

Möbius (Fractional Linear) Transformations have two major properties that need to be proven - conformality and invariance of clines. Let's do conformality first. We need to show that angles are preserved. To do this, we recall three facts. First, that

$$\angle pqr = \arg\left(\frac{p-q}{r-q}\right)$$

Second, that to measure the "angle" between curves, we measure the angle between their tangent lines at the point of intersection. We need this second idea because Möbius Transformations do not preserve lines. Finally, we will need the idea from calculus that tangent lines are limits of secant lines.

So, here's our plan: let $f(z) = \frac{az+b}{cz+d}$ let z be a particular point in \mathbb{C} with $cz+d \neq 0$ and let θ and φ be the directions of two lines from z . Then, points on these two lines are given by $z + te^{i\theta}$ and $z + te^{i\varphi}$, respectively, where t is real. To measure the angle between the images of these two lines, we need to calculate

$$\lim_{t \rightarrow 0} \frac{f(z + te^{i\theta}) - f(z)}{f(z + te^{i\varphi}) - f(z)}$$

The argument of this limit will then be the angle between the tangent lines to the curves which are the images of the original two lines under f .

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{f(z + te^{i\theta}) - f(z)}{f(z + te^{i\varphi}) - f(z)} &= \lim_{t \rightarrow 0} \frac{\frac{a(z+te^{i\theta})+b}{c(z+te^{i\theta})+d} - \frac{az+b}{cz+d}}{\frac{a(z+te^{i\varphi})+b}{c(z+te^{i\varphi})+d} - \frac{az+b}{cz+d}} \\ &= \lim_{t \rightarrow 0} \frac{\frac{(a(z+te^{i\theta})+b)(cz+d) - (az+b)(c(z+te^{i\theta})+d)}{(c(z+te^{i\theta})+d)(cz+d)}}{\frac{(a(z+te^{i\varphi})+b)(cz+d) - (az+b)(c(z+te^{i\varphi})+d)}{(c(z+te^{i\varphi})+d)(cz+d)}} \\ &= \lim_{t \rightarrow 0} \frac{(ate^{i\theta}(cz+d) - cte^{i\theta}(az+b))(cz+d)(c(z+te^{i\varphi})+d)}{(ate^{i\varphi}(cz+d) - cte^{i\varphi}(az+b))(cz+d)(c(z+te^{i\theta})+d)} \\ &= \lim_{t \rightarrow 0} \frac{e^{i\theta}(a(cz+d) - c(az+b))(c(z+te^{i\varphi})+d)}{e^{i\varphi}(a(cz+d) - c(az+b))(c(z+te^{i\theta})+d)} \\ &= \frac{e^{i\theta}(ad-bc)(cz+d)}{e^{i\varphi}(ad-bc)(cz+d)} = e^{i(\theta-\varphi)} \end{aligned}$$

The argument of this limit is, therefore, $\theta - \varphi$ which is the same angle as that made by the original two lines. Thus, f is conformal, except possibly where $z = \infty$ or where $f(z) = \infty$. To show conformality at those places, we need to define what is meant by "angles at infinity." The most reasonable thing to do is to simply *define* the angle between two curves meeting at infinity to be the angle between the two tangent lines at zero of the curves produced by composing the two curves with $f(z) = 1/z$. For lines, this definition turns out to be the same as measuring the angle between the two lines at the *other* place where they meet. With this definition, conformality of Möbius Transformations at infinity is automatic.

To show that Möbius Transformations preserve clines, we will show that the equation $\text{Im}([z, a, b, c]) = 0$ defines a cline, where a, b, c are any three complex numbers and $[z, a, b, c]$ is the cross ratio. The fact that Möbius Transformations preserve cross ratios then gives us immediately that they preserve clines. We will prove this in a slightly different form, namely, that

$$\text{Im}\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) = 0$$

defines a cline. These two forms are equivalent because $[z, a, b, c]$ is a Möbius Transformation in z .

$$\begin{aligned} 2i\text{Im}\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) &= \frac{\alpha z + \beta}{\gamma z + \delta} - \frac{\overline{\alpha z + \beta}}{\overline{\gamma z + \delta}} \\ &= \frac{(\alpha z + \beta)(\overline{\gamma z + \delta}) - (\overline{\alpha z + \beta})(\gamma z + \delta)}{(\gamma z + \delta)(\overline{\gamma z + \delta})} \end{aligned}$$

Since we are setting this imaginary part to 0, we may work with the numerator only. So, the equation we are concerned with is

$$\begin{aligned} 0 &= (\alpha z + \beta)(\overline{\gamma z + \delta}) - (\overline{\alpha z + \beta})(\gamma z + \delta) \\ &= z\overline{z}(\alpha\overline{\gamma} - \overline{\alpha}\gamma) + z(\alpha\overline{\delta} - \gamma\overline{\beta}) + \overline{z}(\beta\overline{\gamma} - \overline{\beta}\delta) + (\beta\overline{\delta} - \overline{\beta}\delta) \\ &= 2i|z|^2\text{Im}(\alpha\overline{\gamma}) + 2i\text{Im}(z(\alpha\overline{\delta} - \gamma\overline{\beta})) + 2i\text{Im}(\beta\overline{\delta}) \end{aligned}$$

For convenience now, let $z = x + iy$, $A = \text{Im}(\alpha\overline{\gamma})$, $\alpha\overline{\delta} - \gamma\overline{\beta} = \omega = B_1 + B_2i$, $C = \text{Im}(\beta\overline{\delta})$. Then, our equation (after dividing by $2i$) becomes

$$\begin{aligned} 0 &= A(x^2 + y^2) + \text{Im}(\omega z) + C \\ &= A(x^2 + y^2) + B_2x - B_1y + C \end{aligned}$$

If $A = 0$, this is a line and if $A \neq 0$ this is a circle centered at $(-B_2/2A, B_1/2A)$.