Million Dollar Mathematics

Holly Trietsch

Holly Trietsch graduated Summa Cum Laude from Ball State in May 2003 with a teaching major in Mathematics and a minor in Spanish. She is currently teaching mathematics at Carmel High School in Carmel, IN. Dr. John Lorch was her honors thesis advisor.

In David Hilbert's lecture in 1900, he presented 15 difficult problems that guided much of the research of mathematicians in the last century. Today, only 12 of these 15 problems have been solved. In honor of this great mathematician and to celebrate mathematics of the new millennium, the Clay Mathematics Institute of Cambridge, Massachusetts is offering a one million dollar prize for the solution to any one of the "Millennium Prize Problems" [7]. These problems, like Hilbert's problems, are long-standing mathematical questions that still have not been solved after many years of serious attempts by different experts. For my honors thesis project, I chose to research and study three of these Millennium Prize Problems: the Riemann Hypothesis, the P versus NP problem, and the Birch and Swinnerton-Dyer Conjecture. My entire honors thesis investigation of these very difficult problems is meant to explain the statements of each problem, provide background information, and explore related examples to establish a foundation about some of most significant and interesting mathematical problems of the new millennium.

The Riemann Hypothesis is one of Hilbert's unsolved problems and is now widely regarded as the most important open problem in pure mathematics. The Riemann Hypothesis is a proposition about the zeros of the Riemann zeta function, developed by G.F.B. Riemann (1826-1866):

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots ,$$

for a complex number s [3]. While the series version of the zeta function given above is only valid when $\mathrm{Re}(s) > 1$, the zeta function may be extended to the whole complex plane except $s = 1$ through analytic continuation. The zeros of the zeta function are the complex numbers s for which $\zeta(s) = 0$. The zeta function has zeros occurring at the negative even integers $\{\cdots, -6, -4, -2\}$; these are referred to as the trivial zeros [3]. All of the other zeros are called the nontrivial zeros and occur in the critical strip $0 < \text{Re}(s) < 1$ for complex number s [1]. The Riemann Hypothesis states that all zeros of the zeta function within the critical strip lie on the line $\text{Re}(s) = 1/2$ [3]. In 1986, it was shown that the first 1.5 billion nontrivial zeros are in fact on this line $\text{Re}(s) = 1/2$ [6]. Since the behavior of this Riemann zeta function is very closely related to the frequency of prime numbers, we will gain more knowledge about the distribution of primes if the Riemann Hypothesis is proved, in particular about the error term in the Prime Number Theorem.

The P versus NP problem is the biggest open question in computer science. P stands for polynomial time and is a class of languages solvable in polynomial time, where a language is simply a problem with a "yes/no" answer [4]. Polynomial time is when the execution time of a computation is no more than a polynomial function of the problem size. NP stands for nondeterministic polynomial time and is a class of languages that have a polynomial time verification algorithm [4]. In other words, a proposed solution to an NP problem can be checked in polynomial time, but to solve the problem may take longer than polynomial time. The P versus NP problem involves trying to prove whether these two classes of languages are in fact different or are actually the same. If it is proven that $P \neq NP$, then many important problems will be proven to be intractable (not solvable in polynomial time) [4]. If $P = NP$, then numerous problems will have much faster solution algorithms than what is now known. However, this would also jeopardize the security of public-key cryptography [2].

The Birch and Swinnerton-Dyer Conjecture deals with trying to determine the number of rational points on a given elliptic curve. In studying this conjecture, it is important to use the Hasse Principle which states that to answer any question about integer equations, reducing the equation modulo p (where p is prime) can be very useful [5]. Using this idea, we try to count the number Np of solutions modulo p to an equation determining an elliptic curve. Therefore, Np is the number of pairs $(x, y) \pmod{p}$ that satisfy an equation of the form equation $y^2 = x^3 + ax + b$. The Birch and Swinnerton-Dyer Conjecture resulted from studying the behavior of $P(x) = \prod_{p \leq x} \frac{Np}{p}$. From their numerical evidence, they conjectured that an elliptic curve of genus 1 has infinitely many rational points if and only if $P(x)$ goes to infinity as x increases [5]. This conjecture, if proven, would "fill the gap" between what is known about rational curves of genus 0 and curves of genus ≥ 2.

References

[1] J.B. Conway, Functions of One Complex Variables, Springer-Verlag (1978).

[2] S. Cook, *The P versus NP problem*:

`http://www.claymath.org/Millennium_Prize_Problems/P_vs_NP/_objects/Official_Problem_Description.pdf`

[3] A. Granville, *Prime Possibilities and Quantum Chaos*, Mathematical Sciences Research Institute Emissary Magazine (Spring 2002) 12–18.

[4] V. Ramachandran, *Lecture: The P versus NP Problem*, University of Texas at Austin (2001).

[5] F. Rodriguez-Villegas, *Lecture: The Birch and Swinnerton-Dyer Conjecture*, University of Texas at Austin (2001).

[6] `http://www.utm.edu/research/primes/notes/rh.html`

[7] `http://www.claymath.org`