1a. Let \( f(z) = \alpha z + \beta \). Then, using the formulae we discussed in class, \( \alpha = e^{i\pi/4} \) and \( \beta = (1+i)(1-\alpha) \). Hence, \( f(z) = e^{i\pi/4}z + (1+i)(1-e^{i\pi/4}) \)

- OR -

Think of this in three steps: translate \( 1+i \) to the origin; rotate around origin; translate origin back to \( 1+i \). The formula that does this is

\[
f(z) = e^{i\pi/4}(z - (1+i)) + (1+i)
\]

1b. \( f(2+i) = e^{i\pi/4}((2+i) - (1+i)) + (1+i) = e^{i\pi/4} + (1+i) = (1+i)(1+\sqrt{2}/2) \)

2a. This is in the form of the equation for a line, so we only need to locate a couple of points on the line in order to describe it: \( z = -1 \) is clearly on the curve since \( (1+2i)(-1)+2i = 1 \). Now, we can add any multiple of the conjugate of \( 1+2i \) to this point and we won’t change the imaginary part (since \( (1+2i)(1-2i) \) is real. So, \( -1 + (1-2i) = -2i \) is also on the line. Thus, this curve is the line between \(-1\) and \(-2i\).

- OR -

Let \( z = x+iy \). Then, after squaring both sides, we get:

\[
|z-i|^2 = |z+i|^2
\]

\[
|z|^2 + 2i|z| + |i|^2 = |z|^2 + 2i|z| + |i|^2
\]

So, this curve is \( y = 0 \) (the real axis).
3a. Using the notation we used in class \((f(z) = \alpha z + \beta)\), convert \(\alpha = i\) to polar coordinates to get \(e^{\pi/2}\). This implies that this transformation is a 90° counterclockwise rotation around its fixed point, which is

\[
\frac{\beta}{1 - \alpha} = \frac{1}{1 - i} = \frac{1 + i}{2}
\]

3b. Now, \(\alpha = 5e^{i0}\) so this is a pure homothety (stretch by a factor of 5) around its fixed point, which is

\[
\frac{\beta}{1 - \alpha} = \frac{-i}{4}
\]

3c. In this case, \(\alpha = \sqrt{2}e^{i\pi/4}\) so this is a combination of a homothety (stretch by a factor of \(\sqrt{2}\)) and a counterclockwise rotation of 45° around its fixed point, which is

\[
\frac{\beta}{1 - \alpha} = \frac{2}{1 - (1 + i)} = 2i
\]

4. Using the hint, observe that if \(z\) satisfies the second equation, then \(z - 1 = e^{i\theta}\) for some angle \(\theta\), so that \(z = 1 + e^{i\theta}\). If we can show that all of these except \(z = 0\) \((\theta = \pi/2)\) satisfy the first equation, then we’re done.

\[
\text{Im} \left( \frac{i}{1 + e^{i\theta}} \right) = \text{Im} \left( \frac{i(1 + e^{-i\theta})}{|1 + e^{i\theta}|^2} \right) = \text{Im} \left( \frac{\sin \theta + i(1 + \cos \theta)}{(1 + \cos \theta)^2 + \sin^2 \theta} \right) = \text{Im} \left( \frac{\sin \theta + i(1 + \cos \theta)}{2 \cos \theta + 2} \right) = \frac{1 + \cos \theta}{2 \cos \theta + 2} = \frac{1}{2}
\]

as long as \(1 + \cos \theta \neq 0\).

- OR -

Let \(z = x + iy\). Then, the second equation becomes (after squaring) \((x - 1)^2 + y^2 = 1\), while the first equation becomes

\[
\frac{1}{2} = \text{Im} \left( \frac{i}{x + iy} \right) = \text{Im} \left( \frac{i(x - iy)}{x^2 + y^2} \right) = \text{Im} \left( \frac{y + ix}{x^2 + y^2} \right) = \frac{x}{x^2 + y^2}
\]
Now, whenever $x^2 + y^2 \neq 0$, we can cross-multiply this equation to yield

\[
\begin{align*}
2x &= x^2 + y^2 \\
1 &= x^2 - 2x + 1 + y^2 \\
1 &= (x - 1)^2 + y^2
\end{align*}
\]