Linear recurrence relations with constant coefficients

Recall that a linear recurrence relation with constant coefficients $c_1, c_2, \cdots, c_k$ $(c_k \neq 0)$ of *degree* k and with *control term* $F(n)$ has the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n) \qquad (n \geq k).$$

It follows from the general recursion theorem that for every string of initial values $a_0, a_1, \cdots, a_{k-1}$ there is exactly one sequence $\{a_n\}$ that satisfies the above recurrence relation *and* matches the given initial conditions. Consequently, if no initial conditions are imposed, there will always be an infinite set of solutions.

The theorems below will show how to find *all* solutions to such a recurrence relation provided the control term is of the special form $F(n) = q(n)s^n$ for some polynomial $q(n)$ and some constant s. More general control terms are discussed in advanced text books on difference equations, such as *An Introduction to Difference Equations* by Saber N. Elaydi (Springer Verlag 1996).

We first consider the case where $F(n) = 0$ for all n. The recurrence relation is then called *homogeneous*. Here is the solution formula:

Theorem 1 [The homogeneous case: finding all solutions]

Let $c_1, c_2, \cdots, c_k$ be constants with $c_k \neq 0$. Consider the homogeneous linear recurrence relation

$$(*) \qquad a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \qquad (n \geq k).$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_t$ be the *distinct* solutions to the *characteristic equation*

$$\lambda^k = c_1 \lambda^{k-1} + c_2 \lambda^{k-2} + \cdots + c_k.$$

Then a sequence $\{a_n\}$ is a solution to $(*)$ if and only if it is of the form

$$a_n = p_1(n)\lambda_1^n + p_2(n)\lambda_2^n + \cdots + p_t(n)\lambda_t^n$$

with polynomials $p_1(n), p_2(n), \cdots, p_t(n)$ such that the degree of $p_i(n)$ is less than the multiplicity of λ_i for all $1 \leq i \leq t$.

Note: The coefficients of the polynomials $p_i(n)$ are determined by $a_0, a_1, \cdots a_{k-1}$.

Example

The characteristic equation to the recurrence relation

$$a_n = a_{n-1} + \frac{5}{12}a_{n-2} - \frac{29}{54}a_{n-3} - \frac{1}{27}a_{n-4} + \frac{2}{27}a_{n-5}$$

is given by

$$\lambda^5 = \lambda^4 + \frac{5}{12}\lambda^3 - \frac{29}{54}\lambda^2 - \frac{1}{27}\lambda + \frac{2}{27},$$

which is equivalent to

$$(3\lambda - 2)^3(2\lambda + 1)^2 = 0.$$

Hence, $\lambda_1 = 2/3$ (with multiplicity 3) and $\lambda_2 = -1/2$ (with multiplicity 2) and the generic solution to $(*)$ is given by

$$a_n = (\alpha_0 + \alpha_1 n + \alpha_2 n^2)\left(\frac{2}{3}\right)^n + (\alpha_3 + \alpha_4 n)\left(-\frac{1}{2}\right)^n.$$

If one is given initial conditions, like $a_0 = 1$, $a_1 = -2$, $a_2 = 3$, $a_3 = 0$, and $a_4 = 6$, then $\alpha_0, \cdots, \alpha_4$ can be determined by solving the appropriate system of five linear equations in five unknowns resulting from taking $n = 0, 1, \cdots, 4$ in the formula defining a_n.

Proof [of Theorem 1]

There are many different ways to establish this result. The proof that we will sketch here uses some knowledge of linear algebra. We will illustrate the general proof at an example of degree 5. First suppose that $\{a_n\}$ is a solution to

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_5 a_{n-5} \quad (n \geq 5).$$

We can rewrite this in matrix form as follows:

$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \\ a_{n-3} \\ a_{n-4} \end{bmatrix} = \underbrace{\begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} a_{n-1} \\ a_{n-2} \\ a_{n-3} \\ a_{n-4} \\ a_{n-5} \end{bmatrix}.$$

So, the matrix A effects a shift in any consecutive string of length 5 taken from the infinite sequence $\{a_n\}$. This allows us to compute all values of $\{a_n\}$ from $a_0, a_1, \cdots, a_4$ by iterating the matrix A:

$$(1) \qquad \begin{bmatrix} a_{n+4} \\ a_{n+3} \\ a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix} = \underbrace{A \cdot A \cdot A \cdots A}_{n\text{-times}} \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}.$$

Also notice that the matrix A is invertible, since $c_5 \neq 0$. We therefore can create a "history" $a_{-1}, a_{-2}, a_{-3}, \cdots$ for our sequence $\{a_n\}$ that follows the same recurrence relation $(*)$ by simply applying A^{-1} rather than A.

However, the form that the matrix A is currently in does not allow for convenient iteration. A change of basis will help us out here. Recall that if B is any 5×5 invertible matrix, then the linear transformation represented by A in the standard basis, will be represented by $J = B^{-1}AB$ in the basis made up from the column vectors of B. To find a good choice for B, we compute the eigenvalues of the matrix A (by expanding across the first row):

$$\begin{aligned}
0 = \det\,(A - \lambda I) &= \det \begin{bmatrix} c_1 - \lambda & c_2 & c_3 & c_4 & c_5 \\ 1 & -\lambda & 0 & 0 & 0 \\ 0 & 1 & -\lambda & 0 & 0 \\ 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1 & -\lambda \end{bmatrix} \\
&= (c_1 - \lambda)(-\lambda)^4 - c_2(-\lambda)^3 + c_3(-\lambda)^2 - c_4(-\lambda) + c_5 \\
&= (-1)^5\Big(\lambda^5 - c_1\lambda^4 - c_2\lambda^3 - c_3\lambda^2 - c_4\lambda - c_5\Big).
\end{aligned}$$

Observe that the characteristic equation of the matrix A is exactly the characteristic equation of the recurrence relation $(*)$. (Hence the name!) To stay within a specific example, say this polynomial factors as follows:

$$(-1)^5\Big(\lambda^5 - c_1\lambda^4 - c_2\lambda^3 - c_3\lambda^2 - c_4\lambda - c_5\Big) = (-1)^5(\lambda - \lambda_1)^2(\lambda - \lambda_2)^3.$$

The *Jordan Normal Form Theorem* now implies that we can choose B such that J has the form

$$J = \left[\begin{array}{cc|ccc} \lambda_1 & 0 & 0 & 0 & 0 \\ 1 & \lambda_1 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 1 & \lambda_2 & 0 \\ 0 & 0 & 0 & 1 & \lambda_2 \end{array}\right].$$

(In general, the Jordan canonical form has square blocks on its diagonal and zeros elsewhere. Each square block has one eigenvalue on its main diagonal and the number 1 on its subdiagonal and is of size equal to the multiplicity of that eigenvalue.) With a brief induction you can verify that the matrix J can be iterated quite easily ($n \geq 5$):

$$J^n = \underbrace{J \cdot J \cdots J}_{n\text{-times}} = \left[\begin{array}{cc|ccc} \binom{n}{0}\lambda_1^n & 0 & 0 & 0 & 0 \\ \binom{n}{1}\lambda_1^{n-1} & \binom{n}{0}\lambda_1^n & 0 & 0 & 0 \\ \hline 0 & 0 & \binom{n}{0}\lambda_2^n & 0 & 0 \\ 0 & 0 & \binom{n}{1}\lambda_2^{n-1} & \binom{n}{0}\lambda_2^n & 0 \\ 0 & 0 & \binom{n}{2}\lambda_2^{n-2} & \binom{n}{1}\lambda_2^{n-1} & \binom{n}{0}\lambda_2^n \end{array}\right]$$

Since $A = BJB^{-1}$ that helps us iterate the matrix A:

$$A^n = A \cdot A \cdot A \cdots A = (BJB^{-1}) \cdot (BJB^{-1}) \cdot (BJB^{-1}) \cdots (BJB^{-1}) = BJ^nB^{-1}.$$

Observe that the expression

$$f(n) = \left(\begin{array}{c} n \\ r \end{array} \right) = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!},$$

when viewed as a polynomial in the variable n, has degree r. Also,

$$\left(\begin{array}{c} n \\ r \end{array} \right) \lambda_i^{n-r} = \frac{\left(\begin{array}{c} n \\ r \end{array} \right)}{\lambda_i^r} \lambda_i^n.$$

Therefore, if we substitute BJ^nB^{-1} for A^n in (1) (and focus on its last row) we get

$$a_n = \Big(\alpha_0 + \alpha_1 n\Big)\lambda_1^n + \left(\alpha_2 + \alpha_3 n + \alpha_4 n^2\right)\lambda_2^n \quad (n \geq 5), \tag{3}$$

with some coefficients $\alpha_0, \cdots, \alpha_4$. On the face of it, (3) holds only for $n \geq 5$. However, we could have used the initial conditions $a_{-1}, a_{-2}, \cdots, a_{-5}$ in all of the above, rather than $a_4, a_3, \cdots, a_0$, and still arrive at a formula that is of the form (3). (Because such a shift in perspective substitutes $n+5$ for n in (3), which then turns into another formula of the same type.) Hence, formula (3) actually holds for all $n \geq 0$. This shows that every solution is of the proposed form.

Next, we prove that every sequence of this form is a solution. As part of the above discussion we see that the five initial conditions $a_0, \cdots, a_4$ of any solution $\{a_n\}$ to $(*)$ determine the five coefficients $\alpha_0, \cdots, \alpha_4$ in the formula for $\{a_n\}$, and they do so by a linear transformation. Moreover, this transformation is injective, because the α's determine the entire sequence $\{a_n\}$. Now, every linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^5$ which is injective, must also be surjective. Consequently, if we are given any numbers $\alpha_0, \cdots, \alpha_4$, whatsoever, we can find numbers $a_0, \cdots, a_4$ by the inverse of this transformation, and then produce an infinite sequence $\{a_n\}$ by applying the recurrence rule $(*)$ whose formula will be

$$a_n = \Big(\alpha_0 + \alpha_1 n\Big)\lambda_1^n + \left(\alpha_2 + \alpha_3 n + \alpha_4 n^2\right)\lambda_2^n \quad (n \geq 0).$$

In short, every sequence of this form is a solution to $(*)$.

Next, we will examine how to find *one particular* solution to a non-homogeneous recurrence relation of the specific type $F(n) = q(n)s^n$:

Theorem 2 [Finding *one* particular solution]

Let constants $c_1, c_2, \cdots, c_k$ ($c_k \neq 0$) be given, along with a constant s and a polynomial $q(n)$. Then the recurrence relation

$$(**) \qquad a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + q(n)s^n \qquad (n \geq k)$$

(among other solutions) always has a solution of the form

$$b_n = n^m p(n) s^n,$$

where $m \geq 0$ is the multiplicity of s in the characteristic equation

$$\lambda^k = c_1 \lambda^{k-1} + c_2 \lambda^{k-2} + \cdots + c_k$$

for the corresponding homogeneous equation and $p(n)$ is a polynomial whose degree is less than or equal to the degree of $q(n)$.

Note: This particular solution cannot accommodate arbitrary initial conditions. So, it might not be the solution you were looking for. In this case, you need to find *all* solutions using Theorem 3 below and pick the one you actually wanted.

Example

Consider

$$a_n = 11a_{n-1} - 39a_{n-2} + 45a_{n-3} + (1 + 2n + n^2 + 5n^3)3^n.$$

Since the characteristic equation to the corresponding homogeneous problem is

$$\lambda^3 = 11\lambda^2 - 39\lambda + 45,$$

which is equivalent to $(\lambda - 3)^2(\lambda - 5) = 0$, we can take

$$b_n = n^2(\beta_0 + \beta_1 n + \beta_2 n^2 + \beta_3 n^3)3^n.$$

The correct coefficients $\beta_0, \cdots, \beta_3$ can be determined by substituting $\{b_n\}$ back into the original non-homogeneous equation:

$$b_n = 11b_{n-1} - 39b_{n-2} + 45b_{n-3} + (1 + 2n + n^2 + 3n^3)3^n.$$

This will produce *one* among infinitely many solutions. To find them all, including the one you wanted, use Theorem 3.

Proof [of Theorem 2]

Denote the degree (in the variable n) of the polynomial $q(n)$ by d. Then for every n, the function $f(x) = q(n-x)$ is a polynomial of degree d (in the variable x). So, by the exercise below,

$$\sum_{k=0}^{d+1} \binom{d+1}{k} (-1)^k \underbrace{q(n-k)}_{f(k)} = 0 \quad \text{for all } n.$$

Multiplying through with s^n we see that

$$\sum_{k=0}^{d+1} \binom{d+1}{k} \underbrace{(-1)^k s^k}_{(-s)^k} \underbrace{q(n-k)s^{n-k}}_{F(n-k)} = 0 \quad \text{for all } n.$$

But that means that the sequence $\{F(n)\} = \{q(n)s^n\}$ satisfies a homogeneous linear recurrence relation whose characteristic equation is

$$(\S) \qquad \sum_{k=0}^{d+1} \binom{d+1}{k} (-s)^k \lambda^{(d+1)-k} = (\lambda - s)^{d+1} = 0.$$

Now, let $\{a_n\}$ be any solution to the non-homogeneous recurrence relation $(**)$. We claim that the sequence $\{a_n\}$ also satisfies a *homogeneous* linear recurrence relation whose characteristic equation is

$$(\lambda^k - c_1\lambda^{k-1} - c_2\lambda^{k-2} - \cdots - c_k)(\lambda - s)^{d+1} = 0.$$

In order to keep the formalism to a minimum, we will illustrate this with an example that conveys the general idea: Suppose our original non-homogeneous recurrence relation reads

$$(**) \qquad a_n = 7a_{n-1} + (3 + 2n)s^n.$$

That is, $k = 1$, $c_1 = 7$, $d = 1$, $q(n) = 3 + 2n$, and we are claiming that $\{a_n\}$ satisfies a *homogeneous* linear recurrence relation with characteristic equation

$$(\lambda - 7)(\lambda - s)^2 = 0.$$

The above says that the sequence $\{F(n)\}$ satisfies a recurrence relation corresponding to the characteristic equation

$$(\S) \qquad \lambda^2 - 2s\lambda + s^2 = (\lambda - s)^2 = 0,$$

namely

$$F(n) - 2sF(n-1) + s^2F(n-2) = 0.$$

Then

$$\begin{array}{lllllll}
(a_n & -7a_{n-1}) & & & = & F(n) \\
 & -2s(a_{n-1} & -7a_{n-2}) & & = & -2sF(n-1) \\
 & & s^2(a_{n-2} & -7a_{n-3}) & = & s^2F(n-2) \\
\hline
a_n & -(7+2s)a_{n-1} & +(s^2-7)a_{n-2} & -7a_{n-3} & = & 0,
\end{array}$$

which has characteristic equation $(\lambda-7)(\lambda-s)^2 = 0$. (This is best seen by substituting λ^i for a_i *row by row* and factoring out $\lambda^{n-3}(\lambda-7)$.) Hence the claim.

Then we can apply Theorem 1 to find a formula for $\{a_n\}$. We get

$$a_n = \alpha_0 7^n + \Big(\alpha_1 + \alpha_2 n\Big)s^n \text{ if } s \neq 7 \text{ and } m = 0,$$

$$a_n = \Big(\alpha_0 + \alpha_1 n + \alpha_2 n^2\Big)s^n \text{ if } s = 7 \text{ and } m = 1.$$

On the other hand,

$$d_n = \alpha_0 7^n$$

solves the homogeneous part of our recurrence, by Theorem 1. Hence, by Theorem 3 below, $b_n = a_n - d_n$ is again a solution to $(**)$. Notice that in both cases, this solution $\{b_n\}$ has the form $n^m p(n) s^n$, with degree $p(n)$ equal to the degree of $q(n)$ (namely 1). The same argument holds for any non-homogeneous linear recurrence relation of the form $(**)$.

Theorem 3 [The general formula]

Let constants $c_1, c_2, \cdots, c_k$ $(c_k \neq 0)$ and a control sequence $F : I\!N \to I\!R$ be given. Suppose you have found one particular solution $\{b_n\}$ to the non-homogeneous recurrence relation

$$(**) \qquad a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n) \qquad (n \in I\!N).$$

Then the generic solution $\{a_n\}$ to $(**)$ is of the form

$$a_n = d_n + b_n,$$

where $\{d_n\}$ is the generic solution to the corresponding homogeneous recurrence relation

$$(*) \qquad a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \qquad (n \in I\!N).$$

Proof

If a_n is of the stated form, then it satisfies $(**)$:

$$\begin{aligned} a_n &= d_n + b_n \\ &= \big(c_1 d_{n-1} + c_2 d_{n-2} + \cdots + c_k d_{n-k}\big) + \big(c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n)\big) \\ &= c_1(d_{n-1} + b_{n-1}) + c_2(d_{n-2} + b_{n-2}) + \cdots + c_k(d_{n-k} + b_{n-k}) + F(n) \\ &= c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n). \end{aligned}$$

Conversely, if $\{a_n\}$ is *any* solution to $(**)$ then the sequence $\{d_n\}$ defined by $d_n = a_n - b_n$ satisfies the homogeneous recurrence relation $(*)$:

$$\begin{aligned} d_n &= a_n - b_n \\ &= \big(c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)\big) - \big(c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n)\big) \\ &= c_1(a_{n-1} - b_{n-1}) + c_2(a_{n-2} - b_{n-2}) + \cdots + c_k(a_{n-k} - b_{n-k}) \\ &= c_1 d_{n-1} + c_2 d_{n-2} + \cdots + c_k d_{n-k}. \end{aligned}$$

Hence, $a_n = d_n + b_n$ has the proposed form.

Exercise

(a) Show that
$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r c = 0$$
for all $n > 0$ and all constants c. [Simply factor out c.]

(b) Show that
$$\binom{n}{r} r = \binom{n-1}{r-1} n$$
for all $1 \leq r \leq n$. [This is the *Chairperson Identity*.]

(c) Show that for every polynomial $f(x)$ of degree d we have
$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r f(r) = 0,$$
whenever $d < n$.

[Use induction on d. Part (a) serves as the basic step $d = 0$. For the inductive step, assume that $d \geq 1$ and that the statement is true for all polynomials of degree less than d. To show that the statement also holds for polynomials of degree d, you only need to verify that
$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r r^d = 0$$
for all $d < n$. Make use of Part (b) and the inductive hypothesis:
$$\begin{aligned}
\sum_{r=0}^{n} \binom{n}{r} (-1)^r r^d &= \sum_{r=1}^{n} \binom{n}{r} (-1)^r r^d \\
&= \sum_{r=1}^{n} \binom{n-1}{r-1} n(-1)^r r^{d-1} \\
&= \sum_{r=0}^{n-1} \binom{n-1}{r} n(-1)^{r+1} (r+1)^{d-1} \\
&= -n \sum_{r=0}^{n-1} \binom{n-1}{r} (-1)^r (r+1)^{d-1} = 0.]
\end{aligned}$$