3a. Let \( S(z) = [z, 4, i, -1] \) then let \( T(z) = S^{-1}(z) \). Specifically,

\[
S(z) = \frac{(z - i)(4 + 1)}{(z + 1)(4 - i)} = \begin{bmatrix} 5 & -5i \\ 4 - i & 4 - i \end{bmatrix}
\]

\[
T(z) = \begin{bmatrix} 4 - i & 5i \\ -4 + i & 5 \end{bmatrix} = \frac{(4 - i)z + 5i}{(-4 + i)z + 5}
\]

3b. Let \( S_1(z) = [z, 0, i, -i] \) and \( S_2(z) = [z, 0, 1, 2] \). Then, let \( T(z) = S_2^{-1}(S_1(z)) \). Specifically,

\[
S_1(z) = \frac{(z - i)(0 + i)}{(z + i)(0 - i)} = \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}
\]

\[
S_2(z) = \frac{(z - 1)(0 - 2)}{(z - 2)(0 - 1)} = \begin{bmatrix} -2 & 2 \\ -1 & 2 \end{bmatrix}
\]

\[
T(z) = \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} = \begin{bmatrix} 4i & 0 \\ 3i & -1 \end{bmatrix} = \frac{4iz}{3iz - 1}
\]

3c. Repeating the scheme used in 3b., let \( S_1(z) = [z, 1, 2, 3] \) and \( S_2(z) = [z, 2, 3, -1] \). Then, let \( T(z) = S_2^{-1}(S_1(z)) \).

\[
S_1(z) = \frac{(z - 2)(1 - 3)}{(z - 3)(1 - 2)} = \begin{bmatrix} -2 & 4 \\ -1 & 3 \end{bmatrix}
\]

\[
S_2(z) = \frac{(z - 3)(2 + 1)}{(z + 1)(2 - 3)} = \begin{bmatrix} 3 & -9 \\ -1 & -1 \end{bmatrix}
\]

\[
T(z) = \begin{bmatrix} -1 & 9 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -7 & 23 \\ -5 & 13 \end{bmatrix} = \frac{-7z + 23}{-5z + 13}
\]

6. Here are the relevant calculations:

\[
z = \frac{2z}{3z - 1}
\]

\[
3z^2 - z = 2z
\]

\[
3z^2 - 3z = 0
\]

\[
z = 0, 1
\]

\[
z = \frac{3z - 2}{2z - 1}
\]

\[
2z^2 - z = 3z - 2
\]

\[
2z^2 - 4z + 2 = 0
\]

\[
z = 1
\]

\[
z = \frac{-2}{z + 1}
\]

\[
z^2 + z = -2
\]

\[
z^2 + z + 2 = 0
\]

\[
z = \frac{-1 \pm \sqrt{7}i}{2}
\]
\[
z = \frac{-iz}{(1 - i)z - 1}
\]

\[
(1 - i)z^2 - z = -iz
\]

\[
(1 - i)z^2 + (i - 1)z = 0
\]

\[
z = 0, 1
\]

8. Let \( T(z) = (az + b)/(cz + d) \), then the equation for the fixed points of \( T \) is \( cz^2 + (d - a)z - b = 0 \) This has roots

\[
a - d \pm \sqrt{(d - a)^2 + 4bc}
\]

which will be equal to \( \pm 1 \) when \( a - d = 0 \) and \( \sqrt{\frac{4bc}{2c}} \). This happens precisely when \( a = d \) and \( b = c \), hence

\[
T(z) = \frac{az + b}{bz + a}
\]

where \( a^2 - b^2 \neq 0 \) – in other words, \( a \neq \pm b \). The second part (single fixed point at 1) occurs when \( (a - d)/2c = 1 \) and \( (d - a)^2 + 4bc = 0 \). So, \( a - d = 2c \) and hence \( 4c^2 + 4bc = 0 \) and thus \( b = -c \). Hence, this occurs when

\[
T(z) = \frac{az - c}{cz + (a - 2c)}
\]

where \( a(a - 2c) + c^2 = (a - c)^2 \neq 0 \) – in other words, \( a \neq c \).

11. Let \( T(z) = (az + b)/(cz + d) \). Then, \( T \) will have a fixed-point at \( \infty \) if and only \( T(\infty) = \infty \). This occurs precisely when \( c = 0 \). Next, we need to rule out any other fixed points. Using the equation above, (with \( c = 0 \)) we see that the equation for the fixed-points of \( T \) is \( (d - a)z + b = 0 \). This will have a fixed point at \( -b/(d - a) \) unless \( d - a = 0 \). So, we need to assume that \( a = d \), reducing our equation to \( b = 0 \) which will have no solutions if \( b \) is not zero. Hence, all together we have \( c = 0, a = d \neq 0, b \neq 0 \). Hence,

\[
T(z) = \frac{az + b}{a} = z + \frac{b}{a}
\]

which is a translation.

14. There are a number of possibilities for such a theorem – here is one: Let \( x_0 \) and \( y_0 \) be any two distinct points in the plane and let \( x_1 \) and \( y_1 \) be any two other distinct points in the plane such that \( |x_0 - y_0| = |x_1 - y_1| \). Then, there is a Euclidean isometry \( T \) such that \( T(x_0) = x_1 \) and \( T(y_0) = y_1 \).

**Proof:** Let

\[
T(z) = x_1 + (z - x_0)\left(\frac{y_1 - x_1}{y_0 - x_0}\right)
\]

It is easy to check that \( T(x_0) = x_1 \) and \( T(y_0) = y_1 \). Furthermore, \( T \) is clearly in the form \( az + \beta \) so it is a Euclidean similarity. We only need to verify that the length of the \( z \) coefficient is 1, so that \( T \) is an isometry and not merely a similarity. The modulus of the coefficient of \( z \), though, is

\[
\left|\frac{y_1 - x_1}{y_0 - x_0}\right| = \frac{|y_1 - x_1|}{|y_0 - x_0|} = 1
\]

by hypothesis.
16. Three facts are needed here: three distinct points determine a unique cline, clines are invariant in Möbius Geometry, and any three distinct points in the Riemann sphere may be taken to any other three distinct points in the Riemann sphere by some Möbius Transformation. Hence, given two clines, $C_1$ and $C_2$ choose three distinct points $p_1, q_1, r_1$ on $C_1$ and three distinct points $p_2, q_2, r_2$ on $C_2$. Now, construct a Möbius Transformation $T$ such that $T(p_1) = p_2$, $T(q_1) = q_2$, and $T(r_1) = r_2$. Then, $T(C_1) = C_2$ since the image of $T(C_1)$ is a cline containing the three points $p_2, q_2, r_2$ and there is a unique such cline.

17. (a) and (f) are congruent, (b) and (e) are congruent, and (c) and (d) are congruent.

18. Note that symmetry with respect to the unit circle is particularly easy to describe, since $z$ and $z^*$ are positive real multiples of one another, and $|z||z^*| = 1$. Using the first condition, let $z^* = \lambda z$ where $\lambda$ is real. Then, the second condition says that $\lambda |z|^2 = 1$ so that $\lambda = 1/|z|^2 = 1/z\overline{z}$. Hence, $z^* = z/z\overline{z} = 1/z$. Thus,

\[
\begin{align*}
1^* &= 1 \\
(1/2)^* &= 1/(1/2) = 2 \\
(i)^* &= 1/i = i \\
(i/2)^* &= 1/i^2 = 2i \\
(1 + i)^* &= 1/(1 + i) = (1 + i)/2 \\
((1 + i)/2)^* &= 1/(1 + i)/2 = 1 + i
\end{align*}
\]

20. First, use a Euclidean similarity to place the circle $C$ on the unit circle, then rotate about the origin to place $z$ on the real axis. If $z$ is inside the unit circle, then $-1 < z < 1$ so that $z = \cos \theta$ for some $0 < \theta < \pi$. Then, the points $B$ and $D$ are given by $e^{i\theta}$ and $e^{-i\theta}$, respectively. Hence, the lines tangent to the circle through $B$ and $D$ have equations $\text{Im}(ie^{-i\theta}(z - e^{i\theta})) = 0$ and $\text{Im}(ie^{i\theta}(z - e^{-i\theta})) = 0$, respectively. Placing these equations in the form necessary for applying the intersection formulae we derived earlier, we get

\[
\begin{align*}
0 &= \text{Im}(ie^{-i\theta}z - i) \\
0 &= \text{Im}(ie^{i\theta}z - i)
\end{align*}
\]

These two lines intersect in the point

\[
\frac{i e^{-i\theta} \text{Im}(-i) - i e^{i\theta} \text{Im}(-i)}{\text{Im}(i e^{-i\theta} e^{i\theta})} = \frac{i e^{i\theta} - i e^{-i\theta}}{\text{Im}(e^{-2i\theta})} = \frac{-2 \sin \theta}{-2 \sin 2\theta} = \frac{-2 \sin \theta}{2 \sin \theta \cos \theta} = \frac{1}{\cos \theta} = \frac{1}{z}
\]

Note that this could also have been derived by similar triangles, using $OzB$ and $OBz^*$. Now, using three convenient points on the unit circle $(1, -1, i)$ we verify that (since $z$ is...
real)

\[ [1/z, 1, -1, i] = \frac{(1/z + 1)(1 - i)}{(1/z - i)(1 + 1)} = \frac{(1 + z)(1 - i)}{2(1 - iz)} \]

\[ = \frac{(1 + z)(1 + i)}{2(1 + iz)} \]

\[ = -i(1 + z)(1 + i) \]

\[ = i(z + 1)(1 - i) \]

\[ = \frac{2(z - i)}{2(z - iz)} \]

\[ = \frac{2(z + iz)}{2(z - iz)} \]

\[ = \frac{z + 1}{z - 1} \]

22. Let the original line \( C \) have the equation \( \text{Im}((z - \beta)e^{-i\theta}) = 0 \). Then, the Euclidean isometry \( T(z) = (z - \beta)e^{-i\theta} \) takes \( C \) to the real axis. Then, use the three points \( 1, 0, \infty \) on the real axis to calculate the cross-ratios in the definitions of symmetry. That is, \( [z^*, 1, 0, \infty] = [z, 1, 0, \infty] \). Next, note that \( [z, 1, 0, \infty] = z \) so in this case \( z^* = \overline{z} \). Invariance of symmetry now implies that, with respect to the original line \( C \), \( z^* = T^{-1}Tz \) or

\[ z^* = \beta + e^{i\theta}((\overline{z} - \overline{\beta})e^{i\theta}) \]

\[ = \beta + (z - \beta)e^{2i\theta} \]

Now, we need to calculate the mirror-image of \( z \) across the original line \( C \) to verify that it is equal to \( z^* \) - recall that the distance from \( z \) to \( C \) is given by \( \text{Im}((z - \beta)e^{-i\theta}) \) and that the mirror-image of \( z \) is then given by

\[ z + 2\text{Im}((z - \beta)e^{-i\theta})(-ie^{i\theta}) \]

since \(-ie^{i\theta}\) is a unit-length complex number perpendicular to the direction of \( C \) (note that we use this one instead of \( ie^{i\theta} \) since \( z + \text{Im}((z - \beta)e^{-i\theta})(-ie^{i\theta}) \) is on the line \( C \) - the other direction moves us farther away from the line rather than across it). So, we calculate

\[ z - 2i\text{Im}((z - \beta)e^{i\theta})e^{i\theta} = z - [(z - \beta)e^{-i\theta} - (\overline{z} - \overline{\beta})e^{i\theta}]e^{i\theta} \]

\[ = \beta + (\overline{z} - \overline{\beta})e^{2i\theta} \]

24. Let \( C \) be a line and \( z \) and \( z^* \) symmetric with respect to \( C \). We want to prove that a circle \( C' \) passing through \( z \) will also pass through \( z^* \) if and only if \( C' \) intersects \( C \) orthogonally. First, use a Möbius Transformation to move \( C \) to the real axis. Then, use horizontal translation and homothety to move \( z \) to \( i \) and \( z^* \) to \( -i \) (recall that exercise 22 above shows that symmetry with respect to the real axis is conjugation). The invariance of symmetry and angle measure mean that if the result is true in this configuration, it is true in general. \( C' \) is assumed to go through \( z = i \). It will also go through \( z^* = -i \) if and only if its center is on the real axis, which will be true if and only if it intersects the real axis with vertical tangents (i.e. orthogonally).