Angle preserving linear transformations

There are various ways to say "a linear transformation preserves angles." In this hand-out, we present two of them. We show that a linear transformation preserves angles if and only if it stretches the length of every vector by some fixed positive number λ, which, in turn, occurs if and only if the dot product gets stretched by λ^2.

We first prove that the latter two conditions are equivalent.

Lemma. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and λ a positive real number.

Then the following two statements are equivalent:

(i) $L(\mathbf{v}) \cdot L(\mathbf{w}) = \lambda^2(\mathbf{v} \cdot \mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$;

(ii) $||L(\mathbf{v})|| = \lambda||\mathbf{v}||$ for all $\mathbf{v} \in \mathbb{R}^n$.

Proof. Suppose $L(\mathbf{v}) \cdot L(\mathbf{w}) = \lambda^2(\mathbf{v} \cdot \mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Then

$$||L(\mathbf{v})||^2 = L(\mathbf{v}) \cdot L(\mathbf{v}) = \lambda^2 \mathbf{v} \cdot \mathbf{v} = \lambda^2 ||\mathbf{v}||^2$$

for all $\mathbf{v} \in \mathbb{R}^n$. Since $\lambda > 0$, this implies that $||L(\mathbf{v})|| = \lambda||\mathbf{v}||$ for all $\mathbf{v} \in \mathbb{R}^n$.

Conversely, if $||L(\mathbf{u})|| = \lambda||\mathbf{u}||$ for all $\mathbf{u} \in \mathbb{R}^n$, then for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we have

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= \frac{1}{4}||\mathbf{v} + \mathbf{w}||^2 - \frac{1}{4}||\mathbf{v} - \mathbf{w}||^2 \\
&= \frac{1}{4}\frac{1}{\lambda^2}||L(\mathbf{v} + \mathbf{w})||^2 - \frac{1}{4}\frac{1}{\lambda^2}||L(\mathbf{v} - \mathbf{w})||^2 \\
&= \frac{1}{\lambda^2}\left(\frac{1}{4}||L(\mathbf{v}) + L(\mathbf{w})||^2 - \frac{1}{4}||L(\mathbf{v}) - L(\mathbf{w})||^2\right) \\
&= \frac{1}{\lambda^2} L(\mathbf{v}) \cdot L(\mathbf{w}).
\end{aligned}$$

$\square$

Theorem. Suppose $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation.

Then the following two statements are equivalent:

(i) There is a $\lambda > 0$ such that $L(\mathbf{v}) \cdot L(\mathbf{w}) = \lambda^2(\mathbf{v} \cdot \mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$;

(ii) L preserves angles between non-zero vectors.

Proof. Suppose that $L(\mathbf{v}) \cdot L(\mathbf{w}) = \lambda^2(\mathbf{v} \cdot \mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Let θ be the angle between two non-zero vectors $\mathbf{v}$ and $\mathbf{w}$, and let $\tilde{\theta}$ be the angle between $L(\mathbf{v})$ and $L(\mathbf{w})$. Then, using above lemma,

$$\cos\tilde{\theta} = \frac{L(\mathbf{v}) \cdot L(\mathbf{w})}{||L(\mathbf{v})||\ ||L(\mathbf{w})||} = \frac{\lambda^2(\mathbf{v} \cdot \mathbf{w})}{(\lambda||\mathbf{v}||)\ (\lambda||\mathbf{w}||)} = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}||\ ||\mathbf{w}||} = \cos\theta.$$

So, the angles, being in the interval $[0, \pi]$, are equal.

Conversely, let us now assume that L preserves the angles between non-zero vectors. Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Since angles are preserved, the vectors $L(\mathbf{e_1}), L(\mathbf{e}_2), \cdots, L(\mathbf{e}_n)$ are non-zero orthogonal vectors. Consider the matrix A with the following columns

$$A = \left[\frac{L(\mathbf{e}_1)}{||L(\mathbf{e}_1)||}, \frac{L(\mathbf{e}_2)}{||L(\mathbf{e}_2)||}, \cdots, \frac{L(\mathbf{e}_n)}{||L(\mathbf{e}_n)||}\right].$$

It has the property that $A^T A = I$. That is, A is an orthogonal matrix. In particular, $(A^T\mathbf{v}) \cdot (A^T\mathbf{w}) = \mathbf{v}^T A A^T \mathbf{w} = \mathbf{v}^T\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. So that $||A^T\mathbf{v}|| = ||\mathbf{v}||$ for all $\mathbf{v} \in \mathbb{R}^n$. The same properties hold when A^T is replaced by A. Moreover,

$$A\mathbf{e}_i = \frac{L(\mathbf{e}_i)}{||L(\mathbf{e}_i)||}.$$

Consequently, $A^T L(\mathbf{e}_i) = c_i\mathbf{e}_i$ with $c_i = ||L(\mathbf{e}_i)|| > 0$. Therefore, for every $i \neq j$,

$$\begin{aligned}
\frac{(c_i\mathbf{e}_i + c_j\mathbf{e}_j) \cdot c_j\mathbf{e}_j}{(c_i^2 + c_j^2)^{1/2} c_j} &= \frac{(A^T L(\mathbf{e}_i + \mathbf{e}_j)) \cdot A^T L(\mathbf{e}_j)}{||A^T L(\mathbf{e}_i + \mathbf{e}_j)||\ ||A^T L(\mathbf{e}_j)||} \\
&= \frac{L(\mathbf{e}_i + \mathbf{e}_j) \cdot L(\mathbf{e}_j)}{||L(\mathbf{e}_i + \mathbf{e}_j)||\ ||L(\mathbf{e}_j)||} \\
&= \frac{(\mathbf{e}_i + \mathbf{e}_j) \cdot \mathbf{e}_j}{||\mathbf{e}_i + \mathbf{e}_j||\ ||\mathbf{e}_j||} = \frac{1}{\sqrt{2}}, \qquad (*)
\end{aligned}$$

because L preserves angles. Working out the leftmost expression of $(*)$, yields

$$2c_j^2 = c_i^2 + c_j^2.$$

Therefore, $c_i = c_j$ for all i and j. We put $\lambda = c_1 = c_2 = \cdots = c_n$. Then $L(\mathbf{e_i}) = \lambda A\mathbf{e}_i$ for all $i = 1, 2, \cdots, n$. Hence, by linearity of L and A, $L(\mathbf{v}) = \lambda A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$. So, $L(\mathbf{v}) \cdot L(\mathbf{w}) = (\lambda A\mathbf{v}) \cdot (\lambda A\mathbf{w}) = \lambda\mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. □