

Lines: a line in the complex plane is represented by the general equation

$$\operatorname{Im}(\alpha z + \beta) = 0$$

This, however, may be specialized into a number of more geometrically useful forms, like the equation of a line through a point z_0 that makes an angle θ with the real axis:

$$\operatorname{Im}\left(\frac{z - z_0}{e^{i\theta}}\right) = 0$$

or the equation of a line through two points, z_0 and z_1 :

$$\operatorname{Im}\left(\frac{z - z_0}{z_1 - z_0}\right) = 0$$

Let's now calculate the intersection between two lines in general form.

Theorem 1: the lines $\operatorname{Im}(\alpha_1 z + \beta_1) = 0$ and $\operatorname{Im}(\alpha_2 z + \beta_2) = 0$ intersect in the point

$$z_0 = \frac{\overline{\alpha_1} \operatorname{Im} \beta_2 - \overline{\alpha_2} \operatorname{Im} \beta_1}{\operatorname{Im}(\alpha_1 \overline{\alpha_2})}$$

Proof: We will substitute this value into both equations and observe that they are satisfied.

It will suffice to do this for the first equation, since the indicated point is clearly unchanged if the two equations are interchanged (both numerator and denominator change sign).

$$\begin{aligned}
2i\operatorname{Im}(\alpha_1 z_0 + \beta_1) &= \alpha_1 z_0 + \beta_1 - \overline{\alpha_1 z_0 + \beta_1} \\
&= \alpha_1 \frac{\overline{\alpha_1} \operatorname{Im} \beta_2 - \overline{\alpha_2} \operatorname{Im} \beta_1}{\operatorname{Im}(\alpha_1 \overline{\alpha_2})} + \beta_1 \\
&\quad - \overline{\alpha_1} \frac{\alpha_1 \operatorname{Im} \beta_2 - \alpha_2 \operatorname{Im} \beta_1}{\operatorname{Im}(\alpha_1 \overline{\alpha_2})} - \overline{\beta_1} \\
&= \beta_1 - \overline{\beta_1} + \frac{|\alpha_1|^2 \operatorname{Im} \beta_2 - \alpha_1 \overline{\alpha_2} \operatorname{Im} \beta_1 - |\alpha_1|^2 \operatorname{Im} \beta_2 + \overline{\alpha_1} \alpha_2 \operatorname{Im} \beta_1}{\operatorname{Im}(\alpha_1 \overline{\alpha_2})} \\
&= \beta_1 - \overline{\beta_1} + \frac{\overline{\alpha_1} \alpha_2 \operatorname{Im} \beta_1 - \alpha_1 \overline{\alpha_2} \operatorname{Im} \beta_1}{\operatorname{Im}(\alpha_1 \overline{\alpha_2})} \\
&= \beta_1 - \overline{\beta_1} + \frac{\operatorname{Im} \beta_1 (2i \operatorname{Im} \overline{\alpha_1} \alpha_2)}{\operatorname{Im}(\alpha_1 \overline{\alpha_2})} \\
&= \beta_1 - \overline{\beta_1} + \frac{\operatorname{Im} \beta_1 (-2i \operatorname{Im} \alpha_1 \overline{\alpha_2})}{\operatorname{Im}(\alpha_1 \overline{\alpha_2})} \\
&= 2i \operatorname{Im} \beta_1 - 2i \operatorname{Im} \beta_1 = 0
\end{aligned}$$

Next, we will calculate the distance between a point and a line.

Theorem 2: the distance Δ between a point z_0 and a line $\text{Im}(\alpha z + \beta) = 0$ is given by

$$\Delta = \frac{|\text{Im}(\alpha z_0 + \beta)|}{|\alpha|}$$

Proof: We know from Euclidean geometry that the shortest distance from a point to a line occurs along a line segment perpendicular to the line. The line through z_0 perpendicular to the given line has equation

$$\begin{aligned} \text{Im}(i(z - z_0)\alpha) &= \text{Im}(i\alpha z + (-i\alpha z_0)) \\ &= 0 \end{aligned}$$

Using Theorem 1, we see that these two lines intersect in

$$\begin{aligned} \frac{\bar{\alpha}\text{Im}(-i\alpha z_0) - (i\bar{\alpha})\text{Im}\beta}{\text{Im}(\alpha\bar{i}\bar{\alpha})} &= \frac{\bar{\alpha}\text{Im}(-i\alpha z_0) + i\bar{\alpha}\text{Im}\beta}{\text{Im}(-i|\alpha|^2)} \\ &= \frac{-\bar{\alpha}}{|\alpha|^2}\text{Im}(-i\alpha z_0) + i\bar{\alpha}\text{Im}\beta \\ &= \frac{\text{Im}(i\alpha z_0) - i\text{Im}\beta}{\alpha} \end{aligned}$$

and then we calculate that the distance between z_0 and this intersection point is

$$\begin{aligned} \Delta &= \left| z_0 - \frac{\text{Im}(i\alpha z_0) - i\text{Im}\beta}{\alpha} \right| \\ &= \left| \frac{\alpha z_0 - \text{Im}(i\alpha z_0) + i\text{Im}\beta}{\alpha} \right| \\ &= \left| \frac{\alpha z_0 - \text{Re}(\alpha z_0) + i\text{Im}\beta}{\alpha} \right| \\ &= \left| \frac{i\text{Im}\alpha z_0 + i\text{Im}\beta}{\alpha} \right| \\ &= \left| \frac{i\text{Im}(\alpha z_0 + \beta)}{\alpha} \right| \\ &= \frac{|\text{Im}(\alpha z_0 + \beta)|}{|\alpha|} \end{aligned}$$

Circles: a circle centered at z_0 with radius r_0 has equation $|z - z_0| = r_0$.

Let's first calculate the intersection points of a circle with a line.

Theorem 3: The circle $|z - z_0| = r_0$ and the line $\text{Im}(\alpha z + \beta) = 0$ intersect in the points

$$z_0 - ir_0 \left(\frac{|\alpha|}{\alpha} \right) e^{\pm i\theta}$$

where

$$\cos \theta = \frac{\text{Im}(\alpha z_0 + \beta)}{|\alpha| r_0}$$

provided such a θ exists.

Proof: Clearly these points are on the circle. We need only verify that they are also on the indicated line.

$$\begin{aligned} \text{Im} \left(\alpha \left(z_0 - ir_0 e^{\pm i\theta} \frac{|\alpha|}{\alpha} \right) + \beta \right) &= \text{Im}(\alpha z_0 + \beta) - \text{Im} \left(ir_0 e^{\pm i\theta} |\alpha| \right) \\ &= \text{Im}(\alpha z_0 + \beta) - r_0 |\alpha| \text{Im}(ie^{\pm i\theta}) \\ &= \text{Im}(\alpha z_0 + \beta) - r_0 |\alpha| \text{Im}(\mp \sin \theta + i \cos \theta) \\ &= \text{Im}(\alpha z_0 + \beta) - r_0 |\alpha| \cos \theta \\ &= 0 \end{aligned}$$

Next, we will calculate the intersection of two circles.

Theorem 4: The circles $|z - z_0| = r_0$ and $|z - z_1| = r_1$ intersect in the points

$$z_0 + r_0 \left(\frac{z_1 - z_0}{|z_1 - z_0|} \right) e^{\pm i\theta}$$

where

$$\cos \theta = \frac{r_0^2 - r_1^2 + |z_1 - z_0|^2}{2r_0 |z_1 - z_0|}$$

Proof: Squaring, expanding and subtracting the two circle equations yields

$$\bar{z}(z_1 - z_0) + z(\bar{z}_1 - \bar{z}_0) = r_0^2 - r_1^2 - |z_0|^2 + |z_1|^2$$

or,

$$2\operatorname{Re}(z(\bar{z}_1 - \bar{z}_0)) = r_0^2 - r_1^2 - |z_0|^2 + |z_1|^2$$

This can be put into the general form of a line as

$$\operatorname{Im}(z(-i(\bar{z}_1 - \bar{z}_0)) + i(r_0^2 - r_1^2 - |z_0|^2 + |z_1|^2)/2) = 0$$

Using Theorem 3, we get the desired result after noting that

$$\frac{-i|z_1 - z_0| - i(\bar{z}_1 - \bar{z}_0)|z_1 - z_0|}{-i(\bar{z}_1 - \bar{z}_0)} = \frac{|z_1 - z_0|}{z_1 - z_0} = \frac{z_1 - z_0}{|z_1 - z_0|}$$

and that

$$\begin{aligned} & \frac{r_0^2 - r_1^2 + |z_1 - z_0|^2}{2r_0|z_1 - z_0|} \\ &= \frac{r_0^2 - r_1^2 + |z_1|^2 + |z_0|^2 - z_0\bar{z}_1 - z_1\bar{z}_0}{2r_0|z_1 - z_0|} \\ &= \frac{r_0^2 - r_1^2 + |z_1|^2 - |z_0|^2 - 2\operatorname{Re}(z_0\bar{z}_1) + 2\operatorname{Re}(z_0\bar{z}_0)}{2r_0|z_1 - z_0|} \\ &= \frac{\operatorname{Re}(2z_0(-(\bar{z}_1 - \bar{z}_0)) + (r_0^2 - r_1^2 - |z_0|^2 + |z_1|^2))}{2r_0|z_1 - z_0|} \\ &= \frac{\operatorname{Im}(2z_0(-i(\bar{z}_1 - \bar{z}_0)) + i(r_0^2 - r_1^2 - |z_0|^2 + |z_1|^2))}{2r_0|z_1 - z_0|} \end{aligned}$$

Summary: Distance from point z_0 to $\operatorname{Im}(\alpha z + \beta) = 0$ is

$$\frac{|\operatorname{Im}(\alpha z_0 + \beta)|}{|\alpha|}$$

Intersection of $\operatorname{Im}(\alpha_1 z + \beta_1) = 0$ and $\operatorname{Im}(\alpha_2 z + \beta_2) = 0$ is

$$\frac{\bar{\alpha}_1 \operatorname{Im} \beta_2 - \bar{\alpha}_2 \operatorname{Im} \beta_1}{\operatorname{Im}(\alpha_1 \bar{\alpha}_2)}$$

Intersection of $|z - z_0| = r_0$ and $\operatorname{Im}(\alpha z + \beta) = 0$ is

$$z_0 - ir_0 \left(\frac{|\alpha|}{\alpha} \right) e^{\pm i\theta}$$

where

$$\cos \theta = \frac{\operatorname{Im}(\alpha z_0 + \beta)}{|\alpha| r_0}$$

Intersection of $|z - z_0| = r_0$ and $|z - z_1| = r_1$ is

$$z_0 + r_0 \left(\frac{z_1 - z_0}{|z_1 - z_0|} \right) e^{\pm i\theta}$$

where

$$\cos \theta = \frac{r_0^2 - r_1^2 + |z_1 - z_0|^2}{2r_0|z_1 - z_0|}$$