1. No, it’s not a Toy Geometry because the inverse of $f$ is not in the set $G$.

2. Method 1 (coordinates): Use a Euclidean similarity to place the 3 vertices of the triangle at $(0,0)$, $(1,0)$ and $(p,q)$. Then, the three side midpoints are $(1/2,0)$, $((p+1)/2,q/2)$ and $(p/2,q/2)$. The equations of the three perpendicular bisectors are

$$
x = 1/2
$$

$$\quad y - \frac{q}{2} = \frac{1-p}{q}(x - \frac{p+1}{2})
$$

$$\quad y - \frac{q}{2} = -\frac{p}{q}(x - \frac{p}{2})
$$

Substituting $x = 1/2$ into the second equation, we find the coordinates of the intersection point of the first two lines to be

$$(x_0, y_0) = \left(\frac{1}{2}, \frac{p(p-1)+q^2}{2q}\right)
$$

Now, substitute these coordinates into the third equation and verify that

$$y_0 - \frac{q}{2} = \frac{p(p-1)+q^2}{2q} - \frac{q}{2}
$$

$$= \frac{p(p-1)+q^2-q^2}{2q}
$$

$$= \frac{p(p-1)}{2q}
$$

$$= (-\frac{p}{q})(\frac{1-p}{2})
$$

$$= (-\frac{p}{q})(\frac{1}{2} - \frac{p}{2})
$$

$$= -\frac{p}{q}(x_0 - \frac{p}{2})
$$

Hence, all three lines are concurrent.

Method 2 (complex numbers): Use a Euclidean similarity to place the 3 vertices at 0, 1 and $z_0$. Then, the three side midpoints are $1/2$, $z_0/2$, and $(z_0+1)/2$. The equations of the three perpendicular bisectors, then, are

$$0 = \text{Im} \ (i(z - 1/2))
$$

$$0 = \text{Im} \ \left( i\left(\frac{z - ((z_0+1)/2)}{z_0 - 1}\right) \right)
$$

$$0 = \text{Im} \ \left( i\left(\frac{z - (z_0/2)}{z_0}\right) \right)
$$
or, placing them in the form required for the formulae we derived earlier:

\[ 0 = \text{Im} \left( \frac{iz - \frac{-i}{2}}{z_0 - \frac{i}{2}} \right) \]

\[ 0 = \text{Im} \left( \frac{i}{z_0 - \frac{i}{2}} + \frac{-i(z_0 + 1)}{2z_0 - 2} \right) \]

\[ 0 = \text{Im} \left( \frac{i}{z_0} + \frac{-i}{2} \right) \]

Using the first and third equations, we see that they intersect at

\[ w_0 = \frac{-i(-1/2) - \frac{i}{z_0}(-1/2)}{\text{Im} \left( \frac{1}{z_0} \right)} \]

\[ = \frac{i/2 - (i/(2\pi))}{\text{Im} \left( \frac{1}{z_0} \right)} \]

\[ = \frac{2i(i/2 - 1/2\pi)}{2i\text{Im} \left( \frac{1}{z_0} \right)} \]

\[ = \frac{-1 + (1/\pi)}{(1/\pi) - (1/z_0)} \]

\[ = \frac{z_0\pi(1/z_0 - (1/z_0))}{z_0\pi(1/z_0) - (1/z_0)} \]

\[ = \frac{-|z_0|^2 + z_0}{z_0 - \overline{z_0}} \]

Now, substituting this into the second equation (first, multiplying by 2i for convenience, since we’re only interested in whether or not the result is zero), we get

\[ 2i\text{Im} \left( \frac{i}{z_0 - \frac{i}{2}}w_0 + \frac{-i(z_0 + 1)}{2z_0 - 2} \right) = \left( \frac{i}{z_0 - \frac{i}{2}}w_0 + \frac{-i(z_0 + 1)}{2z_0 - 2} \right) \]

\[ = \left( \frac{i}{z_0 - \frac{i}{2}}w_0 - \frac{i(z_0 + 1)}{2z_0 - 2} \right) \]

\[ = \left( \frac{i}{z_0 - \frac{i}{2}} \frac{z_0 - |z_0|^2}{z_0 - \overline{z_0}} + \frac{i(z_0 + 1)}{2z_0 - 2} \right) \]

\[ = \left( \frac{i}{z_0 - \frac{i}{2}} \frac{z_0 - |z_0|^2}{z_0 - \overline{z_0}} + \frac{i(z_0 + 1)}{2z_0 - 2} \right) \]

\[ = \left( \frac{-i}{z_0 - \frac{i}{2}} \frac{z_0 - |z_0|^2}{z_0 - \overline{z_0}} + i(z_0 + 1) \right) \]

\[ = \left( \frac{2i(z_0 - |z_0|^2)(\overline{z_0} - 1)}{2(z_0 - 1)(\overline{z_0} - 1)(z_0 - \overline{z_0})} + \frac{-i(z_0 + 1)(\overline{z_0} - 1)(z_0 - \overline{z_0})}{2(z_0 - 1)(\overline{z_0} - 1)(z_0 - \overline{z_0})} \right) \]

2
Now, just considering the numerator (after dividing out the factor of \(i\)), we get

\[
\begin{aligned}
2(z_0 - |z_0|^2)(\overline{z}_0 - 1) &+ \left( -(z_0 + 1)(\overline{z}_0 - 1)(z_0 - \overline{z}_0) \right) \\
-2(\overline{z}_0 - |z_0|^2)(z_0 - 1) &+ \left( -(z_0 + 1)(z_0 - 1)(z_0 - \overline{z}_0) \right)
\end{aligned}
\]

\[
= 2|z_0|^2 - 2z_0 - 2|z_0|^2\overline{z}_0 + 2|z_0|^2 - 2|z_0|^2 + 2\overline{z}_0 + 2|z_0|^2z_0 - 2|z_0|^2 \\
- (z_0 - \overline{z}_0)(|z_0|^2 - z_0 + \overline{z}_0 - 1 + |z_0|^2 - \overline{z}_0 + z_0 - 1) \\
= 2(\overline{z}_0 - z_0)(1 - |z_0|^2) - 2(z_0 - \overline{z}_0)(|z_0|^2 - 1)
\]

\[
= 0
\]

3. Method 1 (coordinates): Same as 2, except that the three altitude equations are

\[
\begin{aligned}
x &= p \\
y &= \frac{1 - p}{q} x \\
y &= \frac{-p}{q} (x - 1)
\end{aligned}
\]

Substituting \(x = p\) into the second equation, we find the coordinates of the intersection point of the first two lines to be

\[
(x_0, y_0) = (p, \frac{-p(p - 1)}{q})
\]

Now, substitute these coordinates into the third equation and verify that

\[
\begin{aligned}
y_0 &= \frac{-p(p - 1)}{q} \\
&= \left( \frac{-p}{q} \right)(p - 1) \\
&= \frac{-p}{q} (x_0 - 1)
\end{aligned}
\]

Hence, all three lines are concurrent.

Method 2 (complex numbers): same as 2 except that the three equations are

\[
\begin{aligned}
0 &= \text{Im} \left( iz - z_0 \right) \\
0 &= \text{Im} \left( \frac{iz}{z_0 - 1} \right) \\
0 &= \text{Im} \left( \frac{iz - i}{z_0} \right)
\end{aligned}
\]

Placing these in the desired form yields

\[
\begin{aligned}
0 &= \text{Im} \left( iz - iz_0 \right) \\
0 &= \text{Im} \left( \frac{i}{z_0 - 1} z + 0 \right) \\
0 &= \text{Im} \left( \frac{i}{z_0} z - \frac{i}{z_0} \right)
\end{aligned}
\]

3
so that the intersection of the first two lines is

\[
\begin{align*}
\frac{i}{i/(z_0 - 1)} \frac{\text{Im} (iz_0)}{\text{Im}(i(z_0 - 1))} &= \frac{-i/(z_0 - 1))(iz_0 + iz_0)}{(1/(z_0 - 1)) - (1/(z_0))} \\
&= \frac{(z_o - 1)(z_0 + \bar{z}_0)}{(z_0 - 1) - (\bar{z}_0 - 1)} \\
&= \frac{(z_0 - 1)(z_0 + \bar{z}_0)}{z_0 - \bar{z}_0}
\end{align*}
\]

Now, substitute \( w_0 \) into the third equation to get

\[
\text{Im} \left( \frac{i}{z_0 (w_0 - 1)} \right) = \frac{i}{z_0} (z_0 - 1)(z_0 + \bar{z}_0) - z_0 + \bar{z}_0
\]

\[
= \text{Im} \left( \frac{i((z_0 - 1)(z_0 + \bar{z}_0) - z_0 + \bar{z}_0)}{z_0(z_0 - \bar{z}_0)} \right)
\]

\[
= \text{Im} \left( \frac{i(z_0(z_0 + \bar{z}_0) - 2z_0)}{z_0(z_0 - \bar{z}_0)} \right)
\]

\[
= \text{Im} \left( \frac{iz_0(z_0 + \bar{z}_0) - 2}{z_0(z_0 - \bar{z}_0)} \right)
\]

Now, note that the numerator is pure imaginary (since \( z_0 + \bar{z}_0 = 2\text{Re}(z_0) \)) and that the denominator is also pure imaginary (since \( z_0 - \bar{z}_0 = 2i\text{Im}(z_0) \)). Hence, the quotient is real and the imaginary part is 0.

4. First, note that there are three ways to place any given triangle into the standard position that we’ve been using. This implies that if we show that one altitude foot and one orthocenter segment midpoint is on the circumcircle of the midpoint triangle, then we have shown that they all do.

Method 1 (coordinates): put the triangle at \((0,0), (1,0)\) and \((p,q)\), then calculate the equations of the perpendicular bisectors of the midpoint triangle:

\[
x = (2p + 1)/4
\]

\[
y - \frac{q}{4} = \left( \frac{1 - p}{q} \right) \left( x - \frac{p + 1}{4} \right)
\]

\[
y - \frac{q}{4} = \left( \frac{-p}{q} \right) \left( x - \frac{p + 2}{4} \right)
\]

Substituting \( x = (2p + 1)/4 \) into either of the other equations yields a circumcenter of

\[
\left( \frac{2p + 1}{4}, \frac{q^2 - p(p - 1)}{4q} \right)
\]
for the midpoint triangle. Hence, this is the center we’ll be using. We calculate the square of the radius as

\[ r^2 = \left( \frac{2p + 1}{4} - \frac{1}{2} \right)^2 + \left( \frac{q^2 - p(p-1)}{4q} \right)^2 \]

\[ = \left( \frac{2p - 1}{4} \right)^2 + \left( \frac{q^2 - p(p-1)}{4q} \right)^2 \]

\[ = \frac{(2p - 1)^2q^2 + (q^2 - p(p-1))^2}{16q^2} \]

\[ = \frac{4p^2q^2 - 4pq^2 + q^2 + q^4 + p^4 - 2p^3 + p^2 - 2q^2p^2 + 2q^2p}{16q^2} \]

\[ = \frac{q^4 + p^4 + 2p^2q^2 - 2pq^2 - 2p^3 + p^2 + q^2}{16q^2} \]

\[ = \frac{(q^2 + p^2)^2 + (p^2 + q^2)(1 - 2p)}{16q^2} \]

\[ = \frac{(p^2 + q^2)((p - 1)^2 + q^2)}{16q^2} \]

The other two points we need to check are the altitude foot \((p, 0)\) and the orthocenter segment midpoint \(\left( \frac{p}{2}, -\frac{p(p - 1)}{2q} \right)\) (recall that we found the coordinates of the orthocenter in 3). So, we verify that

\[ \left( \frac{2p + 1}{4} - p \right)^2 + \left( \frac{q^2 - p(p-1)}{4q} \right)^2 = \left( \frac{1 - 2p}{4} \right)^2 + \left( \frac{q^2 - p(p-1)}{4q} \right)^2 \]

\[ = r^2 \]

And then check that

\[ \left( \frac{2p + 1}{4} - \frac{p}{2} \right)^2 + \left( \frac{q^2 - p(p-1)}{4q} - \frac{-p(p-1)}{2q} \right)^2 = \left( \frac{1}{4} \right)^2 + \left( \frac{(q^2 - p(p-1)) + 2p(p-1)}{4q} \right)^2 \]

\[ = \frac{q^2 + (q^2 + p(p-1))^2}{16q^2} \]

\[ = \frac{q^2 + q^4 + p^3 - 2p^3 + p^2 + 2q^2p^2 - 2q^2p}{16q^2} \]

\[ = r^2 \]

Method 2 (complex numbers): First, we need to calculate the circumcenter of the midpoint triangle. We will do this by noting that the triangle in standard position may be moved to the midpoint triangle by the following sequence of transformations: first rotate about the origin by \(180^\circ\); then shrink about the origin by a factor of 2; then translate by \((z_0 + 1)/2\). The cumulative effect of all of this is the transformation

\[ f(z) = (z_0 + 1 - z)/2 \]

It is easy to verify that \(f(0) = (z_0 + 1)/2\), \(f(1) = z_0/2\) and \(f(z_0) = 1/2\), and these are, of course, the three midpoints that we observed earlier. The point of this is that the circumcenter of the original triangle (which we calculated earlier) is transformed to the
circumcenter of the midpoint triangle by this transformation, so that the center of the circle that we need to consider is

\[ f\left(-\frac{|z_0|^2 + z_0}{z_0 - \overline{z_0}}\right) = \left(z_0 + 1 - \frac{-|z_0|^2 + z_0}{z_0 - \overline{z_0}}\right)/2 \]

\[ = \frac{(z_0 + 1)(z_0 - \overline{z_0}) + |z_0|^2 - z_0}{2(z_0 - \overline{z_0})} \]

\[ = \frac{z_0^2 - \overline{z_0}}{2(z_0 - \overline{z_0})} \]

Then, the square of the radius of the circle is

\[ r^2 = \left|\frac{z_0^2 - \overline{z_0}}{2(z_0 - \overline{z_0})} - \frac{1}{2}\right|^2 \]

\[ = \left|\frac{z_0^2 - \overline{z_0} - z_0 + \overline{z_0}}{2(z_0 - \overline{z_0})}\right|^2 \]

\[ = \left|\frac{z_0^2 - z_0}{2(z_0 - \overline{z_0})}\right|^2 \]

\[ = \frac{|z_0|^2|z_0 - 1|^2}{16\text{Im}(z_0)^2} \]

We now need to calculate the distance from the center to the two points \(\text{Re}(z_0)\) (the altitude foot) and \(\frac{(z_0 - 1)(z_0 + \overline{z_0})}{2(z_0 - \overline{z_0})}\). We begin with the altitude foot:

\[ |\text{Re}(z_0) - \frac{z_0^2 - \overline{z_0}}{2(z_0 - \overline{z_0})}|^2 = \left|\frac{(z_0 + \overline{z_0})(z_0 - \overline{z_0}) - (z_0^2 - \overline{z_0})}{2(z_0 - \overline{z_0})}\right|^2 \]

\[ = \frac{|z_0^2 - \overline{z_0} - z_0^2 + \overline{z_0}|^2}{16\text{Im}(z_0)^2} \]

\[ = \frac{|z_0(z_0 - 1)|^2}{16\text{Im}(z_0)^2} \]

\[ = \frac{|z_0|^2|z_0 - 1|^2}{16\text{Im}(z_0)^2} \]

\[ = r^2 \]

Next, the orthocenter segment midpoint:

\[ \frac{(z_0 - 1)(z_0 + \overline{z_0}) - (z_0^2 - \overline{z_0})}{2(z_0 - \overline{z_0})} \]

\[ = \frac{|z_0|^2 - z_0 + |z_0|^2 - \overline{z_0} - z_0^2 + \overline{z_0}|^2}{16\text{Im}(z_0)^2} \]

\[ = \frac{|z_0|^2 - z_0|^2}{16\text{Im}(z_0)^2} \]

\[ = \frac{|z_0(z_0 - 1)|^2}{16\text{Im}(z_0)^2} \]

\[ = |z_0|^2|z_0 - 1|^2/16\text{Im}(z_0)^2 \]

\[ = r^2 \]

5. Method 1 (Erlanger-type proof): Note that there is a proof in the book for the case where the line is parallel to one of the coordinate axes. We will use this by observing that \(f(z) = e^{i\theta} f(e^{i\theta} z)\). In other words, if we rotate about the origin, invert, then rotate again, the effect is identical to inversion alone. Hence, to obtain the image of a given line under \(f\), we may rotate the given line until it is parallel to one of the coordinate
axes, invert it (obtaining a circle through the origin, by the book’s proof), then rotate this circle about the origin. The last rotation will still produce a circle through the origin, so we’re done.

Method 2 (complex numbers): given a line that does not pass through the origin, there will be a point \( \beta \neq 0 \) that is as close to the origin as possible. At that point, the line will be perpendicular to the line from the origin to \( \beta \). In other words, the line may be given the equation

\[
\text{Im} \left( i(z - \beta)/\beta \right) = \text{Im} \left( \frac{iz - i}{\beta} \right) = 0
\]

The equation, then, for the image under \( f \) of this line is

\[
\text{Im} \left( \frac{if^{-1}(z)}{\beta} - i \right) = 0
\]

Realizing that \( f^{-1} = f \), and multiplying by 2, we obtain the equation

\[
0 = -i\left( \frac{i}{z\beta} - i \right) = \left( \frac{-i}{z\bar{\beta}} + i \right)
\]

\[
= \frac{1}{z\beta} + \frac{1}{z\bar{\beta}} - 2
\]

Now, multiply both sides of this equation by \( |z|^2|\beta|^2/2 \) to obtain

\[
|z|^2|\beta|^2 = \text{Re} (z\beta)
\]

or

\[
|z|^2 = \frac{\text{Re} (z\beta)}{|\beta|^2} = \text{Re} (\frac{z\beta}{|\beta|^2}) = \text{Re} (\frac{z}{\beta})
\]

which, by a sort of complex “completing the square” (see exercise 16 in chapter 2) can be recognized as equivalent to

\[
|z - \frac{1}{\beta}|^2 = \frac{1}{|\beta|^2}
\]

which is a circle centered at \( 1/\beta \) that passes through the origin.

**General Remarks:** It is worth noting here that in some cases the coordinate approach works very nicely (2 and 3, for example) while the complex approach is cumbersome. However, in 4 and 5, the opposite is true. I didn’t even write out the full coordinate approach to 5 because it’s really quite a mess. The important thing is to know both so that you can switch techniques when one becomes unwieldy.