Some Basic Topological Concepts

Topology is the study of those properties of objects that are preserved under careful deformation. This note summarizes the basic topological terminology and concepts needed to make this vague statement precise.

For the entire discussion, let X be a subset of some Euclidean space $\mathbb{R}^n$. The standard Euclidean distance function

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

for two points $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ of X, has the following

Properties: For all $x, y, z \in X$ we have

(1) $d(x, y) \geqslant 0$;

(2) $d(x, y) = 0$ if and only if $x = y$;

(3) $d(x, y) = d(y, x)$;

(4) $d(x, y) \leqslant d(x, z) + d(z, y)$.

We call X together with the distance function d a *metric space.* In fact, one calls any set X along with any real-valued function $d(x, y)$, which satisfies above properties (1)–(4), a metric space.

Given a point $x \in X$ and a radius $\epsilon > 0$ we call the set

$$N_X(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

a *basic neighborhood of x in X.* A subset $N \subseteq X$ is simply called a *neighborhood of x in X* if it contains some basic neighborhood of x. We call a subset $U \subseteq X$ *open in X* if U is a neighborhood for all of its own points; that is, if for every $x \in U$ there is an $\epsilon > 0$ such that $x \in N_X(x, \epsilon) \subseteq U$. The collection of open sets of X has the following important

Properties:

(1) Both the empty set $\emptyset$ and X itself are open in X;

(2) If U and V are open in X, then so is $U \cap V$;

(3) The union of any collection of open sets of X is open in X.

We call X together with its collection of open sets a *topological space*, or simply a *space*. (Note that the same set U might be open in one space X but not in another.) Again, one calls any set X along with any collection of subsets (called the "open sets" of X), which satisfy above properties (1)–(3), a topological space.

Spaces are the objects of study in topology.

We say that an infinite sequence (x_k) in X *converges* to a point $x \in X$ if any given neighborhood U of x in X contains all but finitely many x_k; that is, if for every $\epsilon > 0$ there is an index K such that $d(x_k, x) < \epsilon$ for all $k \geqslant K$. As usual, we will denote this fact by

$$\lim_{k \to \infty} x_k = x.$$

A subset $C \subseteq X$ is called *closed in* X if sequences from C cannot converge to points of X that are outside of C; that is, if given any sequence (x_k) in C and any $x \in X$ with $\lim_{k \to \infty} x_k = x \in X$, we can always conclude that $x \in C$. Note that a subset of X need not be either, open or closed. Moreover, as it was the case for open sets, the same set C might be closed in one space X but not in another. An important relationship between the concepts of *open* and *closed* is the following

Property: Let $U \subseteq X$. Then U is open in X if and only if $X \setminus U$ is closed in X.

Summary: We observe that the topology of X can be described using either one of the following three methods: (1) the open sets of X; (2) the closed sets of X; or (3) the convergent sequences of X. A property of a (metric) space X which is phrased in terms of any one of these three concepts can therefore also be expressed by any of the other two. A property of the space X which can be described in terms of open sets is called a *topological property* of X.

In topology, the only properties of spaces that we study are topological properties.

Now consider a second subset $Y \subseteq \mathbb{R}^n$. We call a function $f : X \to Y$ *continuous at* $x \in X$, if for all sequences (x_k) in X with $\lim_{k \to \infty} x_k = x$, we have $\lim_{k \to \infty} f(x_k) = f(x)$. If f is continuous at every point x of X, then we call f *continuous*. The two topological spaces X and Y are called *topologically equivalent* (or *homeomorphic*) if there is a bijective continuous function $h : X \to Y$ whose inverse function $h^{-1} : Y \to X$ is also continuous. We call such an h a *homeomorphism*. If X and Y are topologically equivalent, we will write $X \approx Y$. Notice that under the correspondence of a homeomorphism all convergent sequences and their limits correspond. Consequently, if X and Y are topologically equivalent, then the open sets of X and the open sets of Y correspond. Therefore:

Topologically equivalent spaces share all topological properties.

Caution: It is important to note that the concept of topological equivalence through homeomorphism is an intrinsic one. It does not make any reference to the Euclidean space $\mathbb{R}^n$ of which X and Y are subsets. In other words, two topologically equivalent spaces are equivalent when viewed from within.

However, a homeomorphism is sometimes (but not always) the result of a gradual deformation over time of the surrounding Euclidean space. This stronger notion is captured in the following definition.

The two topological spaces X and Y are (*ambient*) *isotopic in* $\mathbb{R}^n$ if there is a continuous function $H : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ such that

(1) at every time $t \in [0,1]$, the function $H_t : \mathbb{R}^n \to \mathbb{R}^n$ defined by $H_t(x) = H(x,t)$ is a homeomorphism;

(2) $H_0(x) = x$ for all $x \in \mathbb{R}^n$;

(3) $H_1(X) = Y$.

This function H is called an *isotopy.*

Since an isotopy represents a homeomorphism between X and its image $H_t(X)$ at any given time t during the deformation, we remark the following:

> If one space can be deformed into another by an isotopy, then the two spaces are topologically equivalent and consequently share all topological properties.

However, X and Y might be topologically equivalent <u>without</u> being isotopic in $\mathbb{R}^n$. Moreover, the concept of isotopy depends on the choice of the ambient space $\mathbb{R}^n$. For example, two spaces X and Y that are subsets of the Euclidean plane might not be isotopic as subsets of $\mathbb{R}^2$ but might be isotopic within $\mathbb{R}^3$.

Compactness and Connectedness

Two of the most important topological properties are compactness and connectedness, to whose definition we come next.

We say that the space X is *compact* if from every sequence (x_k) in X we can extract a subsequence (x'_k) which converges to some point x in X; that is, if sequences in X cannot help but "crowd" some point of X. (Recall that a sequence (x'_k) is called a subsequence of a sequence (x_k), if (x'_k) is obtained from (x_k) by possibly omitting finitely or infinitely many members.)

For subsets of $\mathbb{R}^n$ there is an alternative way of characterizing compactness:

Theorem. A subset of $\mathbb{R}^n$ is compact if and only if it is closed in $\mathbb{R}^n$ and bounded.

(X is *bounded* if there is a radius $r > 0$ such that all points of X lie within a distance less than r from the origin.)

The space X is called *disconnected* if it can be written as $X = A \cup B$ with two disjoint non-empty sets A and B such that no sequence of A converges to a point of B and no sequence of B converges to a point of A. If X is not disconnected, then we call it *connected*.

Connectedness is a somewhat subtle concept. However, it is often easy to verify that a space is path connected. We say that X is *path connected*, if for every two points x and y in X there is a continuous function $p : [0, 1] \to X$ with $p(0) = x$ and $p(1) = y$. (We call p a *path* from x to y.)

Since the notions of compactness and connectedness are topological properties, they are preserved under homeomorphism. Interestingly enough, an even stronger statement is true:

Theorem. Suppose $f : X \to Y$ is a continuous <u>onto</u> function, but not necessarily a homeomorphism.

(1) If X is compact, then so is $f(X) = Y$.

(2) If X is connected, then so is $f(X) = Y$.

(3) If X is path connected, then so is $f(X) = Y$.

In short: The continuous image of a compact [resp. connected, path connected] space is compact [resp. connected, path connected].

We close by comparing the two concepts of connectedness:

Theorem.

(1) If the space X is path connected, then X is also connected.

(2) Conversely, suppose a space X is connected. Then X is path connected if and only if every point x of X has some path connected neighborhood N in X.