We introduce a method for producing a variety of $p^r \times p^s$ magic rectangles using $\mathbb{Z}_p$-linear transformations. This adds significantly to the collection of known magic rectangles with non-coprime dimensions.

**Keywords:** magic rectangle; magic square; combinatorial design

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1. Introduction

Our purpose is to introduce a linear-algebraic construction for magic rectangles of size $p^r \times p^s$, where $p$ is prime and $1 \leq r \leq s$. For fixed $p$, $r$, and $s$, this method allows for the production of many different magic rectangles. It follows that, for many non-coprime dimensions, this construction will significantly augment the sparse collection of “known” magic rectangles.

A **magic rectangle** of size $m \times n$ is a rectangular array containing integers $0, 1, \ldots, mn - 1$ such that the row sums are constant and the column sums are constant. (The constants are different if $m \neq n$.) When $m = n$ and both diagonal sums are equal to the row/column sums, one obtains a **magic square**. The construction of magic squares is a venerable pastime, dating back to the Lo-Shu square of ancient China (ca 650 BCE). A variety of magic square constructions may be found in [1]; connections among magic squares, magic rectangles, orthogonal latin squares, and statistical design are described in [6], [9], and [10].

Efforts to construct magic rectangles are much more recent and far less numerous than those for their square counterparts. Sun [11] showed that a magic rectangle of order $m \times n$ exists if and only if $m$ and $n$ have the same parity, are both larger than 1, and are not both 2. (We declare such sizes $m \times n$ to be **admissible**.) Similar (partial) results were achieved independently in [2], and were further refined in [7]. Other constructions of magic rectangles may be found in [3], [4], and [5]. Invariably these constructions have a combinatorial flavor and produce a single example (and its obvious permutations) for each admissible size. We take a different approach: instead of a combinatorial method producing one example for each of a large set of sizes, we introduce a linear-algebraic method that will produce a variety of examples for a rather more limited set of sizes, namely $p^r \times p^s$ where $p$ is prime. This, together with the magic rectangle product theorem given in [2], will increase the number of magic rectangle examples for many admissible sizes $m \times n$ where $\gcd(m, n) > 1$.

The paper is organized as follows: The construction is given in Section 2, conditions we must place upon certain aspects of the construction are developed in Section 3, and in Section 4 we show that these conditions can be satisfied for each
size of the form \( p^r \times p^s \) where \( r < s \). (The case \( r = s \) is addressed in Section 3.) Along the way we will have occasion to perform arithmetic in both \( \mathbb{Z} \) and \( \mathbb{Z}_p \). Arithmetic occurring in \( \mathbb{Z}_p \) with final result regarded as an integer will be indicated by angle brackets. For example, if \( p = 5 \) then \( \langle 2 \cdot 3 + 1 \rangle_5 = 2 \in \mathbb{Z} \).

2. Construction

Let \( p \) be prime and \( 1 \leq r \leq s \). In this section we propose a method for producing magic rectangles of size \( p^r \times p^s \) using \( \mathbb{Z}_p \)-linear operators.

Locations in a \( p^r \times p^s \) magic rectangle can be described by elements of the vector space \( \mathbb{Z}_p^{r+s} \). Rows are enumerated from the top, beginning with 0 and ending with \( p^r - 1 \); columns from left to right beginning with 0 and ending with \( p^s - 1 \). By expressing each row number in base \( p \) we can identify row locations with \( \mathbb{Z}_p^r \). Similarly we can identify column locations with \( \mathbb{Z}_p^s \), and therefore any grid location (row, column) may be identified with an element of \( \mathbb{Z}_p^r \times \mathbb{Z}_p^s = \mathbb{Z}_p^{r+s} \). By way of illustration, the symbol “25” in Figure 1 lies in location 01101 \( \in \mathbb{Z}_2^{2+3} \), where the first two entries indicate the row and the last three entries indicate the column.

\[
\begin{array}{cccccccc}
0 & 1 & 0 & 1 & 2 & 0 & 5 & 17 & 27 \\
28 & 2 & 2 & 8 & 19 & 25 & 13 & 7 \\
11 & 1 & 21 & 31 & 4 & 14 & 26 & 16 \\
23 & 29 & 9 & 3 & 24 & 18 & 6 & 12 \\
\end{array}
\]

Figure 1. A magic rectangle of size \( 2^2 \times 2^3 \).

Symbols in a \( p^r \times p^s \) magic rectangle can also be described by elements of \( \mathbb{Z}_p^{r+s} \). These symbols consist of the numbers \( \{0, 1, \ldots, p^{r+s} - 1\} \). Each symbol \( \Lambda \) has a unique base-\( p \) expansion

\[
\Lambda = \lambda_{p^{r+s} - 1} p^{r+s-1} + \lambda_{p^{r+s} - 2} p^{r+s-2} + \cdots + \lambda_p \cdot p^1 + \lambda_1 \cdot p^0,
\]

where \( \lambda_j \in \{0, 1, \ldots, p-1\} \) for each \( j \in \{0, 1, \ldots, r+s-1\} \). Therefore we can make the identification

\[
\Lambda \leftrightarrow (\lambda_{p^{r+s} - 1}, \ldots, \lambda_p) \in \mathbb{Z}_p^{r+s}.
\]

We will find it convenient to write \( \Lambda = (\Lambda_r, \Lambda_s) \), where

\[
\Lambda_r = (\lambda_{p^{r+s} - 1}, \ldots, \lambda_p) \quad \text{and} \quad \Lambda_s = (\lambda_{p^{r+s} - 1}, \ldots, \lambda_p).
\]

For example, if \( p = 2, r = 2, \) and \( s = 3 \) as in Figure 1, then \( \Lambda = 25 \) corresponds to \( (\Lambda_r, \Lambda_s) = (11, 001) \). We will routinely view these vectors, both for positions and symbols, as column vectors via transpose.

The magic rectangle construction is as follows. Let \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) be an \( (r+s) \times (r+s) \) matrix with entries in \( \mathbb{Z}_p \), expressed in block form, where the sizes of \( A, B, C, D \) are \( r \times r, r \times s, s \times r, \) and \( s \times s \), respectively. Define a linear mapping \( T_M : \mathbb{Z}_p^{r+s} \rightarrow \mathbb{Z}_p^{r+s} \) by \( T_M(\Lambda) = MA \). The mapping \( T_M \) uniquely determines a \( p^r \times p^s \) array with entries in \( \{0, 1, \ldots, p^{r+s} - 1\} \) by declaring \( T_M(\Lambda) \) to be the
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array location housing the number with base-$p$ representation $\Lambda$ (as described in the previous paragraphs). When $p = 2$, $r = 2$, and $s = 3$, the matrix

$$M = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1
\end{bmatrix}$$

determines the magic rectangle shown in Figure 1. To see that the number 25 is sent to the correct location, recall that $\Lambda = 11001$ and so $M\Lambda = 01101$, which is indeed the location housing 25 as described above.

3. Conditions

In the construction of Section 2, clearly the matrix $M$ must be nonsingular so that each location houses a unique number. In this section we develop a list of conditions on the matrix $M$ that guarantee a magic rectangle. These conditions are summarized in Theorem 3.4. Throughout we let the matrix $M$, submatrices $A, B, C, D$, and the mapping $T_M$ be as in Section 2.

Lemma 3.1: If $M$ and $A$ are nonsingular and $B$ has full rank then the rectangular array of numbers determined by $T_M$ has magic rows.

Proof: Let $\mu \in \mathbb{Z}_p^r$. A number $\Lambda = (\Lambda_r, \Lambda_s)$ lies in the $\mu$-th row of the array if and only if

$$A\Lambda_r = \mu - B\Lambda_s. \quad (1)$$

Since $B$ is of full rank its kernel as a linear operator $\mathbb{Z}_p^s \to \mathbb{Z}_p^r$ is of size $p^{r-s}$, so as $\Lambda_s$ ranges over $\mathbb{Z}_p^s$ the right-hand side of (1) achieves each element of $\mathbb{Z}_p^r$ exactly $p^{r-s}$ times. This implies, since $A$ is nonsingular, that $\Lambda_r$ ranges over $\mathbb{Z}_p^r$ exactly $p^{r-s}$ times. Regarding $\Lambda_s$ and $\Lambda_r$ as integers following Section 2, that is,

$$\Lambda_r \longleftrightarrow p^s(\lambda_{p^{r-1}} p^{r-1} + \cdots + \lambda_p p^0) \text{ and } \Lambda_s \longleftrightarrow \lambda_{p^{s-1}} p^{s-1} + \cdots + \lambda_1 p^0,$$

we have that the sum of all the integers in the $\mu$-th row is

$$p^{r-s}[p^s(0 + 1 + \cdots + p^{r-1})] + [0 + 1 + \cdots (p^s - 1)] = \frac{p^s(p^{s+r} - 1)}{2}$$

where the left summand represents the sum of all the $\Lambda_r$, each accounted for $p^{r-s}$ times, and the right summand represents the sum of all the $\Lambda_s$, each accounted for once. This sum is independent of $\mu$; the conclusion follows. \qed

Next we seek conditions on $M$ that ensure magic columns in a rectangle determined by $T_M$. Let $\nu \in \mathbb{Z}_p^s$. A number $\Lambda = (\Lambda_r, \Lambda_s)$ lies in the $\nu$-th column exactly when

$$DA_s = \nu - CA_r. \quad (2)$$

If we assume that $C$ and $D$ are of full rank, then as $\Lambda_r$ ranges over $\mathbb{Z}_p^r$ exactly once we have that $\Lambda_s$ ranges over $\mathbb{Z}_p^s$ a total of $p^{r-s}$ times. If $r = s$ then, arguing
just as in Lemma 3.1, we conclude that the array has magic columns. Therefore, if 
\( r = s \) and \( M, A, B, C, D \) are all of full rank then the array determined by \( T_M \) is a
magic rectangle. Matrices satisfying these conditions exist except when \( p = 2 \) and
\( r = s = 1 \) (e.g., see Proposition 2.4 of [8]), and exist in great abundance as \( p \) and
\( r = s \) become large.

On the other hand, problems arise in (2) when \( r < s \). In this case \( p^{r-s} < 1 \), thus
stifling our attempt to mimic the proof of Lemma 3.1. However, we can say that
the sum of the numbers in the \( \nu \)-th column is
\[ p^s(0 + 1 + \cdots + p^{r-1}) + S_{\nu}, \tag{3} \]
where the left summand is the integer sum of the \( \Lambda_r \)'s , while \( S_{\nu} \) denotes the
integer sum (possibly depending on \( \nu \)) of the corresponding \( \Lambda_s \)'s. Since the magic
sum along columns must be \( p^s(p^{r+s} - 1)/2 \) (obtained by adding all of the symbols
together and dividing by the number of columns), we conclude from (3) that we
require \( S_{\nu} = p^s(p^{s-1} - 1)/2 \). In the remainder of this section we introduce conditions
on \( C \) and \( D \) that yield this value of \( S_{\nu} \).

In arguments that follow we will be considering a \( \mathbb{Z}_p \)-matrix \( W \) of size \( s \times r \) with
block form
\[ W = \begin{bmatrix} U_r \\ U_{s-r} \end{bmatrix}, \tag{4} \]
where \( U_r \) and \( U_{s-r} \) are of sizes \( r \times r \) and \( (s - r) \times r \), respectively. Observe that if
\( U_r \) is nonsingular then the column span of \( W \) is identical to the column span of
\( \begin{bmatrix} I_r \\ V \end{bmatrix} \), where \( V = U_{s-r}U_r^{-1} \) and \( I_r \) is the \( r \times r \) identity matrix. We let \( \vec{v}(1), \ldots, \vec{v}(r) \)
denote the columns of \( V \) and
\[ \vec{v}(r) = \begin{bmatrix} v_s-r-1 \\ \vdots \\ v_0 \end{bmatrix}. \]

**Lemma 3.2:** If \( a, b \in \mathbb{Z}_p \) with \( a \neq 0 \) then the mapping \( x \mapsto ax + b \) is a bijection
of \( \mathbb{Z}_p \) onto itself.

**Lemma 3.3:** Assume \( r < s \). Let \( \mu \in \mathbb{Z}_p^s \) and let \( W \) be as in (4) above where
\( U_r \) is nonsingular and the last column \( \vec{v}(r) \) of \( V \) consists of nonzero entries. Define
\( F : \mathbb{Z}_p^r \to \mathbb{Z}_p^s \) by \( F(x) = \mu + Wx \). Regarding elements of \( \mathbb{Z}_p^s \) as base-\( p \) representations
of integers \( 0, 1, \ldots, p^s - 1 \), we have
\[ \sum_{x \in \mathbb{Z}_p^r} F(x) = \frac{p^s(p^s - 1)}{2}, \]
where the sum is taken over the integers.

**Proof:** Since \( W \) has full rank our sum is exactly the sum over all vectors in the
image of \( F \). These vectors lie in the \( \mu \)-translate of the \( \mathbb{Z}_p \) column span of \( \begin{bmatrix} I_r \\ V \end{bmatrix} \).
that is, they have the form
\[
\begin{pmatrix}
\mu_{s-1} \\
\vdots \\
\vdots \\
\mu_0
\end{pmatrix}
+ \begin{pmatrix}
\lambda_{s-1} \\
\vdots \\
\lambda_{s-r+1} \\
\alpha
\end{pmatrix},
\]
where \(\mu_{s-1}, \ldots, \mu_0\) are the components of \(\mu\), \(\lambda_{s-1}, \ldots, \lambda_{s-r+1}, \alpha\in\mathbb{Z}_p\), and \(\vec{w} = \lambda_{s-1}\vec{v}^{(1)} + \ldots + \lambda_{s-r+1}\vec{v}^{(r)}\). Regarding these vectors as base-\(p\) representations of integers (Section 2), the corresponding integers have the form
\[
\left[ \sum_{j=0}^{s-r-1} (\mu_j + w_j + \alpha v_j)p^j + (\mu_{s-r} + \alpha)p^s - r + 1 \right] + \sum_{j=1}^{r-1} (\mu_{s-r+j} + \lambda_{s-r+j})p^{s-r+j},
\]
where \(w_0, \ldots, w_{s-r-1}\) are the components of \(\vec{w}\).

We need to add these integers as \(\lambda_{s-1}, \ldots, \lambda_{s-r+1}, \alpha\) vary. Begin by holding the \(\lambda\)'s fixed and adding the expressions in (5) as \(\alpha\) to vary over \(\mathbb{Z}_p\). Via Lemma 3.2 (note that \(v_0, \ldots, v_{s-r-1}\) are nonzero) and elementary addition formulas we obtain
\[
\sum_{\alpha=0}^{p-1} \left[ \left( \sum_{j=0}^{s-r-1} (\mu_j + w_j + \alpha v_j)p^j \right) + (\mu_{s-r} + \alpha)p^s - r + 1 \right] + \sum_{j=1}^{r-1} (\mu_{s-r+j} + \lambda_{s-r+j})p^{s-r+j+1}
\]
\[
= \sum_{\alpha=0}^{p-1} \sum_{j=0}^{s-r} (\alpha)p^j + \sum_{j=1}^{r-1} (\mu_{s-r+j} + \lambda_{s-r+j})p^{s-r+j+1}
\]
\[
= p(p^{s-r+1} - 1) + \sum_{j=1}^{r-1} (\mu_{s-r+j} + \lambda_{s-r+j})p^{s-r+j+1}.
\]

Next, adding in (6) by letting \(\lambda_{s-r+1}, \ldots, \lambda_{s-1}\) vary (with a total of \(p^{r-1}\) summands) we use Lemma 3.2 again to yield
\[
\frac{p^r(p^{s-r+1} - 1)}{2} + p^{s-r+2} \sum_{\lambda_{s-r+1}=0}^{p-1} \cdots \sum_{\lambda_1=0}^{p-1} \sum_{j=1}^{r-1} (\mu_{s-r+j} + \lambda_{s-r+j})p^{j-1}
\]
\[
= \frac{p^r(p^{s-r+1} - 1)}{2} + p^{s-r+2} \left( \frac{p^{r-1}(p^{r-1} - 1)}{2} \right)
\]
\[
= \frac{p^r(p^s - 1)}{2}.
\]

This proves the lemma. \(\Box\)

\textbf{Theorem 3.4:} The matrix \(M\) determines a \(p^r \times p^s\) magic rectangle according to the construction given in Section 2 provided that the matrices \(M, A, B, C, D\) are of full rank over \(\mathbb{Z}_p\) and that when \(r < s\) the matrix \(D^{-1}C = \begin{bmatrix} U_r \\ U_{s-r} \end{bmatrix}\) satisfies
(a) \( U_r \) is an \( r \times r \) nonsingular matrix over \( \mathbb{Z}_p \), and

(b) the last column of \( V = U_{s-r}U_r^{-1} \) consists entirely of nonzero entries.

**Proof:** The case \( r = s \) was addressed earlier in the section, so we assume \( r < s \).

The fact that \( M \) is nonsingular ensures that each location of the array is populated by a unique number. Moreover, Lemma 3.1 guarantees that the array will have magic rows provided that \( A \) and \( B \) are of full rank. Finally, in order for the array to have magic columns we require that \( S_\nu = p^r(p^s - 1)/2 \) for each \( \nu \in \mathbb{Z}_p^s \) where, by (2) and (3), \( S_\nu \) is the (integer) sum of all \( \Lambda_s \) satisfying \( \Lambda_s = D^{-1}\nu - D^{-1}CA_r \) as \( \Lambda_r \) ranges over \( \mathbb{Z}_p^r \). According to Lemma 3.3, \( S_\nu \) has the required value when \( C, D \) are of full rank and \(-D^{-1}C\) (or equivalently \( D^{-1}C\)) satisfies the conditions listed in items (a) and (b) above. \( \square \)

### 4. Existence and Consequences

Theorem 3.4 sets forth conditions on the matrix \( M \) (see Section 2) that will guarantee a \( p^r \times p^s \) magic rectangle. Observe that these conditions are open, so that as \( p, r, \) and \( s \) increase we should be able to find many matrices \( M \) that satisfy the conditions, perhaps by random search. These will correspond to many magic rectangles. In this section we establish the existence of such matrices \( M \) when \( r < s \).

(The case \( r = s \) was addressed in Section 3.) The following notation will be used:

- \( 0_{m,n} \) denotes the \( m \times n \) matrix consisting entirely of 0’s.
- \( J_{m,n} \) denotes the \( m \times n \) matrix with 1’s in the last (rightmost) column and 0’s elsewhere.
- \( U_m \) denotes the \( m \times m \) matrix with 1’s on the super-diagonal (immediately above the main diagonal) and 0’s elsewhere.
- \( E_{m,n} \) denotes the \( m \times n \) matrix with a 1 in the \((n, 1)\)-position (\( n \)-th row; first column) and 0’s elsewhere.

**Theorem 4.1:** Matrices \( M \) with entries in \( \mathbb{Z}_p \) satisfying the conditions of Theorem 3.4 exist for all primes \( p \) and all integers \( r, s \) with \( 1 \leq r < s \).

**Proof:** When \( p = 2 \) the matrix

\[
M = \begin{bmatrix}
I_r & U_r & E_{r,s-r} \\
I_r & I_r & 0_{r,s-r} \\
J_{s-r,r} & 0_{s-r,r} & I_{s-r}
\end{bmatrix}
\]

satisfies the conditions of Theorem 3.4, where the division into submatrices \( A, B, C, D \) is indicated by double lines. Similarly, when \( p > 2 \) the matrix

\[
M = \begin{bmatrix}
I_r & -I_r & 0_{r,s-r} \\
I_r & I_r & 0_{r,s-r} \\
J_{s-r,r} & 0_{s-r,r} & I_{s-r}
\end{bmatrix}
\]

satisfies the conditions. \( \square \)
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For example, when \( p = 3, r = 2, \) and \( s = 3 \) the matrix \( M \) given in the proof of Theorem 4.1 gives rise to the following magic rectangle:

\[
\begin{bmatrix}
\end{bmatrix}
\]

The magic rectangle in Figure 1 is generated from a matrix not in the list presented in the proof of Theorem 4.1.

In conclusion, we indicate how these results augment the existing known magic rectangles in sizes other than \( p \times r \times s \). It is known (e.g., see [2]) that given magic rectangles of sizes \( m_1 \times n_1 \) and \( m_2 \times n_2 \), one can construct a magic rectangle of size \( m_1 \times n_1 \), \( m_2 \times n_2 \). Now suppose that \( m \times n \) is an admissible size with \( m = \alpha p_1^r_1 p_2^r_2 \cdots p_k^r_k \) and \( n = \beta p_1^s_1 p_2^s_2 \cdots p_k^s_k \) where \( p_1, \ldots, p_k \) are prime, \( 1 \leq r_j \leq s_j \) for \( 1 \leq j \leq k \), and \( \alpha \times \beta \) is an admissible magic rectangle size. Using the aforementioned product theorem, magic rectangles of size \( m \times n \) can be generated by a single \( \alpha \times \beta \) magic rectangle together with \( p^{r_j} \times p^{s_j} \) rectangles generated by our construction.

References