1. Let \( z \) be a point on a cycle \( C \) (horocycle, hyperbolic circle or hypercycle). In every case, the best way to do this is to rotate the unit disk so that \( C \) has its (Euclidean) center on the positive real axis, then use an appropriate transformation (parabolic, elliptic or hyperbolic, respectively) leaving the new cycle invariant that moves the point until it is on the positive real axis as well. The tangent hyperbolic line, now, is the unique cline through \( z \) and \( z^* \) whose center is on the positive real axis (or is the imaginary axis if \( z = 0 \)).

Note, however, that this is assuming a special case of Exercise 7 – that any point on a cycle may be taken to any other point on the same cycle by an isometry leaving the cycle invariant. This is sufficiently important that I’ll deal with it specifically in Exercise 7.

2. Simply draw the circumcircle for the corresponding Euclidean triangle with the same vertices, then observe that a Euclidean circle is a hyperbolic cycle of some flavor.

6. There are many possible solutions here – I’ll give two different approaches: the first approach will be to solve for the coefficients of the fractional linear transformation directly, while the second will use normal forms.

Rotation about \( 1/2 \) (method 1): the other fixed point will be 2 (the point symmetric to \( 1/2 \) with respect to the unit circle), so we are looking for elliptic transformations with these two fixed points. The two fixed points of \( f(z) = (az+b)/(cz+d) \) satisfy \( cz^2 + (d-a)z - b = 0 \). Since \( c \neq 0 \) in this case, we may assume that \( c = 1 \) which implies that \( d - a = -5/2 \) (the negative sum of the roots) and \( -b = 1 \) (the product of the roots). Thus, a Möbius Transformation fixing 2 and \( 1/2 \) may be written as

\[
f(z) = \frac{az - 1}{z + (a - 5/2)}
\]

And we need now to choose \( a \) to make \( f \) elliptic (this will force \( f \) to leave the unit circle invariant, since the unit circle is a Steiner Circle of the Second Kind with respect to \( 1/2 \) and 2). This will happen when \( 0 \leq K < 1 \) where

\[
K = \frac{(2a - (5/2))^2}{4(a(a - 5/2) + 1)} = \frac{4a^2 - 10a + (25/4)}{4a^2 - 10a + 4} = 1 + \frac{9/4}{4a^2 - 10a + 4}
\]

This will happen in turn when \( 4a^2 - 10a + 4 \) is a real number less than or equal to \( -9/4 \). Equivalently, \( a \) is a root of \( 4a^2 - 10a + \alpha \) where \( \alpha \geq 25/4 \), so that

\[
a = \frac{5}{4} \pm \beta i
\]

where \( \beta = \sqrt{16\alpha - 100} \) and thus may be any real number. Hence, we have

\[
f(z) = \frac{((5/4) + \beta i)z - 1}{z + ((-5/4) + \beta i)}
\]

Parallel Displacement with ideal point \(-1\) (method 1): a single fixed point of \(-1\) means that \( cz^2 + (d-a)z - b = 0 \) has a double root of \(-1\). Again, \( c \neq 0 \) so we may assume that

\[
\]
$c = 1$, in which case $d - a = 2$ (negative sum of the roots) and $-b = 1$ (product of the roots). So, a Möbius Transformation with a single fixed point at -1 may be written as

$$f(z) = \frac{az - 1}{z + (2 + a)}$$

Now, we need to choose $a$ so that $f$ will fix the unit disk (that is, so that it will be a hyperbolic isometry). Recall that hyperbolic isometries may always be put into the form

$$\frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}$$

where $|\alpha|^2 - |\beta|^2 = 1$. Now, the only way to put $f$ into that form is to multiply all coefficients by a pure imaginary number, in which case we see that $ai = (2 + a)i$ or $a + \pi = -2$ which implies that $\Re a = -1$. Thus,

$$f(z) = \frac{(-1 + \beta i)z - 1}{z + (1 + \beta i)}$$

where $\beta$ is any nonzero real number ($\beta = 0$ makes $f$ constant).

Hyperbolic translation from $-i$ to $i$ (method 1): fixed points of $\pm i$ mean that $cz^2 + (d - a)z - b = 0$ has roots $\pm i$. Again, $c \neq 0$ so we may assume that $c = 1$, in which case $d - a = 0$ (negative sum of the roots) and $-b = 1$ (product of the roots). So, a Möbius Transformation with fixed points at $\pm i$ may be written as

$$f(z) = \frac{az - 1}{z + a}$$

Now, we need to choose $a$ so that $f$ is hyperbolic (this will force $f$ to leave the unit circle invariant, since the unit circle is a Steiner Circle of the First Kind with respect to $\pm i$). This will happen when $K > 1$ where

$$K = \frac{(2a)^2}{4(a^2 + 1)} = \frac{4a^2}{4a^2 + 4} = 1 - \frac{4}{4a^2 + 4}$$

This will, in turn happen whenever $a^2 + 1$ is negative real, i.e., whenever $a = \beta i$ where $|\beta| > 1$. Any choice of $a$ will give a hyperbolic element of the hyperbolic group with fixed points $\pm i$. However, the problem also specified that the isometry translate from $-i$ to $i$ – that is, that $-i$ be repulsive and $i$ be attractive. Calculating $f(0) = -1/(\beta i) = i/\beta$ indicates that we need to choose the positive $\beta$ to give the indicated character to $f$. Hence, $\beta > 1$ and

$$f(z) = \frac{\beta i z - 1}{z + \beta i}$$

Rotation about $1/2$ (method 2): First, translate $1/2$ to the origin, then rotate about the origin, then translate the origin back to $1/2$. In matrix terms, calculate

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} = \begin{pmatrix} e^{i\theta} - (1/4) & -(e^{i\theta} + 1)/2 \\ (e^{i\theta} - 1)/2 & 1 - (e^{i\theta}/4) \end{pmatrix}$$

It is not at all obvious that this produces the same result as method 1, but multiplying by a $\textit{very}$ carefully-chosen constant will show them to be identical.
Parallel Displacement with ideal point $-1$ (method 2): this one is harder because we don’t have any “standard” parabolic members of the hyperbolic isometry group. The easiest way to produce such a transformation is to use the trace-square classification method on an arbitrary member of the hyperbolic group and work backwards, like this:

$$g(z) = \frac{\alpha z + \beta}{\beta z + \overline{\alpha}}$$

where $|\alpha|^2 - |\beta|^2 = 1$. For this to be parabolic, we need $K = 1$, where

$$K = \frac{(\alpha + \overline{\alpha})^2}{4} = (2\text{Re} \alpha)^2/4 = (\text{Re} \alpha)^2$$

Hence, we need the real part of $\alpha$ to be $\pm 1$. So, one particular example of a parabolic member of the hyperbolic group would be (choosing $\beta = 1$ arbitrarily)

$$g(z) = (1 + i)z + 1$$

This particular element has fixed point which is a root of $z^2 - 2iz - 1 = 0$ or $i$. Furthermore, at this point we can see that we can produce more parabolic members of the hyperbolic group with fixed point $i$ by simply ensuring that $\beta^2 - 2\text{Im} \alpha z - \beta = 0$ has a double root at $i$ – in other words, by ensuring that $\beta/\beta = 1 = \text{Im} \alpha/\beta$. The first condition means that $\beta$ is real, while the second means that $\alpha = \pm 1 + \beta i$ (note that this automatically implies that $|\alpha|^2 - |\beta|^2 = 1$). So, the collection of all parabolic transformations in the hyperbolic group with fixed point $i$ is the set of all

$$g_{\beta}(z) = \frac{(1 + \beta i)z + \beta}{\beta z + (1 - \beta i)}$$

where $\beta \neq 0$ is real.

Now that we know what the parabolic members of the hyperbolic group that fix $i$ look like, we can rotate $-1$ to $i$, apply one of these, and rotate back:

$$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + \beta i & \beta \\ \beta & 1 - \beta i \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + \beta i & \beta i \\ -\beta i & 1 - \beta i \end{pmatrix}$$

Again, multiplying by a very carefully-chosen constant will show this to be identical to the method 1 result.

Hyperbolic translation from $-i$ to $i$ (method 2): we can produce all hyperbolic Möbius Transformations with fixed points $\pm i$ by taking a hyperbolic Möbius Transformation $T$ with fixed points 0 and $\infty$, and then calculating $S^{-1}TS$ where $S$ is a Möbius Transformation taking $i$ to $\infty$ and $-i$ to 0. The result is given by

$$\begin{pmatrix} -i & -i \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} = \begin{pmatrix} -i(\beta + 1) & \beta - 1 \\ 1 - \beta & -i(\beta + 1) \end{pmatrix}$$

where $\beta > 1$. As observed above, this is automatically in the hyperbolic group, since the unit circle is a Steiner Circle of the First Kind with respect to $\pm i$ and thus is preserved by any hyperbolic transformation fixing those two points. Thus,

$$f(z) = \frac{-i(\beta + 1)iz + \beta - 1}{(1 - \beta)z - i(\beta + 1)}$$