# HYPERBOLIC STRUCTURES ON BRANCHED COVERS OVER HYPERBOLIC LINKS

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Abstract. Using a result of Tian concerning deformation of negatively curved metrics to Einstein metrics, we conclude that, for any fixed link with hyperbolic complement, there is a class of irregular branched covering spaces, branched over that link, effectively detectable by their branching indices, which consists entirely of closed hyperbolic manifolds.

**Section 0 - Introduction.** One of the elusive components of the Thurston Geometrization Conjecture is the the conjecture that all 3-dimensional closed manifolds with a Riemannian metric of negative sectional curvature admit a Riemannian metric of constant negative sectional curvature – the "negatively curved implies hyperbolic" conjecture. This conjecture is known to be false in dimensions higher than 3 (see [G-Th]), but is still very much a viable conjecture in dimension 3 (and is of course true in dimension 2).

One of the easiest ways to construct 3-manifolds admitting negatively curved metrics is to take a hyperbolic orbifold with singular locus a link and consider *sufficiently branched* covers over this orbifold, that is, branched covers whose branching indices over a given component of the singular locus of the orbifold are greater than or equal to the order of the isotropy group of that component. These are precisely the branched covers on which the cone manifold structure lifted from the base orbifold has all cone angles $> 2\pi$ (see [Ho] and [Jo] for discussions of cone manifolds). These cone manifold structures are singular metrics which may be smoothed to Riemannian metrics of negative sectional curvature. This construction is of particular importance to 3-dimensional topology in light of the existence of *universal links*, that is, links such that all closed, orientable 3-manifolds are obtainable as branched covers over that fixed link. Furthermore, all universal links are the singular locus of a hyperbolic orbifold, so this construction potentially has considerable bearing on the classification problem for 3-manifolds.

The purpose of this paper is to combine this smoothing technique, by way of some explicit pinching constant computations, with a result of Tian (see [Ti]) to show

**Theorem 3.2.** *Let $(M, L)$ be a 3-manifold and a link such that $M - L$ admits a hyperbolic metric of finite volume. Let $L_1, \dots, L_q$ be the components of $L$. Let $(\hat{M}, \hat{L})$ be a branched cover over $(M, L)$ with minimum branching index $n_i$ over $L_i$ and maximum branching index $N_i$ over $L_i$. Denote by $Q_i$ the number of components of branching locus over $L_i$ with*

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*branching index not equal to* $n_i$ *(counted with the appropriate multiplicity in the case of longitudinal wrapping). Then, there exist integers* $(m_1, \ldots, m_q)$ *and functions* $(K_1, \ldots, K_q)$ *with* $K_j : \mathbb{Z} \to \mathbb{R}$ *and* $K_j(i) > 1$*, only depending on* $(M, L)$*, such that* $(\hat{M}, \hat{L})$ *admits a hyperbolic metric if* $n_i \geq m_i$ *and* $N_i \leq n_i K_i(Q_i)$ *for all* $i = 1, \ldots, q$*.*

It is a straightforward consequence of Thurston's Hyperbolic Dehn Surgery Theorem (see [Th]) that if all the branching indices over each component are equal (for example, in the case of regular branched covers), and greater than some fixed constant on each component, then the cover is hyperbolic, but this is the first result giving combinatorial conditions on the branched covering map under which irregular branched covers must be hyperbolic (although admittedly this is far from a constructive proof – we have no idea what order of magnitude the $m$'s and $K$'s have).

We will also note a simplified version of Theorem 3.2 which is implied by the conjecture of the existence of a universal constant $K > 1$ such that any 3-manifold with curvature between $-K$ and $-1$ is hyperbolic.

**Section 1 - Smoothing of Hyperbolic Cone Manifolds.**

We will begin by making the following

**Definition.** *A hyperbolic cone manifold is a metric space obtained as the quotient space of a disjoint union of a collection of geodesic n-simplices in* $\mathbb{H}^n$ *by an isometric pairing of codimension-one faces in such a combinatorial fashion that the underlying topological space is a manifold and that the induced triangulation on the underlying space is uniformly locally finite.*

Such a metric space admits a smooth Riemannian metric of constant negative sectional curvature on the union of the top-dimensional cells and the codimension-1 cells. On each codimension-2 cell, the structure is completely described by an angle, which is the sum of the dihedral angles around all of the codimension-2 simplicial faces which are identified to give the cell. The *cone locus* is the closure of all the codimension-2 cells for which this angle is not $2\pi$ (the Riemannian metric may be extended smoothly over all cells whose angle is $2\pi$.

A hyperbolic orbifold may be thought of simply as a hyperbolic cone manifold in which all cone angles are of the form $2\pi/k$ for some integer $k$.

We will restrict ourselves in this paper to 3-dimensional cone manifolds in which the cone locus is a link (i.e., a codimension-2 submanifold) to avoid problems with the smoothing results and with the branched covering maps themselves (branched coverings over non-manifold branch sets need not be manifolds).

On these cone manifolds, we may smooth the metric by the following procedure: take disjoint metrically regular neighborhoods of the components of the cone locus (which must be solid tori or solid Klein bottles – the obvious symmetry of the solid torus case yields the solid Klein bottle case as well). On the complement of the singular core geodesic within each of these solid tori, we have a Riemannian metric of constant negative sectional curvature given (in Fermi coordinates – polar coordinates on each disk cross section with $t$ measuring distance along the core geodesic) by

$$ds^2 = \cosh^2(r)\, dt^2 + dr^2 + (\varphi/2\pi)^2 \sinh^2(r)\, d\theta^2$$

where $\varphi$ is the cone angle of the central core geodesic. Suppose that the radius of the regular neighborhood is $r_0$ so that the above is valid for $0 < r < r_0$. We will now consider metrics of the form

$$ds^2 = f^2(r)dt^2 + dr^2 + g^2(r)d\theta^2$$

and observe that the sectional curvatures in the principal directions relative to the given coordinate system are $-f''(r)/f(r)$, $-g''(r)/g(r)$ and $-f'(r)g'(r)/f(r)g(r)$. We also observe that this metric extends across the central core geodesic whenever $g(0) = 0$, $f(0) \neq 0$ and $g'(0) = 1$. If, in addition, all of the even derivatives of $g$ and the odd derivatives of $f$ vanish at 0, and $f, g$ are $C^\infty$, the extended metric will also be $C^\infty$. Thus, if we can find functions $f, g$ on some interval $[-R_0, R_0]$ which are both increasing and convex such that $f$ is even, $g$ is odd, $f(0) \neq 0$, $g'(0) = 1$, $f, g$ and all their derivatives at $R_0$ match $\sinh(r), (\varphi/2\pi)\cosh(r)$ (respectively) and all their derivatives at $r_0$, we will have found a smooth Riemannian metric of negative sectional curvature on the solid torus which may be extended smoothly to the complement of the solid torus. This can clearly always be done provided $\varphi > 2\pi$. In Section 3, we will find $f, g$ of a particularly nice form in which we can calculate bounds on the sectional curvature of the metric. For now, we will simply note that we have proved

**Lemma 1.1.** *An hyperbolic cone manifold with cone locus a link and all cone angles $> 2\pi$ admits a metric of negative sectional curvature.*

and observe that we have in fact constructed this metric in an explicit way which will allow us to make curvature calculations.

**Section 2 - Einstein Metrics.**

The key result by which we will obtain our hyperbolic structures is the following theorem of Tian [Ti] (see [BGS] or [Th] for a discussion of the "Margulis number" mentioned in the statement):

**Theorem 2.1.** *Let $M$ be a negatively curved Riemannian 3-manifold and $\eta$ a Margulis number for negatively curved 3-manifolds. Denote by $M_\eta$ the $\eta$-thin piece of $M$ and by $R(M)$ the trace-free Ricci curvature of $M$ (in dimension 3, the sum of the Ricci tensor and twice the metric tensor). Then, there is a universal constant $\epsilon$ such that if $M$ satisfies*

1. *$M_\eta$ is a disjoint union of convex neighborhoods $\{C_\alpha\}$ of closed geodesics $\{\gamma_\alpha\}$ with length $\leq 2\eta$ such that the normal injectivity radius of $\gamma_\alpha$ in $C_\alpha$ is greater than 1.*
2. *let $P_\alpha : C_\alpha \to \mathbb{R}$ be a smooth function such that $P_\alpha$ is equal to $\eta$ near the boundary of $C_\alpha$ and $P_\alpha(y)$ is equal to the injectivity radius at $y$ whenever this is less than $\eta/2$ (such $P_\alpha$ always exist). We require that for some choice of $P_\alpha$,*

$$\int_{C_\alpha} \frac{1}{P_\alpha} \mid R(M) \mid^2 \, dV \leq \epsilon$$

   *holds for each $\alpha$.*
3. *all sectional curvatures of $M$ lie between $-1 - \epsilon$ and $-1 + \epsilon$.*
4. *$\int_M \mid R(M) \mid^2 dV \leq \epsilon^2$*

*Then, $M$ admits a negatively curved Einstein metric.*

In fact, Tian's result is rather stronger than this, allowing dimensions other than 3, norms other than the $L^2$ norm, and controlling the amount of deviation of the Einstein metric (and its first three derivatives) from the original negatively curved metric. However, the statement here is all we will in fact need, once we have made the observation that the constant "1" in the first hypothesis is arbitrary and any *a priori* lower bound on the normal injectivity radius will do. In fact, we may take this bound to be $\eta$, in which case the first hypothesis is vacuous in the case at hand, i.e., closed 3-manifolds, in which the $\eta$-thin piece always consists of neighborhoods of short geodesics (length at most $2\eta$) of radius at least $\eta$.

Now, recall that in dimension 3, Einstein manifolds have constant sectional curvature, and thus, if we can verify hypotheses (2), (3) and (4) of Theorem 2.1 for certain smoothings of branched covers with constrained branching indices, where these constraints are of the form of the hypotheses of Theorem 3.2, we will have proven the desired result.

We also note here, for reference in the next section, that, relative to an orthonormal frame, the entries in the $3 \times 3$ matrix for $R(M)$ are all between $-4\epsilon$ and $4\epsilon$ if all sectional curvatures are pinched between $-1-\epsilon$ and $-1+\epsilon$. This follows from the fact that the Ricci tensor may be recovered by polarization from its associated quadratic form $Q(u) = \mathrm{Ric}(u,u)$ and that $\mathrm{Ric}(u,u)$ is simply $< u, u >$ multiplied by the sum of the sectional curvatures of any 2 orthogonal planes containing $u$.

**Section 3 - Pinching Computations.** In this section, we will use some particular smoothings of cone metrics for which we can estimate curvature bounds and the asymptotic behavior of such bounds. These bounds will enable us to complete the proof of Theorem 3.2.

**Lemma 3.1.** *If $(M, L)$ is a (manifold,link) pair such that $M - L$ admits a hyperbolic metric of finite volume, then, for each component, $L_i$ of $L$ there is a positive integer $m_i$ and a function $K_i : [1, \infty) \to [0, \infty)$ such that any branched cover over $M$, branched over $L$ with minimum branching index $n_i \geq m_i$ over $L_i$ and maximum branching index $N_i$ over $L_i$ admits a negatively curved metric which has constant sectional curvature outside of a regular neighborhood of the branching locus and everywhere has sectional curvature pinched between $-1-\epsilon$ and $-1+\epsilon$ where $\epsilon = \max_i K_i(N_i/n_i)$ and $\lim_{x\to 1} K_i(x) = 0$ for all $i$.*

*Proof.* We begin by choosing $m_i$ and $R_i$ so that there is a hyperbolic cone manifold structure on $(M, L)$ with cone locus contained in $L$, such that the cone angle on $L_i$ is $2\pi/m_i$ and there is a regular neighborhood of $L$ consisting of radius-$R_i$ neighborhoods about each $L_i$. We further stipulate that there must exist cone manifold structures on $(M, L)$ with any cone angle $\varphi_i < 2\pi/m_i$ on $L_i$ (this is conjecturally always the case for $m_i$ as above, but in any event, we may increase $m_i$ until this is true) in which the $R_i$-neighborhoods of the $L_i$ still form a regular neighborhood (one can always reduce $R_i$ to achieve this). The Hyperbolic Dehn Surgery Theorem guarantees that some choice of $m_i$ and $R_i$ is possible. One would like to have small $m_i$ values and larger $R_i$ values and, in general, one can trade-off one for the other. It is unclear which one would produce the sharpest bounds.

Now, let $(\hat{M}, \hat{L})$ be a branched cover of $(M, L)$ satisfying our hypotheses. Let $\hat{L}_j$ be some component of the preimage of $L_i$ with branching index $k_j$ $(n_i \leq k_j \leq N_i)$. By the construction of $m_i$ and the hypothesis that $n_i \geq m_i$ there exists a cone metric on $(M, L)$ with cone angle $2\pi/n_i$ at $L_i$. Let us smooth the cone metric on $\hat{M}$ obtained by lifting this cone metric on $M$ in a $R_i$-neighborhood of $\hat{L}_j$. The cone angle at $\hat{L}_j$ is $2\pi k_j/n_i$ and so (using coordinates as in Section 1) the metric is

$$ds^2 = \cosh(r)^2\, dt^2 + dr^2 + (k_j/n_i)^2 \sinh^2(r)\, d\theta^2$$

where $t \in [0, L)$, $\theta \in [0, 2\pi)$, $r \in [0, R_i)$ and $L$ is the length of $\hat{L}_j$. We need to construct functions $f$ and $g$ with domain $[0, \hat{R})$ which satisfy the following hypotheses:

(1) $f, g$ are $C^\infty$, increasing, concave.
(2) $f$ is even, $f(0) = 1$, $f$ and all its derivatives at $\hat{R}$ are equal to $\cosh(r)$ and all its derivatives at $R_i$.
(3) $g$ is odd, $g'(0) = 1$, $g$ and all its derivatives at $\hat{R}$ are equal to $(k_j/n_i)\sinh(r)$ and all its derivatives at $R_i$.

Let us set $f(r) = \cosh(\alpha(r))$ and $g(r) = (k_j/n_i)\sinh(\alpha(r))$. This choice lets us easily bound the sectional curvature of the resulting metric, since the sectional curvatures of the principal (coordinate) directions become $(\alpha'(r))^2$, $(\alpha'(r))^2 + \alpha''(r)\tanh(\alpha(r))$ and $(\alpha'(r))^2 + \alpha''(r)\coth(\alpha(r))$. In particular, if $\alpha''$ is nonnegative, all sectional curvatures are pinched between $\min(\alpha'(r))^2$ and $\max(\alpha'(r))^2 + \alpha''(r)\coth(r)$. Furthermore, the conditions on $f$ and $g$ all reduce to the insistence that $\alpha''$ be a positive $C^\infty$ bump function with support contained in $[0, \hat{R})$ and that $\alpha'(0) = n_i/k_j$, $\alpha'(\hat{R}) = 1$, $\alpha(0) = 0$, $\alpha(\hat{R}) = R_i$. We will return to these conditions momentarily, but first let us obtain more workable estimates for the curvature bounds.

Since $\alpha'' \geq 0$, the minimum of $\alpha'$ is attained at 0, so the lower bound is $(n_i/k_j)^2$. Also, $\alpha' \geq n_i/k_j$, together with $\alpha(0) = 0$ implies that $\alpha(r) \geq n_i r/k_j$, which in turn implies that $\coth(\alpha(r)) \leq \coth(n_i r/k_j)$. In particular,

$$(\alpha'(r))^2 + (\alpha''(r))\coth(\alpha(r)) \leq 1 + (\alpha''(r))\coth(\frac{n_i r}{k_j})$$

Now, when the right-hand side of this equation is maximized, we have that

$$\alpha'''(r)\coth(\frac{n_i r}{k_j}) = \frac{n_i}{k_j}\alpha''(r)\operatorname{csch}^2(\frac{n_i r}{k_j})$$

and thus

$$\alpha''(r) = \frac{n_i}{k_j}\alpha'''(r)\cosh(\frac{n_i r}{k_j})\sinh(\frac{n_i r}{k_j})$$

Thus, using the fact that $(n_i/k_j) \leq \alpha'(r) \leq 1$ and hence $(k_j R_i/n_i) \leq \hat{R} \leq R_i$, we obtain that

$$\begin{aligned}
\max \alpha''(r)\coth(\alpha(r)) &\leq \max \alpha''(r)\coth(\frac{n_i r}{k_j}) \\
&\leq \max \frac{n_i}{k_j}\alpha'''(r)\cosh^2(\frac{n_i r}{k_j}) \\
&\leq \frac{n_i}{k_j}\cosh^2(\frac{n_i \hat{R}}{k_j})\max \alpha'''(r) \\
&\leq \frac{n_i}{k_j}\cosh^2(R_i)\max \alpha'''(r)
\end{aligned}$$

In particular, the sectional curvature of the smooth metric obtained in this way is pinched between $(n_i/k_j)^2$ and $1 + (n_i/k_j)\cosh^2(R_i)\max\alpha'''(r)$.

So, to obtain asymptotic estimates on pinching for these metrics, we must consider bounds on $\max\alpha'''(r)$ where $\alpha''$ is a positive $C^\infty$ bump function with support in $[0, \hat{R}]$, $\alpha(0) = 0$, $\alpha'(0) = n_i/k_j$, $\alpha(\hat{R}) = R_i$, and $\alpha'(\hat{R}) = 1$. Set $\varphi(r) = \alpha''(r)$ and let us translate the constraints on $\alpha$ into constraints on $\varphi$. The first-derivative constraints on $\alpha$ become

$$\int_0^{\hat{R}} \varphi(r)\,dr = 1 - \frac{n_i}{k_j}$$

while the constraints on the values of $\alpha$ itself become

$$\int_0^{\hat{R}} \left[\frac{n_i}{k_j} + \int_0^s \varphi(r)\,dr\right] ds = R_i$$

These constraints can be satisfied in general by the following scheme: let $\psi$ be a fixed bump function with support in $[0, 1]$, area $A$ and $\int_0^1 \int_0^s \psi(r)\,dr\,ds = B$. Then, $\varphi(r) = K_1\psi(K_2 r)$ has support in $[0, 1/K_2]$, area $K_1 A/K_2$ and iterated integral $K_1 B/K_2^2$. Thus, if we set

$$K_2 = \frac{\frac{n_i}{k_j} + (1 - \frac{n_i}{k_j})\frac{B}{A}}{R_i}$$

and

$$K_1 = K_2(1 - \frac{n_i}{k_j})\frac{B}{A}$$

all the constraints on $\varphi$ are satisfied and

$$\alpha(r) = \int_0^r \left[\frac{n_i}{k_j} + \int_0^s \varphi(t)\,dt\right] ds$$

is the required function as above and we have that the sectional curvature is bounded between $(n_i/k_j)^2$ and

$$1 + K(\frac{k_j}{n_i} - 1)\frac{\cosh^2 R_i}{R_i^2}(\frac{B}{A})(\frac{n_i}{k_j} + (1 - \frac{n_i}{k_j})\frac{B}{A})^2$$

where $K$ is $\max\psi'$. Now, letting $n_i/k_j \to 1$, we obtain the desired asymptotic result, by setting $K_i(x)$ to be the radius of the symmetric neighborhood of 1 which is obtained by rescaling the interval

$$[(\frac{1}{x})^2, 1 + (\frac{KB}{A})(\frac{\cosh_i^R}{R_i^2})(x-1)(\frac{1}{x} + (\frac{x-1}{x})\frac{B}{A})^2]$$

□

With this lemma in hand, we will proceed to the proof of

**Theorem 3.2.** *Let $(M, L)$ be a 3-manifold and a link such that $M - L$ admits a hyperbolic metric of finite volume. Let $L_1, \ldots, L_q$ be the components of $L$. Let $(\hat{M}, \hat{L})$ be a branched cover over $(M, L)$ with minimum branching index $n_i$ over $L_i$ and maximum branching index $N_i$ over $L_i$. Denote by $Q_i$ the number of components of branching locus over $L_i$ with branching index not equal to $n_i$ (counted with the appropriate multiplicity in the case of longitudinal wrapping). Then, there exist integers $(m_1, \ldots, m_q)$ and functions $(K_1, \ldots, K_q)$ with $K_j : \mathbb{Z} \to \mathbb{R}$ and $K_j(i) > 1$, only depending on $(M, L)$, such that $(\hat{M}, \hat{L})$ admits a hyperbolic metric if $n_i \geq m_i$ and $N_i \leq n_i K_i(Q_i)$ for all $i = 1, \ldots, q$.*

*Proof.* Using Lemma 3.1, it is sufficient to show that there exists $\epsilon(Q_i)$ such that whenever the metrics constructed in Lemma 3.1 have sectional curvature pinched between $-1-\epsilon(Q_i)$ and $-1+\epsilon(Q_i)$, hypotheses (2), (3) and (4) of Theorem 2.1 are satisfied.

Hypothesis (3) is clear, and (4) is not much harder, once we observe that the integral in (4) is bounded by product of the maximum of $\mid R(M) \mid^2$ with the volume of the support of $R(M)$. The former is uniformly bounded (as observed earlier) and so we need to estimate the latter, which is the volume of all of the tubular neighborhoods of components of branching locus for which the branching index is strictly greater than $n_i$ (there are $Q_i$ of these). Next, we note that as a sequence of hyperbolic cone metrics tends to the complete metric on the complement of the cone locus, the length of each component of the cone locus tends to zero. Thus, there is a global maximum, for all of the cone metrics with cone angle $\leq 2\pi/m_i$ on the length of the cone geodesics. Thus, there is an *a priori* bound on the volume of the tubular neighborhood of the cone geodesic which will be lifted and smoothed. It suffices, then, to estimate the volume of the smoothed tubular neighborhood in terms of the volume of the singular tubular neighborhood. We observe that (with notation as in the proof of Lemma 3.1 - $\ell$ is the length of the core geodesic)

$$\mathrm{Vol}_{\mathrm{smooth}} = 2\pi\ell\frac{k_j}{n_i}\int_0^{\hat{R}} \cosh(\alpha(\hat{r}))\sinh(\alpha(\hat{r}))\,d\hat{r}$$

whereas

$$\mathrm{Vol}_{\mathrm{cone}} = 2\pi\ell\frac{k_j}{n_i}\int_0^{R} \cosh r \sinh r\,dr$$

but

$$\begin{aligned}\int_0^R \cosh r\sinh r\,dr &= \int_0^{\hat{R}} \cosh(\alpha(\hat{r}))\sinh(\alpha(\hat{r}))\alpha'(\hat{r})\,d\hat{r}\\ &\geq (\min(\alpha'(\hat{r})))\int_0^{\hat{R}} \cosh(\alpha(\hat{r}))\sinh(\alpha(\hat{r}))\,d\hat{r}\\ &= \frac{n_i}{k_j}\int_0^{\hat{R}} \cosh(\alpha(\hat{r}))\sinh(\alpha(\hat{r}))\,d\hat{r}\end{aligned}$$

so that $\mathrm{Vol}_{\mathrm{smooth}} \leq \frac{N_i}{n_i}\mathrm{Vol}_{\mathrm{cone}}$ and thus we have bounded the integral in hypothesis (4) by a function of $Q_i$ and $N_i/n_i$ which tends to 0 as $N_i/n_i$ tends to 1.

Hypothesis (2) is handled in a very similar way – we only note that the worst possible case for this hypothesis occurs when the core geodesic of one of the smoothing neighborhoods is "short," but that in this case, the symmetry of the neighborhood allows us to trivially integrate along the geodesic (the $t$ direction) and around the geodesic (the $\theta$ direction) and that the short length essentially cancels the $1/P_\alpha$ term in the integrand. □

We also note that if one had a universal pinching constant $\epsilon$ such that any manifold with sectional curvature between $-1-\epsilon$ and $-1+\epsilon$ could be deformed to a constant curvature metric, then one could eliminate the dependence on $Q_i$ in the previous theorem.

It is difficult to make direct use of this theorem to show that any particular branched cover is hyperbolic, since there are so many undetermined universal constants involved, but it does provide the first sufficient combinatorial conditions under which an irregular branched cover may be shown to be hyperbolic.

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