1. Let $T$ be a Möbius Transformation with distinct fixed points $p$ and $q$, and let $Sz = (z - p)/(z - q)$. Then, the notation I used in class is that $STS^{-1}$ has fixed points 0 and $\infty$ and so, has the form $STS^{-1}(z) = \lambda z$ for some complex number $\lambda$. The equivalent statement in this exercise is $STz = \lambda Sz$. The notation I used in class is a bit easier to see what’s actually going on, whereas this notation is easier for this particular proof. We want to show that, if $z$ is any complex number other than $p$ or $q$, that $\lambda = [Tz, z, p, q]$. Since cross-ratio is invariant under Möbius Transformations, we have

$$[Tz, z, p, q] = [STz, Sz, 0, \infty]$$

$$= \frac{(STz - 0)(Sz - \infty)}{(STz - \infty)(Sz - 0)}$$

$$= \frac{STz}{Sz}$$

$$= \frac{\lambda Sz}{Sz}$$

$$= \lambda$$

Note that the second line is really to be interpreted as a limit.

2. Let $T$ be a Möbius transformation with one fixed point, $p$, and let $Rz = 1/(z - p)$. The notation I used in class is that $RTR^{-1}z = z + \beta$, whereas the (equivalent) notation in this exercise is $RTz = Rz + \beta$. Substituting $z = \infty$ into this equation gives $RT(\infty) = \beta$, whereas substituting $z = z_0 = T^{-1}(\infty)$ gives $0 = Rz_0 + \beta$ or $\beta = -R(z_0)$.

3a. Fixed points are given by solutions to

$$z = \frac{z}{2z - 1}$$

$$2z^2 - z = z$$

$$2z^2 - 2z = 0$$

$$z = 0, 1$$

Using exercise 1, with $z = \infty$, we see that

$$\alpha = [1/2, \infty, 0, 1]$$

$$= \frac{(1/2 - 0)(\infty - 1)}{(1/2 - 1)(\infty - 0)}$$

$$= \frac{1/2 - 0}{1/2 - 1}$$

$$= -1$$

so that the transformation is elliptic with rotation angle $\pi$. Note that using $z = \infty$ is particularly easy, since two of the terms in the cross-ratio always cancel. The Steiner circles of the second kind (with respect to 0 and 1), therefore, are left invariant, while the Steiner circles of the first kind are permuted.
Alternatively, calculate
\[ K = \frac{0^2}{4(-1)} = 0 \]
\[ \alpha = (2K - 1) \pm 2\sqrt{K^2 - K} = -1 \]

3b. Fixed points are given by solutions to
\[ z = \frac{3z - 4}{z - 1} \]
\[ z^2 - z = 3z - 4 \]
\[ z^2 - 4z + 4 = 0 \]
\[ z = 2 \]

Using exercise 2, with \( p = 2 \), we see that
\[ \beta = \frac{1}{3 - 2} = 1 \]
so that the transformation is parabolic with Degenerate Steiner circles tangent and perpendicular to the real axis.

Alternatively, calculate
\[ K = \frac{2^2}{4(1)} = 1 \]
so that \( T \) is parabolic.

3c. Fixed points are given by solutions to
\[ z = \frac{z}{-z + 2} \]
\[ -z^2 + 2z = z \]
\[ z^2 - z = 0 \]
\[ z = 0, 1 \]

Using exercise 1, with \( z = \infty \), we see that
\[ \alpha = [-1, \infty, 0, 1] \]
\[ = \frac{(-1 - 0)(\infty - 1)}{(-1 - 1)(\infty - 0)} \]
\[ = \frac{-1 - 0}{-1 - 1} \]
\[ = 1/2 \]
so that the transformation is hyperbolic with scale factor 1/2. The Steiner circles of the first kind (with respect to 0 and 1), therefore, are left invariant, while the Steiner circles of the second kind are permuted. To see which fixed point is attractive and which is repulsive, take a non-fixed-point to begin and repeatedly evaluate \( T \) on that point. The sequence will always converge to the attractive fixed-point. For example, begin with \(-1\):
\[ T(-1) = -1/3 \]
\[ T(-1/3) = -1/7 \]
\[ T(-1/7) = -1/15 \]
\[ T(-1/15) = -1/33 \ldots \]
which is converging to 0. Hence, 0 is the attractive fixed point. Another way of seeing this is that the scale factor we calculated is less than 1. This means that the fixed-point we called $p$ (i.e., the first one in the cross-ratio) is attractive. A scale factor greater than one indicates that $q$ is attractive.

Alternatively, calculate

$$K = \frac{3^2}{4(2)} = \frac{9}{8}$$

$$\alpha = (2K - 1) \pm 2\sqrt{K^2 - K} = \frac{5}{4} \pm 2\sqrt{\frac{9}{64}} = 1/2, 2$$

3d. Fixed points are given by solutions to

$$z = \frac{-z}{(1 + i)z - i}$$

$$(1 + i)z^2 - iz = -z$$

$$(1 + i)z^2 - (i - 1)z = 0$$

$$z = 0, i$$

Using exercise 1, with $z = \infty$, we see that

$$\alpha = [(i - 1)/2, \infty, 0, i]$$

$$= ((i - 1)/2 - 0)(\infty - i)$$

$$= (i - 1)/2 - 0$$

$$= i - 1$$

$$= (i - 1)(i - 1)$$

$$= -2i$$

$$= \frac{-2i}{2} = i$$

so that the transformation is elliptic with rotation angle $\pi/2$. The Steiner circles of the second kind (with respect to 0 and $i$), therefore, are left invariant, while the Steiner circles of the first kind are permuted.

Alternatively, calculate

$$K = \frac{(-i - 1)^2}{4(i)} = 1/2$$

$$\alpha = (2K - 1) \pm 2\sqrt{K^2 - K} = \pm i$$

5. $Tz = e^{i\theta}z + b$ is elliptic when $\theta \neq 0$ and parabolic when $\theta = 0$. It is never hyperbolic or
loxodromic. One quick way to see this is to calculate

\[
K = \frac{(1 + e^{i\theta})^2}{4e^{i\theta}}
\]

\[
= \frac{1 + 2e^{i\theta} + e^{2i\theta}}{4e^{i\theta}}
\]

\[
= \frac{e^{-i\theta} + 2 + e^{i\theta}}{4}
\]

\[
= \frac{2\cos \theta + 2}{4}
\]

\[
= \frac{\cos \theta + 1}{2}
\]

This is clearly between 0 and 1 always and is equal to 1 precisely when \( \theta = 0 \).

6. Inversion is elliptic with rotation angle \( \pi \) – the only possibility for a transformation that is its own inverse.

7a. If \( Tz = (az + b)/(cz + d) \) is elliptic with \( T(\infty) = 0 \) and fixed points \( i \), and \(-i\), then we have the following requirements on \( a, b, c, d \) (only using the last three conditions):

\[
a = 0
\]

\[
ai + b = di - c
\]

\[
b - ai = -c - di
\]

Together, these imply that \( a = d = 0 \) and \( b = -c \). These automatically force \( T \) to be elliptic with rotation angle \( \pi \), since \( K = 0 \). So, \( Tz = b/ -bz = -1/z \)

Alternatively, use exercise 1 to calculate the normal form as \( \alpha = [0, \infty, i, -i] = -1 \) and then calculate \( T \) using the indicated fixed points and normal form as follows:

\[
\left( \begin{array}{cc} i & i \\ -1 & 1 \end{array} \right) \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -i \\ 1 & i \end{array} \right) = \left( \begin{array}{cc} 0 & -2 \\ 2 & 0 \end{array} \right)
\]

7b. If \( T \) is parabolic with fixed point \( 1 + i \) and \( T(\infty) = 100 \), then the normal form is \( z \mapsto z + (1/(99 - i)) \). Using the normal form and indicated fixed point, we then calculate

\[
\left( \begin{array}{cc} -1 - i & -1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{cc} 99 - i & 1 \\ 0 & 99 - i \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ 1 & -1 - i \end{array} \right) = \left( \begin{array}{cc} -100 & 2i \\ -1 & -98 + 2i \end{array} \right)
\]

So that

\[
Tz = \frac{100z - 2i}{z + 98 - 2i}
\]

Note the rather unusual form of the middle matrix, which was chosen to make the calculation easier.

7c. We can use exercise 1 to calculate that \( \alpha = [20, \infty, 5, 10] = 3/2 \). Then, calculate \( T \) using the indicated fixed points and normal form as follows:

\[
\left( \begin{array}{cc} -10 & 5 \\ -1 & 1 \end{array} \right) \left( \begin{array}{cc} 3 & 0 \\ 0 & 2 \end{array} \right) \left( \begin{array}{cc} 1 & -5 \\ 1 & -10 \end{array} \right) = \left( \begin{array}{cc} -20 & 50 \\ -1 & -5 \end{array} \right)
\]

So that

\[
Tz = \frac{20z - 50}{z + 5}
\]
8. The value of the limit is the attractive fixed point for all \( z \) except for the repulsive fixed point. Here is a proof: Let \( T \) be a hyperbolic or loxodromic transformation with attractive fixed point \( q \) and repulsive fixed point \( p \). Then, \( T = S^{-1}RS \) where \( Sz = (z - p)/(z - q) \), \( Rz = \alpha z \) and \( |\alpha| > 1 \). Note that \( T^n = (S^{-1}RS)^n = S^{-1}R^nS \). Thus, the limit

\[
\lim_{n \to \infty} T^n z = \lim_{n \to \infty} S^{-1} R^n S z = S^{-1} \left( \lim_{n \to \infty} R^n (Sz) \right)
\]

Now, \( R^n(Sz) = \alpha^n Sz \), so that \( \lim_{n \to \infty} R^n(Sz) = \infty \) unless \( Sz = 0 \). Applying \( S^{-1} \) to this, we see that \( \lim_{n \to \infty} T^n z = S^{-1}(\infty) = q \) unless \( Sz = 0 \) in which case \( z = p \). So, the limit is \( q \) for all \( z \) except for \( z = p \).

12. Since Möbius Transformations preserve clines and perpendicularity, we may place \( p \) and \( q \) anywhere we like by using a suitable Möbius Transformation and check the situation out there. Let us assume, then, that \( p = 0 \) and \( q = \infty \). We need to show that the only clines perpendicular to all lines through the origin are circles centered at the origin. Let \( C \) be such a cline. Clearly no line can be perpendicular to all lines through the origin, so we may assume that \( C \) is a circle. If a line and circle intersect perpendicularly, then the line is a diameter of the circle. Hence, all lines through the origin that intersect \( C \) are diameters of \( C \). Let \( z \) and \( w \) be two distinct, non-antipodal points on \( C \) and let \( \ell \) and \( m \) be the lines through the origin and \( z \) and \( w \), respectively. Then, \( \ell \) and \( m \) are both diameters of \( C \), hence the center of \( C \) is the intersection of \( \ell \) and \( m \), which is the origin. Hence, \( C \) is a circle centered at the origin, that is, a Steiner Circle of the Second Kind.

15a. If \( C_1 \) and \( C_2 \) intersect in two points \( x \) and \( y \), then send \( x \) to \( \infty \) and \( y \) to \( 0 \) via a Möbius Transformation, so that the images of \( C_1 \) and \( C_2 \) are lines \( \ell \) and \( m \) through the origin. No line can be perpendicular to both \( \ell \) and \( m \), so a cline perpendicular to both must be a circle. Furthermore, \( \ell \) and \( m \) must both be diameters of this circle, so the circle must be centered at the origin. In other words, the set of all clines perpendicular to both \( C_1 \) and \( C_2 \) is the set of Steiner Circles of the Second Kind with respect to \( x \) and \( y \).

15b. Suppose \( C_1 \) and \( C_2 \) are tangent at the point \( x \). Send \( x \) to \( \infty \) via a Möbius Transformation and then the images of \( C_1 \) and \( C_2 \) will be two parallel lines \( \ell \) and \( m \). No circles can intersect both \( \ell \) and \( m \) perpendicularly, since then \( \ell \) and \( m \) would be two parallel diameters of a circle. The only possibility, then, is that the image clines in question are lines perpendicular to both \( \ell \) and \( m \). That is, the set of all clines perpendicular to both \( C_1 \) and \( C_2 \) is the set of Degenerate Steiner Circles through \( x \) perpendicular to \( C_1 \).

15c. If \( C_1 \) and \( C_2 \) do not intersect, then Exercise 14 (not assigned) says that there exist \( z \) and \( z^* \) which are simultaneously symmetric with respect to \( C_1 \) and \( C_2 \). Any cline through \( z \) and \( z^* \) will thus intersect both \( C_1 \) and \( C_2 \) perpendicularly. In other words, \( C_1 \) and \( C_2 \) are Steiner Circles of the Second Kind with respect to \( z \) and \( z^* \), and thus the clines perpendicular to both are the Steiner Circles of the First Kind with respect to \( z \) and \( z^* \).

Alternatively, if \( C_1 \) and \( C_2 \) do not intersect, let \( x,y \) and \( z \) be any three distinct points on \( C_2 \). Use a Möbius Transformation to send \( x \) to \( \infty \), \( y \) to \( 0 \) and \( z \) to \( 1 \), so that \( C_2 \) is transformed to the real axis and \( C_1 \) is transformed to a circle not intersecting the real axis. Use a further Möbius Transformation preserving the real axis to move the center of the image of \( C_1 \) to the positive imaginary axis, say, centered at \( ki \) with radius \( r < k \). One cline that is perpendicular to both of these images is the imaginary axis. Another
is the circle centered at 0 with radius $R$, where $R$ is chosen to make the intersection with the image of $C_1$ perpendicular. Note that, as $R$ increases from $k - r$ to $k + r$, this intersection angle goes from 0 to $\pi$, hence it is equal to $\pi/2$ at some point in between. This point can be calculated if necessary, but it’s really only necessary to observe that it exists. The clines perpendicular to the images of $C_1$ and $C_2$, then, will be the Steiner Circles of the First Kind with respect to $Ri$ and $-Ri$, the two points of intersection of the two clines constructed to be perpendicular to the images of $C_1$ and $C_2$.

16. Put the various cases of 15 together. The two nonintersecting clines produce a family of Steiner circles of the First Kind that are their common perpendiculars. Move the picture via Möbius Transformations so as to make these Steiner circles lines through the origin. Then, the images of the original nonintersecting clines are concentric circles around the origin, while the third has an image that is some cline that intersects both of these. If it is a line, then there is clearly a unique perpendicular for all three. Hence, assume that it is a circle. A circle will always have a unique diameter that passes through the origin unless it is centered at the origin. But, this circle can’t be concentric with the first two since it intersects them. Hence, the unique diameter of the third circle that passes through the origin is a unique perpendicular for all three.