{-1}, de^{-1}a^{-1}, ca^{-1}e^{-1}, bece^{-1} \rangle$$

for the fundamental group of the 6_2^2 link complement and note that it is generated by a and b (which are meridians of the two components) and thus we need only specify φ on these two elements. We will use the orbifold structure in which a has cone angle $2\pi/3$ and b has cone angle π .

There are two distinct types of associated 2-manifolds here, the ones corresponding to components of the cone locus that cover the a component and the b component, respectively. For the former, as before, we set

$$\begin{aligned} F' = & \text{fix}(\varphi(b^2)) && \cap && \text{fix}(\varphi(c^2)) \\ & \cap \text{fix}(\varphi(a^3))\varphi(ec^{-1}) && \cap && \text{fix}(\varphi(a^3))\varphi(b^{-1}a^{-1}ec^{-1}) \\ & \cap \text{fix}(\varphi(d^2))\varphi(a^{-1}ec^{-1}) && \cap && \text{fix}(\varphi(d^2))\varphi(ec^{-1}) \end{aligned}$$

and let

$$G' = \{j \mid \text{orbit}(\langle \varphi(ce^{-1}a^{-1}b) \rangle, j) \subset F'\}$$

and compute that the type-II faces run between the vertices labelled i (where $i \in G'$) and $(i\varphi(ce^{-1}d^{-1}))'$

For the components of the cone locus that cover the b component of the 6_2^2 link, we set

$$F'' = \begin{array}{l} \text{fix}(\varphi(a^3)) \quad \cap \quad \text{fix}(\varphi(c^2)) \\ \cap \quad \text{fix}(\varphi(b^2))\varphi(a^{-1}) \quad \cap \quad \text{fix}(\varphi(a^3))\varphi(e^{-1}) \end{array}$$

and let

$$G'' = \{j \mid \text{orbit}(\langle \varphi(ea^{-1}ba^{-1}) \rangle, j) \subset F''\}$$

and we set

$$F''' = \begin{array}{l} \text{fix}(\varphi(d^2)) \quad \cap \quad \text{fix}(\varphi(a^3))\varphi(ae^{-1}) \\ \cap \quad \text{fix}(\varphi(c^2))\varphi(ae^{-1}) \quad \cap \quad \text{fix}(\varphi(a^3))\varphi(c^{-1}ae^{-1}) \end{array}$$

and let

$$G''' = \{j \mid \text{orbit}(\langle \varphi(ea^{-1}ba^{-1}) \rangle, j) \subset F'''\}$$

For covers over the 6_2^2 link, we have the type-II faces running between vertices labelled i and $i\varphi(ea)$ where $i \in G''$ and also between i' and $(i\varphi(d))'$ where $i \in G'''$. We note also that crossing a type-II face around a cone geodesic that covers b reverses the orientation of the geodesic.

We will again apply this procedure to two examples. For the first (10.56 in [Jo3]), we set

$$\begin{aligned} \varphi(a) &= (0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9) \\ \varphi(b) &= (0\ 3)(1\ 4)(2\ 6\ 7\ 5\ 8\ 9) \end{aligned}$$

We find that there are two components of cone locus, one of which (covering b) has associated 2-orbifold a Möbius band with one order-2 singularity and the other of which (covering a) has associated 2-manifold a disk with one cone point. Thus, we have a manifold whose torus decomposition consists of a Seifert-fibered space over the Möbius band with one singular fiber of order 2 and an atoroidal manifold with one cusp.

For our second example, we set

$$\begin{aligned} \varphi(a) &= (0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9) \\ \varphi(b) &= (0\ 1\ 3\ 4)(2\ 6\ 5\ 8)(7\ 9) \end{aligned}$$

This is example 10.49 in [Jo3].

Here, we again have two components of cone locus (both of the 4-cycles in $\varphi(b)$ are on the same component of cone locus) and we find that there are no type-II faces, so that we have an atoroidal manifold (which is in fact hyperbolic – again courtesy of *snapped*).

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