1.7-1 We did this in class.

1.7-3 The hint I gave for this one is really a full proof. Here it is again: use the notation as in Figure 1.7a, and let $J$ denote the center of the Feuerbach circle. First show that $A'$ is equidistant from $E$ and $F$ (show that both $E$ and $F$ are on the circle centered at $A'$ that goes through $B$ and $C$). Note that this makes $A'EF$, $A'BF$ and $A'CE$ isosceles triangles.

Use these three triangles to show that $\angle EFA'$ is congruent to $\angle BAC$. Altogether, this shows that the angle $\angle EJA'$ (and hence $\angle FJA'$) is equal to twice angle $BAC$. Use these (and the other analogous results) to show that

\[
\begin{align*}
\angle DJA' &= 2(\pi - 2C - A) \\
\angle A'JB' &= 2(\pi - A - B) \\
\angle B'JE &= 2(2A + B - \pi) \\
\angle EJF &= 2(\pi - 2A) \\
\angle FJC' &= 2(2A + C - \pi)
\end{align*}
\]

And then calculate $\angle XJY = 2\pi/3$, so that $XYZ$ is equilateral.

1.7-4 Note that the internal and external angle bisectors are perpendicular, and that, while the incenter $I$ is the intersection of the three internal bisectors, each excenter is the intersection of two external and one internal bisector. So, the sides of the triangle $I_aI_bI_c$ are the three external bisectors. The three internal bisectors are then the three altitudes of this triangle and hence the incenter is the orthocenter of $I_aI_bI_c$.

1.7-5 The previous exercise shows that the orthic triangle of $I_aI_bI_c$ is the original triangle $ABC$. So, the 9-point center of $I_aI_bI_c$ is the circumcenter of $ABC$ (namely, $O$). The previous exercise also shows that $I$ is the orthocenter of $I_aI_bI_c$ so that the Euler line of this triangle is $IO$.

1.8-2 The smallest sum is a “degenerate” triangle that has only two vertices: the vertex with the obtuse angle and the foot of the altitude on the opposite side.

1.8-4 If the weights are equal, the strings will make equal angles with each other. This means that the knot will come to rest at the point which is the solution to the Fermat problem.

1.8-5 Imagine a fifth point in the center of the square. Now solve two Fermat problems with the triangles made up of the center and the two top/bottom corners. Here is a calculus-based approach to solving this problem: let’s set up the vertices at $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. Our fifth point, then, is located at $(1/2, 1/2)$ and so we’re trying to find a number $0 < y < 1/2$ that minimizes twice the sum of the distances from $(1/2, y)$ to the three points $(0, 0)$, $(1, 0)$ and $(1/2, 1/2)$. In other words, we are trying to minimize

\[f(y) = 1 - 2y + 4\sqrt{\frac{1}{4} + y^2}\]

Differentiating, we see that

\[f'(y) = -2 + \frac{4y}{\sqrt{\frac{1}{4} + y^2}}\]
and this will be zero when

\[
2y = \sqrt{\frac{1}{4} + y^2}
\]
\[
4y^2 = \frac{1}{4} + y^2
\]
\[
3y^2 = \frac{1}{4}
\]
\[
y = \sqrt{\frac{1}{12}}
\]
\[
y = \frac{1}{2\sqrt{3}}
\]

and, at this point

\[
f(y) = 1 - \frac{2}{2\sqrt{3}} + 4\sqrt{\frac{1}{4} + \frac{1}{12}}
\]
\[
= 1 - \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}}
\]
\[
= 1 + \frac{3}{\sqrt{3}}
\]
\[
= 1 + \sqrt{3}
\]

Technically, we should verify that the endpoints of the interval produce larger function values, but it is readily computed that \( f(0) = 3 \) and \( f(1/2) = 2\sqrt{2} \), both of which are larger than \( 1 + \sqrt{3} \).

1.8-6 For a “very obtuse” triangle, the best point is the vertex where the obtuse angle is. The solution for a convex quadrilateral is the intersection of the diagonals.

1.9-2 Use the equations on page 25 to see that \( \alpha = \beta = \gamma = 40^\circ \) for an equilateral triangle and \( \alpha = 30^\circ, \beta = \gamma = 45^\circ \) for an isosceles right triangle.