1. First, observe that the volume is

\[
\int_{-1}^{1} \int_{-\left(1-x^6\right)^{1/6}}^{\left(1-x^6\right)^{1/6}} \int_{x^2}^{x^2} e^{x^6+y^6} + x^2 \, dz \, dy \, dx
\]

\[= \int_{-1}^{1} \int_{-\left(1-x^6\right)^{1/6}}^{\left(1-x^6\right)^{1/6}} x^2 y^2 e^{x^6+y^6} \, dy \, dx\]

Now, let \( u = x^3 \) and \( v = y^3 \). Then, the Jacobian of \((x, y)\) with respect to \((u, v)\) is

\[\frac{u^{-2/3}v^{-2/3}}{9}\]

so that the integral becomes

\[
\frac{1}{9} \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} e^{u^2+v^2} \, dv \, du
\]

Now, use polar coordinates in the \((u, v)\)-plane to obtain

\[
\frac{1}{9} \int_{0}^{2\pi} \int_{0}^{1} re^{r^2} \, dr \, d\theta = \frac{\pi(e - 1)}{9}
\]

2. The initial setup yields

\[
\int_{0}^{2} \int_{0}^{y} e^{y^2} \, dx \, dy = \int_{0}^{2} e^{y^2} \int_{0}^{y} \, dx \, dy
\]

Integrating then yields (let \( u = y^2 \))

\[
\int_{0}^{2} ye^{y^2} \, dy = \frac{1}{2} \int_{0}^{4} e^u \, du = \frac{1}{2}(e^4 - 1)
\]

3. Change to polar coordinates - then you have

\[
\int_{0}^{2\pi} \int_{0}^{1} r(1-r^2) \, dr \, d\theta = 2\pi \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{2}
\]

4. the surface in question is \( z = \sqrt{1-x^2-y^2} \), so

\[
\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{1-x^2-y^2}}
\]

\[
\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{1-x^2-y^2}}
\]
Hence, the surface area is given by (let \( u = 1 - r^2 \))

\[
\int_{-1/2}^{1/2} \int_{-\sqrt{(1/4)-x^2}}^{\sqrt{(1/4)-x^2}} \left[ 1 + \left( \frac{-x}{\sqrt{1-x^2-y^2}} \right)^2 + \left( \frac{-y}{\sqrt{1-x^2-y^2}} \right)^2 \right] dy \, dx
= \int_{-1/2}^{1/2} \int_{-\sqrt{(1/4)-x^2}}^{\sqrt{(1/4)-x^2}} \frac{1}{\sqrt{1-x^2-y^2}} \, dy \, dx
= \int_0^{2\pi} \int_0^{1/4} \frac{r}{\sqrt{1-r^2}} \, dr \, d\theta
= 2\pi \int_{3/4}^1 \frac{du}{2\sqrt{u}} = 2\pi(1 - \sqrt{3/4}) = \pi(2 - \sqrt{3})
\]

5. Change the order of integration and let \( u = x^3 \) so that you have

\[
\int_0^1 \int_0^{x^2} e^x \, dy \, dx
= \int_0^1 x^2 e^x \, dx
= \frac{1}{3} \int_0^1 e^x \, dx = e - 1
\]

6. The easy parametrization is \( \vec{r}(t) = (2\cos t, \sqrt{2}\sin t, \sqrt{2}\sin t) \) where \( 0 \leq t \leq 2\pi \) (note that this satisfies both equations). Then, \( |\vec{r}'(t)| = \sqrt{4\sin^2 t + 2\cos^2 t + 2\cos^2 t} = 2 \). Thus, the arclength is \( 4\pi \).

7-12. The best way to describe \( R \) is in spherical coordinates: \( 0 \leq \theta \leq 2\pi, \ 0 \leq \rho \leq 1, \ 0 \leq \varphi \leq \pi/4 \). In every case, symmetry indicates that the \( x \) and \( y \) coordinates of the centroid or center of mass are zero, so I’ll concentrate on the \( z \) coordinate. With uniform density, the \( z \) coordinate is

\[
\bar{z} = \frac{\int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho \cos \varphi \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta}{\int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta}
= \frac{\pi/2}{(2\pi/3)} \int_0^{\pi/4} \sin \varphi \, d\varphi
= \frac{3}{16} - 8\sqrt{2}
\]

With density proportional to the distance from the origin, the \( z \) coordinate is

\[
\bar{z} = \frac{\int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \cos \varphi \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta}{\int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^3 \sin \varphi \, d\rho \, d\varphi \, d\theta}
= \frac{2\pi/5}{(\pi/2)} \int_0^{\pi/4} \sin \varphi \, d\varphi
= \frac{2}{10 - 5\sqrt{2}}
\]
With density proportional to the distance from the $z$-axis, the $z$ coordinate is

$$
\bar{z} = \frac{\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\pi/4} \rho^2 \cos \varphi \sin \varphi \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta}{\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\pi/4} \rho^3 \sin^2 \varphi \, d\rho \, d\varphi \, d\theta}
$$

$$
= \frac{(2\pi/5) \int_0^{\pi/4} \sin^2 \varphi \cos \varphi \, d\varphi}{(\pi/2) \int_0^{\pi/4} \sin^2 \varphi \, d\varphi}
$$

$$
= \frac{8\sqrt{2}}{15(\pi - 2)}
$$

The moments of inertia are similar.

13. The multivariable chain rule really says that the matrix of partial derivatives of a composite function is the product of the two matrices of partial derivatives of the component functions.

Thus,

$$
Dh = (Df)(Dg)
$$

Since we know $Dg$, we only need to calculate

$$
Df = \begin{pmatrix}
\ln v & u/v \\
v^2 & 2uv \\
\sin v & u \cos v
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
1 & 2 \\
\sin 1 & \cos 1
\end{pmatrix}
$$

so that

$$
Dh = \begin{pmatrix}
0 & 1 \\
1 & 2 \\
\sin 1 & \cos 1
\end{pmatrix}
\begin{pmatrix}
2 & 3 \\
1 & 0 \\
\cos 1 + 2 \sin 1 & 3 \sin 1
\end{pmatrix}
$$

14. The surface is a level set (contour) of the function $f(x) = xy + yz + zx$ and thus $\nabla f$ is normal to the surface. $\nabla f = (y + z, x + z, y + x) = (2, 2, 2)$ at $(1, 1, 1)$ so the equation of the tangent plane is

$$
2(x - 1) + 2(y - 1) + 2(z - 1) = 0
$$

15. You can either calculate the gradient and sketch it directly or notice that the contours are given by

$$
K = \frac{x}{x^2 + \frac{y^2}{4}}
$$

$$
Kx^2 + \frac{K}{4}y^2 = x
$$

$$
x^2 + \frac{y^2}{4} = \frac{x}{K}
$$

$$
(x - \frac{1}{2K})^2 + \frac{y^2}{4} = \frac{1}{4K^2}
$$

$$
4K^2(x - (1/2K))^2 + K^2y^2 = 1
$$

so that the contours are ellipses centered at $(1/(2K), 0)$ that go through the origin and have an aspect ratio of 2 (they are twice as tall as they are wide). The gradient vector field is perpendicular to these ellipses.