Covering maps, covering transformations, and the fundamental group

Definition. Suppose $\bar{X}$ and X are spaces that are path connected and locally path connected. A continuous function $p : \bar{X} \to X$ is called a *covering map* if for every $x \in X$ there is a path connected open subset N of X (called an *elementary neighborhood* of x) such that (i) $x \in N$; and (ii) for every path component U of $p^{-1}(N)$ we have that $p|_U : U \to N$ is a homeomorphism.

Remark. Note that U in the above definition is necessarily open in $\bar{X}$ since $\bar{X}$ is locally path connected, p is continuous, and N is open in X. For if $\bar{x} \in U$, then $\bar{x} \in p^{-1}(N)$ and $p^{-1}(N)$ is open in $\bar{X}$. So, there is a path connected open subset $V \subseteq \bar{X}$ such that $\bar{x} \in V \subseteq p^{-1}(N)$. However, U is a path component of $p^{-1}(N)$ so that $x \in V \subseteq U$. Hence, U is a neighborhood for all of its points and therefore open.

Example 1. $\bar{X} = \mathbb{R}$, $X = S^1 \subseteq \mathbb{R}^2$, $p(t) = (\cos(2\pi t), \sin(2\pi t))$.

Example 2. $\bar{X} = X = \{z \in \mathbb{C} \mid |z| = 1\}$, $p(z) = z^3$.

Example 3. $\bar{X} = \mathbb{R}^2$, $X = T^2 \subseteq \mathbb{R}^3$ (with toroidal radii $0 < r < R$, centered at the origin), $p(u, v) = ((R + r\cos(2\pi v))\cos(2\pi u), (R + r\cos(2\pi v))\sin(2\pi u), r\sin(2\pi v))$.

Example 4. Represent the projective plane $X = P^2$ as the quotient space of S^2, where $x \sim -x$, i.e., with pairs of diametrically opposite points identified to one point $[x] = \{x, -x\}$. Take $\bar{X} = S^2$ and $p(x) = [x]$.

In class, we proved the following important

Lemma 1 (Unique path lifting). *Let $p : \bar{X} \to X$ be a covering map and let $\bar{x} \in \bar{X}$, $x \in X$ with $p(\bar{x}) = x$. Then for every continuous path $f : [0, 1] \to X$ with $f(0) = x$ there is exactly one continuous path $g : [0, 1] \to \bar{X}$ with $g(0) = \bar{x}$ and $f = p \circ g$.*

A similar argument yields a proof of

Lemma 2 (Unique homotopy lifting). *Let $p : \bar{X} \to X$ be a covering map and let $\bar{x} \in \bar{X}$, $x \in X$ with $p(\bar{x}) = x$. Then for every continuous homotopy $H : [0, 1] \times [0, 1] \to X$ with $H(0, 0) = x$ there is exactly one continuous homotopy $G : [0, 1] \times [0, 1] \to \bar{X}$ with $G(0, 0) = \bar{x}$ and $F = p \circ G$.*

Definition. We call the map $g : ([0, 1], 0) \to (\bar{X}, \bar{x})$ of Lemma 1 the *lift* of the path $f : ([0, 1], 0) \to (X, x)$. Similarly, in Lemma 2, $G : ([0, 1] \times [0, 1], (0, 0)) \to (\bar{X}, \bar{x})$ is called the *lift* of the homotopy $H : ([0, 1] \times [0, 1], (0, 0)) \to (X, x)$.

Based on Lemma 1 and Lemma 2, we proved in class the following

Corollary 1. *Let $p : \bar{X} \to X$ be a covering map. If $\alpha, \beta : [0,1] \to \bar{X}$ are two continuous paths with $\alpha(0) = \beta(0)$ and $[p \circ \alpha] = [p \circ \beta]$, then $[\alpha] = [\beta]$; in particular $\alpha(1) = \beta(1)$.*

This immediately yields

Corollary 2. *Let $p : \bar{X} \to X$ be a covering map and let $\bar{x} \in \bar{X}$, $x \in X$ with $p(\bar{x}) = x$. Then the homomorphism $p_\# : \pi_1(\bar{X}, \bar{x}) \to \pi_1(X, x)$ given by $p_\#([\alpha]) = [p \circ \alpha]$ is injective.*

Remark. In particular, $\pi_1(\bar{X}, \bar{x})$ is isomorphic to the subgroup $p_\#(\pi_1(\bar{X}, \bar{x}))$ of $\pi_1(X, x)$. In turn, $p_\#(\pi_1(\bar{X}, \bar{x}))$ are precisely those elements of $\pi_1(X, x)$, whose loop-representatives lift to *loops* at $\bar{x}$.

Definition. Let $p : \bar{X} \to X$ be a covering map. We call $T : \bar{X} \to \bar{X}$ a *covering transformation* if (i) T is a homeomorphism and (ii) $p = p \circ T$. The set of all covering transformations forms a group under function composition, which is called the *automorphism group* $\mathrm{Aut}(\bar{X} \xrightarrow{p} X)$.

Example 5. With the covering maps defined as above, we have $\mathrm{Aut}(\mathbb{R} \xrightarrow{p} S^1) \approx \mathbb{Z}$, since every covering transformation is of the form $T(t) = t + n$ for some $n \in \mathbb{Z}$. Similarly, $\mathrm{Aut}(\mathbb{R}^2 \xrightarrow{p} T^2) \approx \mathbb{Z} \times \mathbb{Z}$, since every covering transformation is of the form $T(u, v) = (u + n, v + m)$ for some $n, m \in \mathbb{Z}$.

Remark. Recall that for a subgroup H of a group G we define the *normalizer* of H in G by $\mathbb{N}_G(H) = \{g \in G \mid gH = Hg\}$; it is the largest subgroup of G in which H is normal.

We now come to an important result, which connects covering space theory to fundamental groups:

Theorem. *Let $p : \bar{X} \to X$ be a covering map and let $\bar{x} \in \bar{X}$, $x \in X$ with $p(\bar{x}) = x$. Consider $G = \pi_1(X, x)$ and $H = p_\#(\pi_1(\bar{X}, \bar{x}))$. Then there is a surjective homomorphism $\phi : \mathbb{N}_G(H) \to Aut(\bar{X} \xrightarrow{p} X)$ whose kernel is equal to H. In particular, $Aut(\bar{X} \xrightarrow{p} X) \approx \mathbb{N}_G(H)/H$.*

Proof. For $[\alpha] \in \mathbb{N}_G(H)$ and $\bar{y} \in \bar{X}$ we define $\phi([\alpha])(\bar{y})$ as follows: choose any continuous path $f : [0,1] \to \bar{X}$ with $f(0) = \bar{x}$ and $f(1) = \bar{y}$. Let $\tilde{\alpha} : [0,1] \to \bar{X}$ be the lift of $\alpha : [0,1] \to X$ with $\tilde{\alpha}(0) = \bar{x}$ and let $f' : [0,1] \to \bar{X}$ be the lift of $p \circ f : [0,1] \to X$ with $f'(0) = \tilde{\alpha}(1)$. We define $\phi([\alpha])(\bar{y}) = f'(1)$. (You might want to sketch a diagram at this point.)

We have to show that

(i) ϕ is well-defined;

(ii) $\phi([\alpha]) \in \mathrm{Aut}(\bar{X} \xrightarrow{p} X)$;

(iii) ϕ is a homomorphism;

(iv) ϕ is onto;

(v) ker $\phi = H$.

(i) We wish to show that the definition of ϕ is independent of the choice of f. To this end, let $g : [0,1] \to \bar{X}$ be another continuous path with $g(0) = \bar{x}$ and $g(1) = \bar{y}$. It is our goal to show that $f'(1) = g'(1)$. Put $\bar{z} = \tilde{\alpha}(1)$, then

$$[\ \bar{\tilde{\alpha}}\ ]\pi_1(\bar{X}, \bar{x})[\tilde{\alpha}] = \pi_1(\bar{X}, \bar{z}).$$

Applying $p_\#$ to this equation, we get

$$[\alpha]^{-1}p_\#(\pi_1(\bar{X}, \bar{x}))[\alpha] = p_\#(\pi_1(\bar{X}, \bar{z})).$$

However, by assumption, $[\alpha]^{-1}p_\#(\pi_1(\bar{X}, \bar{x}))[\alpha] = p_\#(\pi_1(\bar{X}, \bar{x}))$, so that

$$p_\#(\pi_1(\bar{X}, \bar{x})) = p_\#(\pi_1(\bar{X}, \bar{z})).$$

This means that the elements of $\pi_1(X, x)$ which lift to loops at $\bar{x}$ are the same as those which lift to loops at $\bar{z}$. Consequently,

$$[p \circ (f \cdot \bar{g})] \in p_\#(\pi_1(\bar{X}, \bar{x})) = p_\#(\pi_1(\bar{X}, \bar{z})).$$

In other words, $[p \circ (f \cdot \bar{g})] = [p \circ \gamma]$ for some $\gamma : [0,1] \to \bar{X}$ with $\gamma(0) = \gamma(1) = \bar{z}$. From Corollary 1 we now learn that $(p \circ f) \cdot (p \circ \bar{g})$ lifts to some *loop* at $\bar{z}$. Since f' and g' are the unique lifts of $p \circ f$ and $p \circ g$ at $\bar{z}$, respectively, we must have $f'(1) = g'(1)$.

(ii) First of all not that, by definition, we have $p \circ \phi([\alpha])(\bar{y}) = p(f'(1)) = p(f(1)) = p(\bar{y})$. If N is an elementary neighborhood of $y = p(\bar{y})$ and U_1 and U_2 the path components of $p^{-1}(N)$ containing $\bar{y}$ and $\phi([\alpha])(\bar{y})$, respectively, we get $\phi([\alpha])(U_1) = U_2$. To see this, all we have to do is choose the path f in the definition of $\phi([\alpha])(\bar{w})$ for $\bar{w} \in U_1$ to always begin with the same path f running from $f(0) = \bar{x}$ to $f(1) = \bar{y}$ and concatenate it with a path h running from $h(0) = \bar{y}$ to $h(1) = \bar{w}$ which stays in U_1. This observation yields continuity of $\phi[\alpha]$. Also, it is clear that $\phi([\bar{\alpha}])$ is the inverse of $\phi[\alpha]$. In summary, $\phi[\alpha] \in \mathrm{Aut}(\bar{X} \xrightarrow{p} X)$.

(iii) Let $[\alpha], [\beta] \in \mathbb{N}_G(H)$ and $\bar{y} \in \bar{X}$. Let $\tilde{\alpha}, \tilde{\beta} : [0,1] \to \bar{X}$ be the lifts of the paths $\alpha, \beta : [0,1] \to X$ with $\tilde{\alpha}(0) = \tilde{\beta}(0) = \bar{x}$, respectively. Choose a path $f : [0,1] \to \bar{X}$ from $f(0) = \bar{x}$ to $f(1) = \bar{y}$. Let f' be the lift of $p \circ f$ with $f'(0) = \tilde{\beta}(1)$. Then $\phi([\beta])(\bar{y}) = f'(1)$. Let $\tilde{\beta}'$ be the lift of $p \circ \tilde{\beta} = \beta$ with $\tilde{\beta}'(0) = \tilde{\alpha}(1)$. Then $\tilde{\alpha} \cdot \tilde{\beta}'$ is the lift of $\alpha \cdot \beta : [0,1] \to X$ which starts at $\bar{x}$. Let f'' be the lift of $p \circ f$ with $f''(0) = \tilde{\alpha} \cdot \tilde{\beta}'(1) = \tilde{\beta}'(1)$. (Again, a sketch of the situation might help here.) Then, by definition,

$$\phi([\alpha] * [\beta])(\bar{y}) = \phi([\alpha \cdot \beta])(\bar{y}) = f''(1).$$

On the other hand, $\tilde{\beta}' \cdot f''$ is now the lift of $p \circ (\tilde{\beta} \cdot f')$ with begins at $\tilde{\alpha}(1)$, so that

$$\phi([\alpha]) \circ \phi([\beta])(\bar{y}) = \phi([\alpha])(f'(1)) = \tilde{\beta}' \cdot f''(1) = f''(1).$$

Hence, $\phi([\alpha] * [\beta]) = \phi([\alpha]) \circ \phi([\beta])$ and ϕ is indeed a homomorphism.

(iv) Let $T \in \text{Aut}(\bar{X} \xrightarrow{p} X)$. Choose any continuous path $\tilde{\alpha} : [0,1] \to \bar{X}$ with $\tilde{\alpha}(0) = \bar{x}$ and $\tilde{\alpha}(1) = T(\bar{x})$. Put $\alpha = p \circ \tilde{\alpha}$. Then $\alpha(0) = p \circ \tilde{\alpha}(0) = p(\bar{x}) = x$ and $\alpha(1) = p \circ \tilde{\alpha}(1) = p \circ T(\bar{x}) = p(\bar{x}) = x$. Therefore, $[\alpha] \in \pi_1(X, x)$.

In fact, $[\alpha] \in \mathbb{N}_G(H)$. To see why, first recall from Part (i) above that

$$[\alpha]^{-1} p_{\#}(\pi_1(\bar{X}, \bar{x}))[\alpha] = p_{\#}(\pi_1(\bar{X}, T(\bar{x}))). \tag{1}$$

On the other hand, since $T : \bar{X} \to \bar{X}$ is a homeomorphism, we know that it induces an isomorphism $T_{\#} : \pi_1(\bar{X}, \bar{x}) \to \pi_1(\bar{X}, T(\bar{x}))$. Consequently,

$$p_{\#}(\pi_1(\bar{X}, \bar{x})) = (p \circ T)_{\#}(\pi_1(\bar{X}, \bar{x})) = p_{\#}(T_{\#}(\pi_1(\bar{X}, \bar{x}))) = p_{\#}(\pi_1(\bar{X}, T(\bar{x}))). \tag{2}$$

Combining Equations (1) and (2) we get

$$[\alpha]^{-1} p_{\#}(\pi_1(\bar{X}, \bar{x}))[\alpha] = p_{\#}(\pi_1(\bar{X}, \bar{x})),$$

which says that $[\alpha] \in \mathbb{N}_G(H)$.

If now $\bar{y} \in \bar{X}$ and $f : [0,1] \to \bar{X}$ is any path with $f(0) = \bar{x}$ and $f(1) = \bar{y}$, consider $f' = T \circ f$. Since $p \circ f' = p \circ (T \circ f) = (p \circ T) \circ f = p \circ f$, we see that f' is the lift of $p \circ f$ with $f'(0) = T \circ f(0) = T(\bar{x}) = \tilde{\alpha}(1)$. Therfore, $\phi([\alpha])(\bar{y}) = f'(1) = T(f(1)) = T(\bar{y})$. Hence, $T = \phi([\alpha])$, proving that ϕ is onto.

(v) Finally, $\phi([\alpha]) = \text{id}_{\bar{X}}$ if and only if $f(1) = f'(1)$, which by unique path lifting can only occur when $f = f'$, that is, when $\tilde{\alpha}(0) = \tilde{\alpha}(1)$. This is the case precisely when α lifts to a *loop* at $\bar{x}$, i.e., when $[\alpha] \in p_{\#}(\pi_1(\bar{X}, \bar{x})) = H$. So, the kernel of ϕ equals H.

□