GEODESIC INTERSECTIONS IN ARITHMETIC HYPERBOLIC 3-MANIFOLDS

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Abstract. It was shown by Chinburg and Reid that there exist closed hyperbolic 3-manifolds in which all closed geodesics are simple. Subsequently, Basmajian and Wolpert showed that almost all quasi-Fuchsian 3-manifolds have all closed geodesics simple and disjoint. The natural conjecture arose that the Chinburg-Reid examples also had disjoint geodesics. Here we show that this conjecture is both almost true (they have no geodesics that intersect except at right angles) and spectacularly false (any pair of closed geodesics admits infinitely many closed geodesics which intersects both geodesics of the pair perpendicularly). The latter statement is shown to be true for all closed arithmetic hyperbolic 3-manifolds.

Section 0 - Introduction

By a *hyperbolic n-manifold* we shall mean a complete orientable n-dimensional Riemannian manifold all of whose sectional curvatures are -1. If M is a hyperbolic 3-manifold, the universal cover of M can be identified with $\mathbb{H}^3$, the upper half-space model of hyperbolic 3-space, and M is realized as $\mathbb{H}^3/\Gamma$ for some Γ a discrete torsion-free subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$. Now $\mathrm{Isom}^+(\mathbb{H}^3)$ can be identified with $\mathrm{PSL}(2,\mathbb{C})$ (which in turn is isomorphic to $\mathrm{PGL}(2,\mathbb{C})$), and Γ is called a Kleinian group. In the sequel we will only be interested in the case when M is closed, in which case Γ is referred to as cocompact.

Of interest to us is the structure of the set of closed geodesics in closed hyperbolic 3-manifolds. The motivation comes from the following. A closed geodesic in a (closed) hyperbolic n-manifold is simple if it has no self-intersections, and non-simple otherwise. In dimension 2 every closed hyperbolic manifold has a non-simple closed geodesic. However in dimension 3 the situation is much more complex. Many closed hyperbolic 3-manifolds contain immersions of totally geodesic surfaces and so there are non-simple closed geodesics. In [JR] examples were given of closed hyperbolic 3-manifolds containing a non-simple closed geodesic but having no immersed totally geodesic surface. However it was shown in [CR] that there exist closed hyperbolic 3-manifolds in which all closed geodesics are simple. Subsequently, Basmajian and Wolpert [BW] showed that almost all 3-manifolds arising as the quotient of $\mathbb{H}^3$

1991 *Mathematics Subject Classification.* Primary: 57M50, Secondary: 30F40.

Key words and phrases. hyperbolic 3-manifold, geodesic, intersection, arithmetic manifold.

First author partially supported by Ball State University Office of Academic Research. Second author is supported by The Royal Society. This author also wishes to thank The University of Aberdeen for its hospitality whilst working on this paper. Both authors wish to thank the referee for helpful suggestions which led to Theorems 4.2 and 4.3

by a quasi-Fuchsian subgroup of $\mathrm{PSL}(2,\mathbb{C})$ have all closed geodesics simple and disjoint. It was shown in [CR] that closed geodesics of the same complex length (see Lemma 2.2 for the definition) were disjoint. The natural conjecture motivated by this was that the examples constructed in [CR] had all closed geodesics disjoint.

The main results here show that this conjecture is both almost true (they have no geodesics that intersect except at right angles) and spectacularly false (any pair of closed geodesics admits infinitely many closed geodesics which intersects both geodesics of the pair perpendicularly). The latter statement is shown to be true for all closed arithmetic hyperbolic 3-manifolds. Like the methods of [CR] and [JR] the methods here rely heavily on the arithmetic techniques.

In the final section of the paper we apply this same technology to arrive at a partial answer to a question posed by Weeks: does every finite-volume hyperbolic manifold have a complex length ℓ such that the collection of all closed geodesics of that length form a link.

Section 1 - Arithmetic Preliminaries

Here we recall some salient points on arithmetic aspects of Kleinian groups.

1.1.

Let k be a field of characteristic different from 2. The standard notation for a quaternion algebra over k is the following. Let a and b be non-zero elements of k. Then $\left(\frac{a,b}{k}\right)$ denotes the quaternion algebra over k with basis $\{1, i, j, ij\}$ subject to $i^2 = a$, $j^2 = b$ and $ij = -ji$. $\left(\frac{a,b}{k}\right)$ is called a Hilbert Symbol for the quaternion algebra.

If now k is a number field, and ν is a valuation of k associated to a real embedding of k, we say a quaternion algebra A over k is *ramified* at ν if $A \otimes_k k_\nu \cong \mathcal{H}$ where $\mathcal{H}$ is the Hamiltonian quaternions over $\mathbb{R}$.

1.2.

Let Γ be a Kleinian group and let $\mathbb{Q}(\mathrm{tr}\,\Gamma)$ denote the trace-field of Γ. When Γ has finite co-volume $\mathbb{Q}(\mathrm{tr}\,\Gamma)$ is a finite extension of $\mathbb{Q}$. Following [R1] and [NR] we define the *invariant trace-field* $k\Gamma$ and *invariant quaternion algebra* $A\Gamma$ of Γ as follows. Let $\Gamma^{(2)} = \mathrm{gp}\{\gamma^2 : \gamma \in \Gamma\}$. Then $k\Gamma = \mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)})$ and $A\Gamma$ is the quaternion algebra over $k\Gamma$ defined by (see [B]):

$$\mathrm{A}\,\Gamma = \{\Sigma a_i\gamma_i \;:\; a_i \in k\Gamma, \gamma_i \in \Gamma^{(2)}\},$$

where all sums are finite, is a quaternion algebra over $k\Gamma$. $k\Gamma$ and $\mathrm{A}\,\Gamma$ are invariants of the commensurability class of Γ. Now $\mathrm{A}\,\Gamma$ can be explicitly determined from Γ, see [HLM]:

Lemma 1.1. *Let Γ be a cocompact Kleinian group for which $k\Gamma = \mathbb{Q}(\mathrm{tr}\,\Gamma)$, and let γ and δ be a pair of non-commuting elements of Γ. Then,*

$$\mathrm{A}\,\Gamma \cong \left(\frac{(\mathrm{tr}^2(\gamma) - 4), (\mathrm{tr}([\gamma, \delta]) - 2)}{k\Gamma}\right).$$

□

1.3.

Recall the definition of arithmetic Kleinian groups, see [Bo], or [V] for details.

Let k be a number field having exactly one complex place. Let B be a quaternion algebra over k which ramifies at all real places of k. Let $\mathcal{O}$ be an order of B and let $\mathcal{O}^1$ be the group of elements of reduced norm 1 in $\mathcal{O}$. Over an embedding $k \hookrightarrow \mathbb{C}$ inducing the complex place of k one may choose an algebra embedding $\rho : B \hookrightarrow M(2, \mathbb{C})$ which restricts to an injection $\rho : \mathcal{O}^1 \hookrightarrow \mathrm{SL}(2, \mathbb{C})$. Let $\mathrm{P} : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{PSL}(2, \mathbb{C})$ be the natural projection. Then $\mathrm{P}\,\rho(\mathcal{O}^1)$ is a Kleinian group of finite covolume. An *arithmetic Kleinian group* Γ is a subgroup of $\mathrm{PSL}(2, \mathbb{C})$ commensurable with a group of the type $\mathrm{P}\,\rho(\mathcal{O}^1)$. We say Γ is *derived from a quaternion algebra* if Γ is actually a subgroup of some $\mathrm{P}\,\rho(\mathcal{O}^1)$. We call $Q = \mathbb{H}^3/\Gamma$ arithmetic or derived from a quaternion algebra if Γ is arithmetic or derived from a quaternion algebra.

It is shown in [MR] that a Kleinian group of finite co-volume is arithmetic if and only if the group $\Gamma^{(2)}$ is derived from a quaternion algebra.

Section 2 - Nonperpendicular Geodesic Intersections

We begin by recalling some relevant facts about traces in $\mathrm{SL}(2, \mathbb{C})$ and their relationship to hyperbolic geometry. Note that we will be working in $\mathrm{SL}(2, \mathbb{C})$ rather than in $\mathrm{PSL}(2, \mathbb{C})$. When we say that an isometry of $\mathbb{H}^3$ is *represented* by a matrix in $\mathrm{SL}(2, \mathbb{C})$, we assume a fixed representation from $\mathrm{Isom}^+(\mathbb{H}^3)$ to $\mathrm{PSL}(2, \mathbb{C})$ and a fixed, consistent lifting of the image group to $\mathrm{SL}(2, \mathbb{C})$.

The first three lemmas allow us to detect intersection of axes of two isometries from their traces and the trace of their commutator.

Lemma 2.1. *Let $a, b \in \mathrm{SL}(2, \mathbb{C})$ have trace different from ± 2 and axes x and y, respectively, in $\mathbb{H}^3$. Let $\varphi \in \mathrm{SL}(2, \mathbb{C})$ represent the unique isometry which takes x to y whose axis intersects both x and y perpendicularly.*

Then,

$$\operatorname{tr} \varphi = \pm \sqrt{2 \pm 2 \sqrt{1 + 4 \frac{\operatorname{tr}[a, b] - 2}{(\operatorname{tr}^2 a - 4)(\operatorname{tr}^2 b - 4)}}}$$

Proof. First, conjugate a and b so that the fixed points of a are ± 1 and the fixed points of b are $\pm \omega$ for some $\omega \in \mathbb{C}$. After this conjugation, φ has the z-axis as its axis and has trace

$$\operatorname{tr} \varphi = \sqrt{\omega} + \frac{1}{\sqrt{\omega}}$$

More explicitly,

$$a = \begin{pmatrix} \frac{\operatorname{tr} a}{2} & \frac{\sqrt{\operatorname{tr}^2 a - 4}}{2} \\ \frac{\sqrt{\operatorname{tr}^2 a - 4}}{2} & \frac{\operatorname{tr} a}{2} \end{pmatrix}$$

$$b = \begin{pmatrix} \frac{\operatorname{tr} b}{2} & \frac{\omega \sqrt{\operatorname{tr}^2 b - 4}}{2} \\ \frac{\sqrt{\operatorname{tr}^2 b - 4}}{2\omega} & \frac{\operatorname{tr} b}{2} \end{pmatrix}$$

$$\varphi = \begin{pmatrix} \sqrt{\omega} & 0 \\ 0 & \frac{1}{\sqrt{\omega}} \end{pmatrix}$$

A direct calculation then shows that

$$\mathrm{tr}[a,b]-2=\frac{(\mathrm{tr}^2 a-4)(\mathrm{tr}^2 b-4)(\omega^2-1)^2}{16\omega^2}$$

and thus,

$$(\omega-\frac{1}{\omega})^2=16\frac{\mathrm{tr}[a,b]-2}{(\mathrm{tr}^2 a-4)(\mathrm{tr}^2 b-4)}$$

so that

$$\begin{aligned}(\mathrm{tr}^2\varphi-2)^2&=(\omega+\frac{1}{\omega})^2=4+(\omega-\frac{1}{\omega})^2\\&=4+16\frac{\mathrm{tr}[a,b]-2}{(\mathrm{tr}^2 a-4)(\mathrm{tr}^2 b-4)}\end{aligned}$$

The conclusion follows upon solving for $\mathrm{tr}\,\varphi$. □

Lemma 2.2. *Let $a\in\mathrm{SL}(2,\mathbb{C})$ represent a loxodromic or elliptic isometry $\alpha\in\mathrm{Isom}^+(\mathbb{H}^3)$. Denote the translation distance of α by ρ and the rotation angle (torsion) by θ. Then, the complex length $\rho+i\theta$ of α, is given by:*

$$\rho+i\theta=2\cosh^{-1}(\frac{\mathrm{tr}\,a}{2})$$

Proof. Conjugate so that a is diagonal, and use the fact that $\cosh^{-1}(z)=\ln(z+\sqrt{z^2-1})$. □

Lemma 2.3. *Let $a,b\in\mathrm{SL}(2,\mathbb{C})$ represent loxodromic or elliptic isometries of $\mathbb{H}^3$ whose axes intersect at an angle θ. Then,*

$$\sin^2\theta=-4\frac{\mathrm{tr}[a,b]-2}{(\mathrm{tr}^2 a-4)(\mathrm{tr}^2 b-4)}$$

In particular, the right-hand side of the above equation is real and positive.

Proof. Combining Lemmas 2.1 and 2.2 (note that since the axes intersect, the isometry φ in Lemma 2.1 is elliptic with axis equal to the common perpendicular to the axes of a and b at their point of intersection), we see that

$$\cos(\frac{\theta}{2})=\frac{\pm\sqrt{2\pm2\sqrt{1+4\frac{\mathrm{tr}[a,b]-2}{(\mathrm{tr}^2 a-4)(\mathrm{tr}^2 b-4)}}}}{2}.$$

Using the standard double-angle formula for cosine yields

$$\cos\theta=\pm\sqrt{1+4\frac{\mathrm{tr}[a,b]-2}{(\mathrm{tr}^2 a-4)(\mathrm{tr}^2 b-4)}}$$

or, on converting from cosine to sine,

$$\sin^2\theta=-4\frac{\mathrm{tr}[a,b]-2}{(\mathrm{tr}^2 a-4)(\mathrm{tr}^2 b-4)}.$$

□

The next two lemmas, combined with the previous lemma, allow us to deduce number-theoretic conditions on the traces of two isometries which are necessary for them to have intersecting axes.

Lemma 2.4. *Let $a, b \in \mathrm{SL}(2, \mathbb{C})$. Then,*

$$\operatorname{tr}[a, b] - 2 = \operatorname{tr}^2 a + \operatorname{tr}^2 b + \operatorname{tr}^2 ab - \operatorname{tr} a \operatorname{tr} b \operatorname{tr} ab - 4$$

Proof. This is a standard trace identity, valid in $SL(2, R)$ for any commutative ring R. $\square$

Lemma 2.5. *Let $a, b \in \mathrm{SL}(2, \mathbb{C})$ represent loxodromic or elliptic isometries of $\mathbb{H}^3$ whose axes intersect at an angle θ. Then,*

$$\operatorname{tr} ab = \frac{\operatorname{tr} a \operatorname{tr} b \pm \cos\theta \sqrt{(\operatorname{tr}^2 a - 4)(\operatorname{tr}^2 b - 4)}}{2}$$

In particular, $(\operatorname{tr}^2 a - 4)(\operatorname{tr}^2 b - 4)\cos^2\theta$ is a perfect square in the trace field of $\langle a, b\rangle$.

Proof. Use Lemma 2.3 to write $\operatorname{tr}[a, b] - 2$ in terms of $\operatorname{tr} a, \operatorname{tr} b$, and θ. Use Lemma 2.4 to write $\operatorname{tr}[a, b] - 2$ in terms of $\operatorname{tr} a, \operatorname{tr} b$, and $\operatorname{tr} ab$. Set these two equal to obtain an equation involving $\operatorname{tr} a, \operatorname{tr} b, \operatorname{tr} ab$, and θ which is quadratic in $\operatorname{tr} ab$. Solve this equation for $\operatorname{tr} ab$ to obtain the desired result. $\square$

Using Lemmas 2.5 and 2.3, together with the explicit determination of a Hilbert symbol given in Lemma 1.1 allows us to derive a condition on the invariant quaternion algebra (in Lemma 2.7 below) which is necessary for the existence of non-perpendicular intersecting geodesics in any of the corresponding Kleinian groups (arithmetic or not).

We will require the following elementary geometric lemma.

Lemma 2.6. *Let G be a Kleinian group of finite co-volume, $g \in G$ be an elliptic element and ℓ the axis of g in $\mathbb{H}^3$. Then the stabilizer of ℓ in G, $\{h \in G \mid h\ell = \ell\}$ is infinite.*

Proof. Let $G_\ell = \{h \in G \mid h\ell = \ell\}$. The axis ℓ admits a collar neighbourhood $C(\ell)$ which is precisely invariant under G_ℓ and with $C(\ell)$ having infinite volume. $C(\ell)$ projects into $\mathbb{H}^3/G$, and since G has finite co-volume, we deduce that G_ℓ must be infinite. $\square$

Lemma 2.7. *Let Γ be a cocompact Kleinian group containing two elements a and b which represent loxodromic or elliptic isometries having axes which intersect at an angle θ where $0 < \theta < \pi/2$. Then, $\mathrm{A}\,\Gamma$ admits a Hilbert symbol having one entry equal to $-\tan^2\theta$. In particular, $\mathrm{A}\,\Gamma$ admits a Hilbert symbol having a real entry.*

Proof.

By Lemma 2.6, and cocompactness we can assume that a and b are loxodromic (an elliptic element in Γ will necessarily have a loxodromic element with the same axis). We will pass to $\Gamma^{(2)}$ for this will not change the angle of intersection of the geodesics. Let $x = a^2$ and $y = b^2$.

Lemma 2.5 implies that

$$\frac{\cos^2\theta(\operatorname{tr}^2 x - 4)(\operatorname{tr}^2 y - 4)}{4} = (\operatorname{tr} xy - \frac{\operatorname{tr} x \operatorname{tr} y}{2})^2.$$

Specifically, we note that the left-hand side of this equation is therefore a square in $k\Gamma$. Lemma 2.3 and Lemma 1.1 imply $\mathrm{A}\,\Gamma$ admits a Hilbert symbol with one entry being $\dfrac{-\sin^2\theta(\operatorname{tr}^2 x-4)(\operatorname{tr}^2 y-4)}{4}$. Since entries of Hilbert symbols may be freely divided by non-zero squares in $k\Gamma$, we may divide by $\dfrac{\cos^2\theta(\operatorname{tr}^2 x-4)(\operatorname{tr}^2 y-4)}{4}$ to obtain a Hilbert symbol entry equal to $-\tan^2\theta$. $\square$

We remark here that the restriction to cocompact does not lose any real information, since the invariant quaternion algebra of a non-cocompact Kleinian group of finite covolume is always the matrix algebra over the invariant trace-field. This always has a Hilbert symbol with a real entry.

In [CR] it was shown that there exist infinitely many commensurability classes of co-compact Kleinian groups all of whose closed geodesics were simple. The key point in [CR] was to exhibit Kleinian groups whose invariant quaternion algebras could not have a Hilbert symbol with one of the entries being real. This was used there to deduce the simplicity of all closed geodesics. All the manifolds constructed were arithmetic. With this we note,

Theorem 2.8. *There are infinitely many commensurability classes of closed hyperbolic 3-manifolds each of which has all closed geodesics simple and no two closed geodesics intersect, except possibly perpendicularly.*

Remark. Note that we shall find subsequently that there are *many* perpendicular intersections in these manifolds.

Proof. As noted above there are infinitely many commensurability classes of arithmetic hyperbolic 3-manifolds whose invariant quaternion algebra does not admit a Hilbert symbol with a real entry, and so all closed geodesics are simple. However, Lemma 2.7 shows that such manifolds can only have perpendicular intersections between closed geodesics. $\square$

Section 3 - Perpendicular Geodesic Intersections

When we consider the problem of perpendicular intersections of closed geodesics in arithmetic manifolds, a very different picture emerges. Here, instead of being able to construct special manifolds which have no such intersecting geodesics, we find that any pair of closed geodesics admits an infinite collection of distinct closed geodesics, each of which intersects both members of the pair perpendicularly.

More precisely, we have the following theorem:

Theorem 3.1. *Let M be a closed arithmetic hyperbolic 3-manifold and let x and y be two closed geodesics in M. Let z be a geodesic arc between x and y which is perpendicular to both (there is one such in each free homotopy class of arcs between x and y). Then, there exists a closed geodesic z' which contains z.*

The proof will require the following lemma. For a proof of the lemma, see for example [GMMR] Lemma 7.1.

Lemma 3.2. *Let Γ be a Kleinian group with trace-field k, all of whose traces lie in R_k the ring of algebraic integers of k. Let a and b be a pair of non-commuting elements of Γ, and let $\mathcal{O} = R_k[1, a, b, a.b]$. Then $\mathcal{O}$ is an order of $\{\Sigma a_i\gamma_i \ : \ a_i \in k, \gamma_i \in \Gamma\}$.* $\square$

Proof of Theorem 3.1. Since the angle of intersection is unchanged on passing to the finite cover of M determined by $\pi_1(M)^{(2)}$ we will assume for convenience that M is derived from a quaternion algebra. Let Γ denote the image of $\pi_1(M)$ in $\mathrm{PSL}(2,\mathbb{C})$ under the faithful discrete representation.

Passing to $\mathbb{H}^3$ the universal cover of M, we take $\tilde{x}$ to be any geodesic which covers x, then let $\tilde{z}$ to be any lift of z which starts at a point of $\tilde{x}$, then take $\tilde{y}$ to be the geodesic covering y which intersects the terminal point of $\tilde{z}$.

Now, let $A, B \in \Gamma$ with axes $\tilde{x}$ and $\tilde{y}$, respectively and let a and b be matrix representatives for A and B in $\mathrm{SL}(2,\mathbb{C})$. Note that $ab-ba$ is an element of $\mathrm{GL}(2,\mathbb{C})$ which has trace 0 and whose image in $\mathrm{PGL}(2,\mathbb{C})$ has the geodesic containing $\tilde{z}$ as its axis (it is elliptic of order two and conjugates a to a^{-1} and b to b^{-1}). We note here Jørgensen's observation ([T] Corollary 5.4.2) that $ab-ba$ defines an involution on $\mathbb{H}^3/\langle a, b\rangle$.

Let $\mathcal{O} = R_{k\Gamma}[1, a, b, a.b]$ be as in Lemma 3.2, and $\mathrm{Norm}(\mathcal{O}) = \{x \in \mathrm{A}\,\Gamma^* \mid x\mathcal{O}x^{-1} = \mathcal{O}\}$. The image $\Gamma(\mathcal{O})$ of $\mathrm{Norm}(\mathcal{O})$ in $\mathrm{PGL}(2,\mathbb{C})$ is an arithmetic Kleinian group ([Bo]). To see this note that $\mathrm{Norm}(\mathcal{O})$ contains $\mathcal{O}^1$, and any element of $\mathrm{Norm}(\mathcal{O})$ normalizes $\mathcal{O}^1$. Therefore $\Gamma(\mathcal{O})$ is a subgroup of the normalizer of $\mathrm{P}\,\mathcal{O}^1$ in $\mathrm{PGL}(2,\mathbb{C})$, which is an arithmetic Kleinian group. Thus $\mathbb{H}^3/\Gamma(\mathcal{O})$ is a closed orbifold commensurable with M. In fact $\Gamma(\mathcal{O})$ coincides with the normalizer of $\mathrm{P}\,\mathcal{O}^1$ in $\mathrm{PGL}(2,\mathbb{C})$. Note that our discussion above shows that $\Gamma(\mathcal{O})$ contains the images of a, b and $ab-ba$.

By Lemma 2.6, there is some loxodromic element $c \in \Gamma(\mathcal{O})$ whose axis is equal to the axis of $ab-ba$. Using the commensurability of $\Gamma(\mathcal{O})$ and Γ, we get a loxodromic element $c' \in \Gamma$ whose axis is equal to the axis of $ab-ba$. Thus, the image of the axis of $ab-ba$ in M is the desired closed geodesic z'. $\square$

A similar idea proves the following.

Theorem 3.3. *Let M be a closed hyperbolic 3-manifold such that $\pi_1(M)$ is 2-generator. Then M contains perpendicularly intersecting geodesics.*

Proof. Let a and b be generators for $\pi_1(M)$. Using the observation of Jørgensen mentioned in the proof of Theorem 3.1, we see that a, b and $ab-ba$ generate the fundamental group of a closed hyperbolic orbifold O, commensurable with M. The argument is completed as in the proof of Theorem 3.1. $\square$

In view of the results of Basmajian and Wolpert mentioned in the Introduction the Theorems of §2 and 3 yield:

Corollary 3.4. *If $M = \mathbb{H}^3/\Gamma$ is a closed hyperbolic 3-manifold all of whose closed geodesics are simple and disjoint, then M is non-arithmetic and if N is commensurable with M then $\pi_1(N)$ is not 2-generator.* $\square$

We now briefly indicate a construction of manifolds which satisfy the conclusion of Corollary 3.4. First recall a version of Margulis' characterization of arithmeticity ([Mar]); a Kleinian group Γ is non-arithmetic if and only if the group

$$\mathrm{Comm}(\Gamma) = \{g \in \mathrm{PSL}(2,\mathbb{C}) : g\Gamma g^{-1} \text{ is commensurable with } \Gamma\}$$

is discrete. Furthermore $\mathrm{Comm}(\Gamma)$ is the unique maximal element in the commensurability class of Γ.

With this and the previous observation of Jørgensen about 2-generator groups we deduce.

Lemma 3.5. *Let $M = \mathbb{H}^3/\Gamma$ be a closed non-arithmetic hyperbolic 3-manifold. If* $\mathrm{Comm}(\Gamma)$ *is torsion-free then* Γ *is not commensurable with a* 2*-generator group.* $\square$

Examples.

Some of the discussion here is taken from [R2] §3. Let K be the knot 9_{32} of [Ro]. $S^3 \setminus K$ admits a complete hyperbolic structure of finite volume with trivial symmetry group ([Ri]). Let $S^3 \setminus K = \mathbb{H}^3/\Gamma_K$. It is also shown in [Ri] that the invariant trace-field of Γ_K has degree 29 and so cannot contain $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$. Then [NR] Proposition 9.1 implies $\mathrm{Comm}(\Gamma_K) = \Gamma_K$. Now Thurston's theory of hyperbolic Dehn surgery [T], and Borel's result ([Bo]) that the set of volumes of arithmetic manifolds is discrete in $\mathbb{R}$ taken together with Lemma 3.5 and the above discussion on K implies that large Dehn surgeries on K satisfy the conclusion of Corollary 3.4.

Section 4 - Application

Jeff Weeks has asked if every closed hyperbolic manifold has a complex length ℓ such that the collection of closed geodesics of length ℓ forms a link. Let us say that such a complex length (or equivalently the corresponding trace, $a = 2\cosh(\ell/2)$) has the *disjoint axis property.* Complex lengths with the disjoint axis property would be useful in calculating the symmetry group of closed hyperbolic manifolds since they provide a "geometrically canonical link" which could be removed, yielding a cusped manifold. The symmetry group of this cusped manifold could then be calculated using existing methods and one could then pass to the subgroup which preserves the meridional system which surgers the cusped manifold back to the original closed manifold.

Implicit in [CR] (and explicit above) is the fact that for the commensurability classes constructed in [CR], *every* complex length which is represented in the manifold has the disjoint axis property. To show that a given trace a has the disjoint axis property, it suffices to show that the algebra does not admit a Hilbert symbol (a^2-4, r) where r is a real element of $k\Gamma$. The algebras constructed in [CR] do not admit any real Hilbert symbol entries at all, and hence every trace has the disjoint axis property. However, here we show that under certain other conditions one can find a trace a with the disjoint axis property even in cases where *every* trace does not.

To state the theorem we make a definition. Let Γ be a Kleinian group. For any a in $\mathbb{Q}(\mathrm{tr}\,\Gamma)$, let $\Gamma(a)$ denote the union (in $\mathbb{H}^3/\Gamma$) of the axes of all of the elements of Γ that have trace a.

Theorem 4.1. *Let Γ be an arithmetic Kleinian group. Let ν and ν' be a pair of distinct complex conjugate finite places of $k\Gamma$. Then, if $\mathrm{A}\,\Gamma$ is ramified at ν and unramified at ν', there exists $a \in k\Gamma$ such that $\Gamma(a)$ is a link.*

Proof. Given Γ, we will actually find $a \in k\Gamma$ such that any group Γ' commensurable with Γ will have $\Gamma'(a')$ a (possibly empty) link for any a' which is the trace of a rational power of an element of trace a. This, together with the construction of a Γ' for which $\Gamma'(a)$ is nonempty, clearly suffices to prove the theorem. Let $k = k\Gamma$ denote the invariant trace-field of Γ and let $\mathcal{P} = \{\nu, \nu_1, \nu_2, \ldots, \nu_n\}$ be the set of finite places at which $\mathrm{A}\,\Gamma$ is ramified. Let $\mathcal{P}' = \mathcal{P} \cup \{\nu'\}$ and let $\mathcal{P}''$ denote the union of $\mathcal{P}'$ and the set of real places of k. For each $\mu \in \mathcal{P}''$ denote by k_μ the completion of k at μ and by $|\cdot|_\mu$ the canonical absolute value at μ. For $\mu \in \mathcal{P}'$, let

π_μ denote a (fixed) uniformizer for k_μ and assume without loss of generality that $\pi_{\nu'}$ is the complex conjugate of π_ν.

Next, for each $\mu \in \mathcal{P}'$ define a_μ to be an element of k_μ such that $a_\mu^2 = 4 + 16\pi_\mu$. Note that, by the Local Square Theorem ([O, p. 159]), $4 + 16\pi_\mu$ is always a square in k_μ, whether k_μ is dyadic or not. Note that $a_\mu^2 - 4$ is not a square in k_μ for any $\mu \in \mathcal{P}'$. Define a_μ to be equal to 1 for all real places μ. Let $\epsilon > 0$ be the smallest of the numbers $|a_\mu^2 - 4\|_\mu/2$ for all $\mu \in \mathcal{P}''$. Now, using continuity of multiplication, let $\epsilon' > 0$ be such that $|x - a_\mu|_\mu < \epsilon'$ implies that $|x^2 - a_\mu^2|_\mu < \epsilon$ for all $\mu \in \mathcal{P}''$.

Then the Very Strong Approximation Theorem ([O], p. 77) implies that we can find an algebraic integer $a \in k$ such that $|a - a_\mu|_\mu < \epsilon'$ for all $\mu \in \mathcal{P}''$. Denote $a^2 - 4$ by β. Then, β is a nonsquare at all places of $\mathcal{P}''$. In particular, it is a nonsquare at all places which ramify $\mathrm{A}\,\Gamma$, so there is a Hilbert symbol (α, β) for $\mathrm{A}\,\Gamma$, see for example [O], p 203. We claim that α cannot be real (under the complex place of k).

Thus suppose that α is real. Then, since $\mathrm{A}\,\Gamma$ is unramified at ν' and ramified at ν, there exist $x, y \in k_{\nu'}$ such that $\alpha x^2 + \beta y^2 = 1$ but there exist no such solutions in k_ν. Now the local square theorem implies that this quadratic equation has solutions in a local field if and only if it has solutions modulo 4π (after first dividing α and β by even powers of π to insure that the equation remains quadratic modulo 4π). Recall also that β is congruent to zero modulo $4\pi_\nu$ and $4\pi_{\nu'}$. Thus, α (again after dividing out any even powers of π_ν or $\pi_{\nu'}$) must be a square modulo $4\pi_{\nu'}$ but a nonsquare modulo $4\pi_\nu$. But this is clearly impossible, since α is invariant under complex conjugation which takes π_ν to $\pi_{\nu'}$.

Our earlier work now shows that no two axes of elements of trace a can intersect.

We note here that if $a = \operatorname{tr} x$ and $c = \operatorname{tr} x^n$ then $(c^2 - 4)/(a^2 - 4)$ is a square in k. Thus, $c^2 - 4$ is in the same square coset of k as β. To see this, conjugate x so that it is diagonal, with λ and λ^{-1} on the diagonal. Then, $a^2 - 4 = (\lambda - \lambda^{-1})^2$ and $c^2 - 4 = (\lambda^n - \lambda^{-n})^2$. So,

$$\begin{aligned}(c^2 - 4)/(a^2 - 4) &= (\lambda^{n-1} + \lambda^{n-3} + \cdots + \lambda^{1-n})^2 \\ &= (\operatorname{tr} x^{n-1} + \operatorname{tr} x^{n-3} + \cdots)^2\end{aligned}$$

where the second sum ends either with $\operatorname{tr} x$ or with 1.

Hence, in any group commensurable with Γ, the axes of any two elements commensurable with elements of trace a cannot intersect. Hence the claim in the first paragraph of this proof.

Now, to show that there exists Γ' commensurable with Γ in which $\Gamma'(a)$ is nonempty, we proceed as follows. Let $\lambda = \sqrt{a^2 - 4}$. The field $k(\lambda)$ splits $A\Gamma$, and so embeds in $A\Gamma$. Let the embedding be generated over k by $\gamma \in A\Gamma$. Since $a^2 - 4$ is an algebraic integer, λ is an algebraic integer, and it follows that we can take γ to be an integer of the quaternion algebra (see [V]). Hence there exists an order $\mathcal{O}$ of $A\Gamma$ containing γ. Then, $P\rho(\mathcal{O}^1)$ is a group commensurable with Γ which contains an element $P\gamma$ of trace a as is required. □

We remark here that this theorem only applies to arithmetic manifolds whose invariant trace field is a degree-two extension of a totally real field. Similar methods allow us to obtain the following, in which the restriction on the invariant trace-field is relaxed, but we are restricted to non-dyadic ideals.

It will be convenient to recall some notation, see [La] p. 20–22. Let k be a number field, $a \in k$, non-zero and let $\mathcal{A}$ be the principal fractional ideal generated by a. As is well-known there is a factorization of $\mathcal{A}$ as $\Pi\mathcal{P}^{r_{\mathcal{P}}}$ (where the product is over all prime ideals) with at most finitely many of the $r_{\mathcal{P}}$ being non-zero. Define $r_{\mathcal{P}}$ to be the *order* of $\mathcal{A}$ at $\mathcal{P}$.

Now suppose that k/F is a finite extension of number fields, and suppose p is a prime ideal of F. Denoting by R_k the ring of integers of k, the ideal pR_k factorizes as a product of k-primes $\mathcal{P}_1^{e_1}\dots\mathcal{P}_g^{e_g}$. The exponent e_i is called *the ramification index* of $\mathcal{P}_i$ over p.

We will also use both of these terms in reference to places rather than primes.

Theorem 4.2. *Let Γ be an arithmetic Kleinian group. Let F be the maximal totally real subfield of $k\Gamma$ and ν, ν' be a pair of distinct finite places of $k\Gamma$ which lie above the finite non-dyadic place ξ of F, each with odd ramification index. Then, if $\mathrm{A}\,\Gamma$ is ramified at ν and unramified at ν', there exists $a \in k\Gamma$ such that $\Gamma(a)$ is a link.*

Proof. The proof will follow the same outline as the proof of Theorem 4.1, but we will insure that β is a nonsquare unit at ν and ν', whence it follows that the order of α at ν must be odd and thus, if $\alpha \in F$, the order of α at ν' must be also odd, which contradicts the assumption that $\mathrm{A}\,\Gamma$ is unramified at ν.

Define $\mathcal{P}$, $\mathcal{P}'$, $\mathcal{P}''$ as in the proof of Theorem 4.1. Do the same for a_μ except for $\mu = \nu$ and $\mu = \nu'$. To define a_ν and $a_{\nu'}$, let q_ν and $q_{\nu'}$ denote fixed nonsquare units in k_ν and $k_{\nu'}$ respectively, not equal to -1. Then, let $a_\nu = 2(q_\nu + 1)/(q_\nu - 1)$ so that $a_\nu^2 - 4 = 4q_\nu/(q_\nu - 1)^2$ which is a nonsquare unit of k_ν. Define $a_{\nu'}$ similarly. Now, define a and β as in the proof of Theorem 4.1: there is again a Hilbert symbol (α, β) for $\mathrm{A}\,\Gamma$ and again we claim that α cannot be real.

Suppose that α is real. Then, $\alpha \in F$ which implies that the orders of α at ν and ν' have the same parity since both ν and ν' have odd ramification index in k/F (the order of α at ν is equal to the product of the order of α at ξ and the ramification index of ν in k/F – similarly for ν'). Now, since ν is non-dyadic, either α or β must be a non-unit at ν (see [O], p. 166). But, β is a unit at ν by construction, so α must be a non-unit at ν. In fact, since α is a nonsquare at ν, α must have odd order at ν. But this implies that α must also have odd order at ν', and since β is a nonsquare unit at ν', it follows that $\mathrm{A}\,\Gamma$ must be ramified at ν', contrary to hypothesis.

The remainder of the proof proceeds as in the proof of Theorem 4.1. $\square$

Similar methods also yield

Theorem 4.3. *Let Γ be an arithmetic Kleinian group. Let F be the maximal totally real subfield of $k\Gamma$ and ν be a finite place of $k\Gamma$ which lies over a finite non-dyadic place ξ of F, with even ramification index. Then, if $\mathrm{A}\,\Gamma$ is ramified at ν, there exists $a \in k\Gamma$ such that $\Gamma(a)$ is a link.*

Proof. The proof is identical to that of Theorem 4.2, but after construction of β and α we note that the order of α at ν must be odd, which is impossible for $\alpha \in F$, since the ramification index of ν in k/F is even. $\square$

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