In these notes, I’ll show that the lift of the map $f(w) = 1/w$ in the Riemann sphere is the reflection in the $xy$-plane.

Let $S$ denote the unit sphere in 3-space,

$$S = \{(x, y, z)|x^2 + y^2 + z^2 = 1\}$$

and let $\mathbb{C}^*$ denote the Riemann sphere,

$$\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$$

Then, stereographic projection from $S$ to $\mathbb{C}^*$ is defined by setting $g(x, y, z)$ equal to the point where the line from $(0,0,1)$ to $(x, y, z)$ intersects the plane $z = 0$ (regarding the plane $z = 0$ as the complex plane $\mathbb{C}$). This is defined for all points on $S$ except $(0,0,1)$. If we then define $g(0,0,1)$ to be $\infty$, then we have a map $g : S \to \mathbb{C}^*$. When $z \neq 1$, we may use similar triangles in the unique plane containing $(0,0,0)$, $(0,0,1)$ and $(x,y,z)$, and easily see that $g$ is given by the formula

$$g(x,y,z) = \begin{cases} 
\frac{x+iy}{1-z} & \text{if } z \neq 1 \\
\infty & \text{if } z = 1
\end{cases}$$

We need to also work out the inverse map, $g^{-1}$. Denote an arbitrary point in $\mathbb{C}$ by $u + iv$, and let $(x,y,z)$ be a point in $S$ such that $g(x,y,z) = u + iv$. Assume for the moment that such a point exists and we will derive expressions for $x$, $y$ and $z$ in terms of $u$ and $v$. At that point, we will know that $(x,y,z)$ exists whenever the expressions are defined (which will turn out to be everywhere).

First, note that the similar triangles argument alluded to above, combined with the fact that $x^2 + y^2 + z^2 = 1$ gives us

$$u^2 + v^2 = \frac{1+z}{1-z}$$

or

$$z = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}$$

whence we see that

$$1 - z = \frac{2}{u^2 + v^2 + 1}$$

so that

$$x = \frac{2u}{u^2 + v^2 + 1}$$

$$y = \frac{2v}{u^2 + v^2 + 1}$$

$$z = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}$$

hence,

$$g^{-1}(u + iv) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right)$$
or, writing it in terms of a complex number $w$,

$$g^{-1}(w) = \left( \frac{2\text{Re}(w)}{|w|^2 + 1}, \frac{2\text{Im}(w)}{|w|^2 + 1}, |w|^2 - 1 \right)$$

It is worth checking that

$$x^2 + y^2 + z^2 = \frac{4u^2 + 4v^2 + (u^2 + v^2 - 1)^2}{(u^2 + v^2 + 1)^2}$$

$$= \frac{4(u^2 + v^2) + (u^2 + v^2)^2 - 2(u^2 + v^2) + 1}{(u^2 + v^2 + 1)^2}$$

$$= \frac{(u^2 + v^2)^2 + 2(u^2 + v^2) + 1}{(u^2 + v^2 + 1)^2}$$

$$= \frac{(u^2 + v^2 + 1)^2}{(u^2 + v^2 + 1)^2}$$

$$= 1$$

Now, remember that the motivation for all this was to figure out what $f(z) = 1/z$ does geometrically. It’s easy to see that

$$f(u + iv) = \frac{1}{u - iv} = \frac{u + iv}{(u - iv)(u + iv)} = \frac{u + iv}{u^2 + v^2}$$

and we’re wanting to see what $g^{-1}(f(g(x, y, z)))$ is. Now we have all the pieces:

$$g^{-1}(f(g(x, y, z))) = g^{-1}(f\left(\frac{x + iy}{1 - z}\right))$$

$$= g^{-1}\left(\frac{x + iy}{x^2 + y^2}\right)$$

$$= g^{-1}\left(\frac{(x + iy)(1 - z)}{x^2 + y^2}\right)$$

$$= g^{-1}\left(\frac{x + iy}{1 - z^2}\right)$$

$$= g^{-1}\left(\frac{x + iy}{1 + z}\right)$$

$$= \left(\frac{2x}{x^2 + y^2} + 1\right), \left(\frac{2y}{x^2 + y^2} + 1\right), \left(\frac{x^2 + y^2}{(1+z)^2} - 1\right)$$

$$= \left(\frac{2x}{1 - z^2} + 1\right), \left(\frac{2y}{1 - z^2} + 1\right), \left(\frac{1 - z^2}{(1+z)^2} - 1\right)$$

$$= \left(\frac{1 - z}{1 + z} + 1\right), \left(\frac{1 - z}{1 + z} + 1\right), \left(\frac{1 - z}{1 + z} + 1\right)$$

$$= (x, y, -z)$$
A quicker way to see this (and one that doesn’t require us to work out an explicit formula
for \( g^{-1} \)) is to observe that \( f(g(x, y, z)) = \frac{x+iy}{1+z} \) (line 5 above). Since \( g(x, y, z) = \frac{x+iy}{1+z} \), this tells us that \( f(g(x, y, z)) = g(x, y, -z) \). However, the explicit formula for \( g^{-1} \) will be useful eventually, so we might as well work it out now!

Note also that reflection in the \( yz \)-plane (changing the sign of the first coordinate) covers \( f(z) = -z \) and that reflection in the \( xz \)-plane (changing the sign of the second coordinate) covers \( f(z) = \overline{z} \). This gives a quick proof of the fact (verified in an exercise) that the antipodal map covers \( f(z) = -1/\overline{z} \) since composing all three of these reflection maps gives that transformation.