The Well-Ordering Principle

The well-ordering principle is a concept which is equivalent to mathematical induction. In your textbook, there is a proof for how the well-ordering principle implies the validity of mathematical induction. However, because of the very way in which we constructed the set of natural numbers and its arithmetic, we deduced, in class, the validity of mathematical induction directly from the axioms of set theory. In this note, we show how mathematical induction, in turn, implies the well-ordering principle.

Theorem [The well-ordering principle].
Every non-empty subset of the natural numbers has a least element.

Proof. Let A be a non-empty subset of $\mathbb{N}$. We wish to show that A has a *least* element, that is, that there is an element $a \in A$ such that $a \leq n$ for all $n \in A$. We will do this by strong induction on the following predicate:

$$P(n) : \text{"If } n \in A\text{, then } A \text{ has a least element."}$$

Basic Step: $P(0)$ is clearly true, since $0 \leq n$ for *all* $n \in \mathbb{N}$.

Strong Inductive Step: We want to show that $[P(0) \wedge P(1) \wedge \cdots \wedge P(n)] \to P(n+1)$. To this end, suppose that $P(0), P(1), \cdots, P(n)$ are all true and that $n+1 \in A$. We consider two cases.

CASE 1: $\neg\exists m(m \in A \wedge m < n+1)$.

In this case, $n+1$ is the least element of A.

CASE 2: $\exists m(m \in A \wedge m < n+1)$.

In this case, since $P(m)$ is true, A has a least element.

Either way, we conclude that $P(n+1)$ is true.

So, by strong mathematical induction, we obtain that $P(n)$ is true for all $n \in \mathbb{N}$. Since A is not empty, we can pick an $n \in A$. Moreover, since $P(n)$ is true, this implies that A has a least element. $\square$