Stereographic Projection

Recall that *stereographic projection* $S : \Sigma \setminus \{N\} \to \mathbb{C}$ from the unit sphere Σ (minus the north pole $N = (0,0,1)$) onto the complex plane is given by

$$S(x,y,h) = \left(\frac{x}{1-h}, \frac{y}{1-h}\right) = \frac{x}{1-h} + \frac{y}{1-h}i.$$

Here, we use the notation $\Sigma = \{(x,y,h) \mid x^2+y^2+h^2 = 1\}$. From Problem 12 of Chapter 3 we know that its inverse $R : \mathbb{C} \to \Sigma \setminus \{N\}$ is given by

$$R(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2}\right). \tag{1}$$

In this note we shall prove the following

Theorem. *Stereographic projection maps circles of the unit sphere, which contain the north pole, to Euclidean straight lines in the complex plane; it maps circles of the unit sphere, which do not contain the north pole, to circles in the complex plane.*

Proof. Let a circle c on the unit sphere Σ be given. Then this circle c is the set of all points (x,y,h) on Σ, which lie on a slicing plane Π. The plane Π has some equation

$$Ax + By + Ch + D = 0, \tag{2}$$

with real numbers A, B, C, D such that $A^2 + B^2 + C^2 \neq 0$. In order to obtain an equation for the projection points $(x,y) \in \mathbb{C}$ of the circle c under stereographic projection, we substitute (1) into Equation (2), which yields

$$A\left(\frac{2x}{1+x^2+y^2}\right) + B\left(\frac{2y}{1+x^2+y^2}\right) + C\left(\frac{x^2+y^2-1}{1+x^2+y^2}\right) + D = 0.$$

Or, after rearranging the terms:

$$A2x + B2y + (x^2+y^2)(C+D) = C - D. \tag{3}$$

If the circle c contains the north pole $N = (0,0,1)$, then, substituting N into Equation (2), we have $C + D = 0$. So, in this case, Equation (3) says that the projection of c is a Euclidean straight line. If, on the other hand, the circle c does not contain the north pole, then $C+D \neq 0$. In this case we can divide Equation (3) by $C+D$ to get

$$x^2 + y^2 + \frac{2A}{C+D}x + \frac{2B}{C+D}y = \frac{C-D}{C+D}.$$

The latter equation, upon completing the square, becomes

$$\left(x + \frac{A}{C+D}\right)^2 + \left(y + \frac{B}{C+D}\right)^2 = \frac{A^2+B^2+C^2-D^2}{(C+D)^2}.$$

This last equation is the equation of a Euclidean circle in the complex plane of radius $R = \sqrt{\frac{A^2+B^2+C^2-D^2}{(C+D)^2}}$, centered at $(-\frac{A}{C+D}, -\frac{B}{C+D})$. Note that R is a real number, because $D^2 < A^2 + B^2 + C^2$: this follows from the fact that the distance from the origin to the plane Π must be less than 1, and from Calculus III, we know that this distance is given by

$$\frac{|D|}{\sqrt{A^2+B^2+C^2}}.$$

$\square$